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3-D Temperature Fields in Laminated Shells Subjected to Thermo-Loads

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Abstract: The temperature fields in the laminated shells were studied, including open cylindrical shells and cylindrical shells, according to the thermal theory. Analytical solution of the temperature in the shells with the known temperature on the surfaces was present. The thinning layer approach was introduced to simplify the three-dimensional heat conduction equation. Firstly, the layered shell was divided into N thinner layers. The governing equation was simplified by replacing the variable r by r_0 in the center line of every thin layer. The general solutions of temperature satisfying the simplified three-dimensional governing equation in single-layered shell were deduced in the cylindrical coordinate system. Then, the temperature and heat flux relationships between the surfaces could be found by transferring matrixes. According to the continuities of temperature and heat flux in the interface of the laminates, the temperature and heat flux relationships of the surfaces were derived. With the temperature condition on the surfaces, the unknown coefficients in the general solution of temperature were obtained. Finally, the effects from the thinning layer approach were eliminated by analyzing different numbers of thin layers. The validity and accuracy of the proposal method were proved from the convergence and comparison studies. And several numerical examples were studied to investigated the temperature effects from surface temperatures, geometric size of the shells and composition of layers.

Keywords: Analytical solution, temperature, laminated Shells, the analytical method, the transfer matrix method.

1 Introduction

The mechanical properties of layered cylindrical shells have attracted considerable research efforts due to their increasing applications in engineering. Owing to the inhomogeneous material properties among the layers, thermal stresses and inhomogeneous deformations emerge in the temperature environment. This is not beneficial to the structural safety.

The research on temperature distribution in layered cylindrical shell attracts a lot of

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interest. Ostrowski et al. [Ostrowski and Jędrysiak (2017)] studied the temperature distribution of a quarter-phase laminates with periodic distributions. Vidal et al. [Vidal, Gallimard, Ranc et al. (2017)] present a method to obtain the temperature distribution of laminates or sandwich beams in a temperature environment whose heat source in a certain position. Kantor et al. [Kantor, Smetankina and Shupikov (2001)] present a method for determination of thermal condition of laminated elements of structures. Norouzi et al. [Norouzi, Niya, Kayhani et al. (2012)] gave an analytical solution of temperