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On the Genesis of the Marshall-Olkin Family of Distributions via the T-X Family Approach: Statistical Modeling

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Abstract: In the last couple of years, there Has been an increased interest among the statisticians to define new families of distributions by adding one or more additional parameter(s) to the baseline distribution. In this regard, a number of families have been introduced and studied. One such example is the Marshall-Olkin family of distributions that is one of the most prominent approaches used to generalize the existing distributions. Whenever, we see a new method, the natural questions come in to mind are (i) what are the genesis of the newly proposed method and (ii) how did the proposed method is obtained. No doubt, the Marshall-Olkin family is a very useful method and has attracted the researchers. But, unfortunately, the authors failed to provide the explanation about the genesis of the method that how this family of distributions is obtained. To address this issue, in this article, an attempt Has been made to provide a straight forward computation about the genesis of the Marshall-Olkin family that somehow completes its derivation. The genesis of the Marshall-Olkin family is based on the T-X family approach. Furthermore, we have showed that other extensions of the Marshall-Olkin family can also be obtained via the T-X family method. Finally, a real-life application form insurance science is presented to illustrate the newly proposed extension of the Marshall-Olkin family.

Keywords: Family of distributions; Marshall-Olkin family; T-X family approach; statistical modeling

1 Introduction

Statistical distributions are very worthwhile in describing and predicting real-world phenomena. In the recent era, several new statistical distributions have been proposed via extending the classical distributions for modelling data in applied areas, particularly, in the practice of insurance and economics [1–7], engineering [8–10] and biological studies [11–15], among others. However, the traditional distributions are not flexible enough to model real world phenomena of nature. Therefore, it is always of interest to obtain the extended versions of the existing distributions for modeling real life data. This has been done through many different approaches [16,17]. One such

approach of generalizing the traditional distributions is the Marshall-Olkin (MO) family approach. The cumulative distribution function (cdf) of the MO family is given by

$$G(x;\sigma;\xi) = \frac{F(x;\xi)}{1-(1-\sigma)[1-F(x;\xi)]}, \quad \sigma > 0, \ x, \ \xi \in \mathbb{R}, \tag{1}$$

where $F(x;\xi)$ is the cdf of the baseline distribution may depend on the vector parameter $\xi \in \mathbb{R}$, and σ is an additional parameter. The probability density function (pdf) corresponding to Eq. (1) is

$$g(x;\sigma,\xi) = \frac{\sigma f(x;\xi)}{\{1-\overline{\sigma}[1-F(x;\xi)]\}^2}, \quad x \in \mathbb{R}, \tag{2}$$

where $\overline{\sigma} = 1 - \sigma$.

The MO family method has been used extensively to generalize the existing distributions. The MO family of distributions is one of the most prominent approaches used to generalize the existing distributions. But the authors did not provide the genesis and the derivation of the MO family and hence the derivation of the paper is not completed. In this article, an effort has been made to provide the genesis and straight forward computation of the MO family. There may also be other methods through which the MO family can be obtained. However, this article offers the derivation of the MO family via the T-X family approach [18].

This paper is organized as follows: The genesis of the MO family is presented in Section 2. Further developments are discussed in Section 3. A special sub-case of the newly proposed family is presented in Section 4. A real-life application to financial sciences is presented in Section 5. Finally, the paper is concluded in the last section.

2 Genesis of the Marshall-Olkin Family of Distributions

This section offers the derivation of the MO family of distributions via the approach of T-X family. Let $v(t)$ be the pdf of a random variable, say T, where $T \in [m,n]$ for $-\infty \leq m < n < \infty$ and let $W[F(x;\xi)]$ be a function of cdf of a random variable, say X, depending on the vector parameter ξ satisfying the conditions given below:

   i $W[F(x;\xi)] \in [m,n]$
   ii $W[F(x;\xi)]$ is differentiable and monotonically increasing, and
   iii $W[F(x;\xi)] \to m$ as $x \to -\infty$ and $W[F(x;\xi)] \to n$ as $x \to \infty$.

Recently, Alzaatreh et al. [18] defined the cdf of the T-X family of distributions by

$$G(x) = \int_m^{W[F(x;\xi)]} v(t)\,dt, \quad x \in \mathbb{R}, \tag{3}$$

where $W[F(x;\xi)]$ satisfies the conditions stated above. The pdf corresponding to Eq. (3) is

$$g(x) = \left\{\frac{\partial}{\partial x}W[F(x;\xi)]\right\} v\{W[F(x;\xi)]\}, \quad x \in \mathbb{R}.$$

Using the T-X idea, several new classes of distributions have been introduced in the literature. Let T follows the exponential distribution with θ, and then its cdf is given by

$$V(t) = 1 - e^{-\theta t}, \quad t \geq 0, \ \theta > 0. \tag{4}$$

If $\theta = 1$, then from Eq. (4), we have

$$v(t) = 1 - e^{-t}, \quad t \geq 0. \tag{5}$$

The pdf corresponding to Eq. (5) is

$$v(t) = e^{-t}, \quad t > 0. \tag{6}$$

If T follows Eq. (6) and setting $W(x; \xi) = -\log\left(1 - \left(\dfrac{F(x; \xi)}{1 - (1 - \sigma)(1 - F(x; \xi))}\right)\right)$ in Eq. (3), we have

$$G(x; \sigma, \xi) = \int_0^{-\log\left(1 - \left(\frac{F(x; \xi)}{1 - (1 - \sigma)(1 - F(x; \xi))}\right)\right)} e^{-t} dt, \quad \sigma > 0, \ x, \ \xi \in \mathbb{R}.$$

On solving, we get

$$G(x; \sigma, \xi) = -\left\{1 - \left(\dfrac{F(x; \xi)}{1 - (1 - \sigma)(1 - F(x; \xi))}\right) - 1\right\}, \quad \sigma > 0, \ x, \ \xi \in \mathbb{R}.$$

$$G(x; \sigma, \xi) = -1 + \left(\dfrac{F(x, \sigma, \xi)}{1 - (1 - \sigma)(1 - F(x; \sigma, \xi))}\right) + 1, \quad \sigma > 0, \ x, \ \xi \in \mathbb{R},$$

$$G(x; \sigma, \xi) = \dfrac{F(x; \sigma, \xi)}{1 - (1 - \sigma)(1 - F(x; \sigma, \xi))}, \quad \sigma > 0, \ x, \ \xi \in \mathbb{R}. \tag{7}$$

The Eq. (7) which is equal to Eq. (1) provides the cdf of the MO family.

3 Other Extensions of the MO Family

Other extensions of the MO family can also be obtained through the T-X approach. For example, the Kumaraswamy version, exponentiated version and alpha power transformed version, etc.

3.1 Exponentiated Version of the MO Family

If $v(t)$ be pdf given by Eq. (6) and letting $W(x) = -\log\left(1 - \left(\dfrac{F(x; \xi)}{1 - (1 - \sigma)(1 - F(x; \xi))}\right)^a\right)$ in Eq. (3), we get the exponentiated Marshall-Olkin (EMO) family given by

$$G(x; a, \sigma, \xi) = \left(\dfrac{F(x; \xi)}{1 - (1 - \sigma)(1 - F(x; \xi))}\right)^a, \quad a, \sigma > 0, \ x, \ \xi \in \mathbb{R}, \tag{8}$$

where $a > 0$ is an additional shape parameter. The density function corresponding to Eq. (8) can easily be obtained simply by differentiation.

3.2 Kumaraswamy Version of the MO Family

This section offers the derivation of the Kumaraswamy version of the MO family of distributions. If $v(t)$ represent the density function Eq. (6) and letting

$$W(x) = -\log\left(1 - \left\{1 - \left(\frac{F(x;\xi)}{1-(1-\sigma)(1-F(x;\xi))}\right)^a\right\}^b\right)$$ in Eq. (3), we get the Kumaraswamy

Marshall-Olkin (Ku-MO) family given by

$$G(x,a,b,\sigma,\xi) = 1 - \left\{1 - \left(\frac{F(x;\xi)}{1-(1-\sigma)(1-F(x;\xi))}\right)^a\right\}^b, \quad a, b, \sigma > 0, x, \xi \in \mathbb{R}. \tag{9}$$

The density function of the Ku-MO family can be obtained by differentiating Eq. (9).

3.3 Alpha Power Transformed Version of the MO Family

If $v(t)$ follows Eq. (6) and letting $W(x) = -\log\left(\dfrac{\alpha^{\frac{F(x;\xi)}{1-(1-\sigma)(1-F(x;\xi))}} - 1}{\alpha - 1}\right)$ in Eq. (3), we get the

alpha power transformed version of the MO family given by

$$G(x;\alpha,\sigma,\xi) = \frac{\alpha^{\frac{F(x;\xi)}{1-(1-\sigma)(1-F(x;\xi))}} - 1}{\alpha - 1}, \quad \sigma, \alpha > 0, \alpha \neq 1, x, \xi \in \mathbb{R}. \tag{10}$$

On differentiating Eq. (10), we get the density function of the alpha power transformed Marshall-Olkin (APTMO) family.

3.4 Exponentiated Version of the Alpha Power Transformed Marshall-Olkin Family

If $v(t)$ follows Eq. (6) and letting $W(x) = -\log\left(\dfrac{\alpha^{\left(\frac{F(x;\xi)}{1-(1-\sigma)(1-F(x;\xi))}\right)^a} - 1}{\alpha - 1}\right)$ in Eq. (3), we get the

alpha power transformed version of the EMO family given by

$$G(x;a,\alpha,\sigma,\xi) = \frac{\alpha^{\left(\frac{F(x;\xi)}{1-(1-\sigma)(1-F(x;\xi))}\right)^a} - 1}{\alpha - 1}, \quad \sigma, \alpha > 0, \alpha \neq 1, x, \xi \in \mathbb{R}. \tag{11}$$

4 A Sub-Model Description of the APTMO Family

In this section, we investigate a special sub-case of the APTMO family. The density function corresponding to Eq. (10) is given by

$$g(x;\alpha,\sigma,\xi) = \frac{\sigma f(x;\xi)}{(\log\alpha)\{1 - \overline{\sigma}[1 - F(x;\xi)]\}^2}\alpha^{\frac{F(x;\xi)}{1-(1-\sigma)(1-F(x;\xi))}}, \quad \alpha, \sigma > 0, \alpha \neq 1, x, \xi \in \mathbb{R}.$$

Consider the distribution and density functions of the Weibull random variable given by $F(x;\xi) = 1 - e^{-\gamma x^\theta}, x, \theta, \gamma > 0$, and $f(x;\xi) = \theta\gamma x^{\alpha-1}e^{-\gamma x^\theta}$, where, $\xi = (\theta,\gamma)$. Then, the cdf of the APTMO-Weibull distribution is given by

$$G(x;\alpha,\sigma,\xi) = \frac{\alpha^{\frac{1-e^{-\gamma x^\theta}}{1-(1-\sigma)e^{-\gamma x^\theta}}} - 1}{\alpha - 1}, \quad \sigma, \alpha > 0, \alpha \neq 1, x, \xi \in \mathbb{R}. \tag{12}$$

The density function corresponding to Eq. (12) is given by

$$g(x;\alpha,\sigma,\xi) = \frac{\sigma\theta\gamma x^{\theta-1}e^{-\gamma x^\theta}}{(\log\alpha)\left[1-\bar{\sigma}e^{-\gamma x^\theta}\right]^2}\alpha^{\left\{\frac{\left(1-e^{-\gamma x^\theta}\right)}{1-(1-\sigma)e^{-\gamma x^\theta}}\right\}}, \quad \alpha,\ \sigma > 0,\ \alpha \neq 1,\ x,\ \xi \in \mathbb{R}. \tag{13}$$

For (i) $\sigma = 0.5$, $\gamma = 1$ and (ii) $\sigma = 1$, $\gamma = 0.5$ and different values of α and θ, plots for the pdf of the APTMO-Weibull distribution are sketched in Fig. 1.
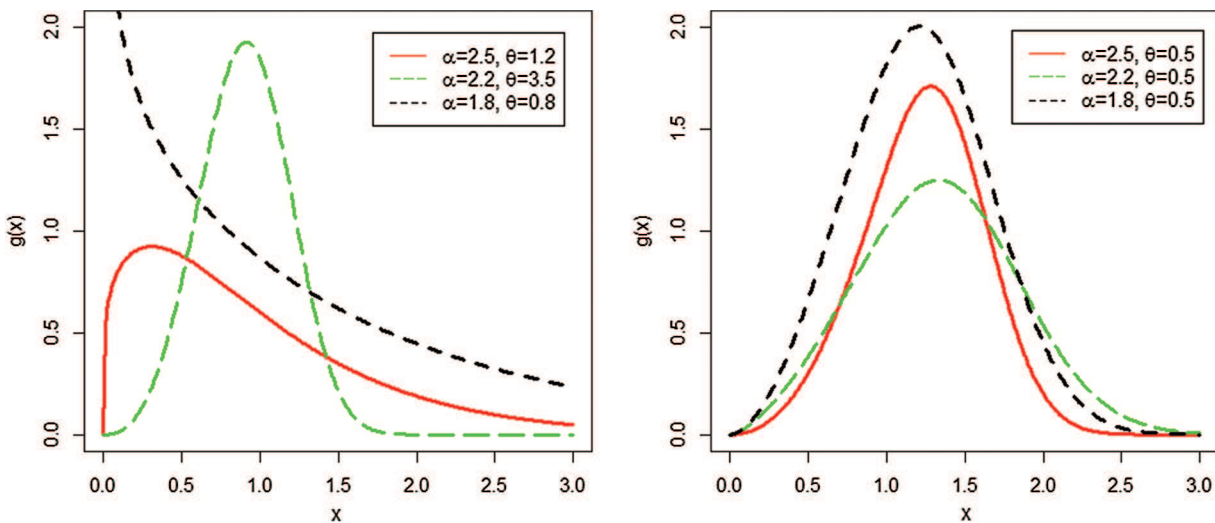

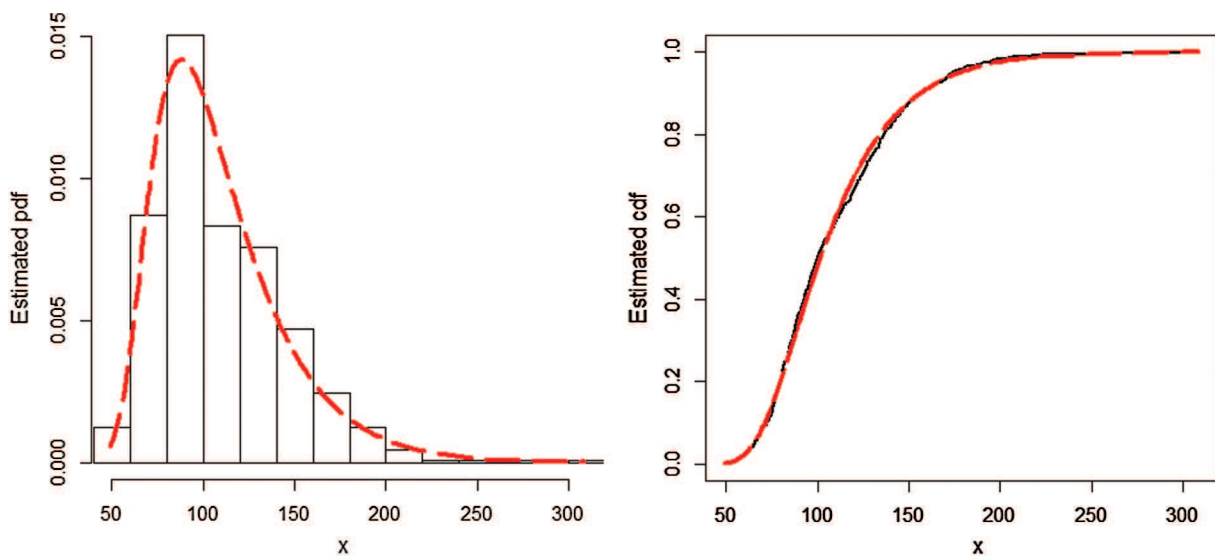**Figure 1:** Density plots of the APTMO-Weibull distribution

Figure 2: The fitted pdf and cdf of the APTMO-Weibull distribution

5 Statistical Modeling

In this section, we demonstrate the flexibility of the APTMO-Weibull distribution by using a heavy-tailed insurance claims data. The data set represents the unemployment insurance initial claims per month from 1971 to 2018, and it is available at https://data.worlddatany-govns8z-xewg.

By applying the APTMO-Weibull distribution to this data, we observed that the APTMO-Weibull model can be used quite effectively to provide the best description of the real phenomena of nature. Corresponding to the insurance claims data, the fitted cdf and density plots of the APTMO-Weibull ($\alpha = 1.2$, $\gamma = 0.8$, $\theta = 0.5$, $\sigma = 0.9$) are presented in Fig. 2. The probability-probability (PP) plot and Kaplan–Meier survival plots are sketched in Fig. 3.

The graphical sketching provided in Fig. 3 indicate that the APTMO-Weibull distribution provides the better fit and could be chosen as an adequate model to analyze the heavy-tailed insurance claims data.
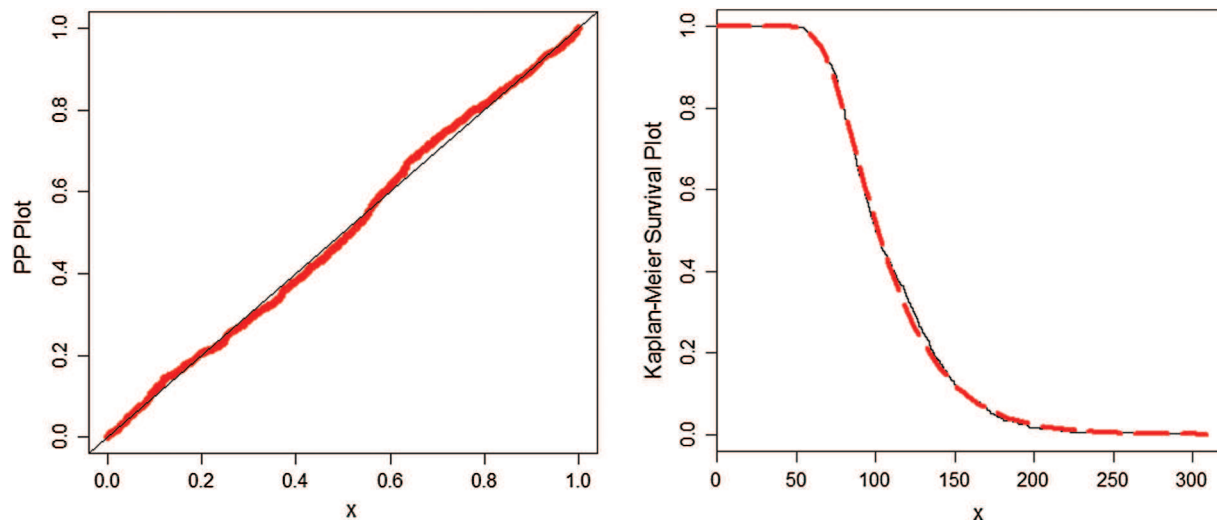

Figure 3: The PP plot and Kaplan–Meier survival plots of the APTMO-Weibull distribution

6 Concluding Remarks

In this article, a straight forward computation of the genesis of the MO family of distributions is provided. It is showed that the MO family of distributions can be obtained via the T-X family approach. It is also showed that the other extensions of the MO family such as the exponentiated version and the Kumaraswamy version can be obtained using the T-X method. Furthermore, a sub-case of the APTMO family is considered and a real data set from the insurance sciences is analyzed. The real-life application show that the newly proposed APTMO-Weibull distribution can be used quite effectively to model data in insurance sciences and other related fields. We hope that the method of genesis provided in this paper will attract the researchers and will use it to obtain the extensions of the other existing family of distributions.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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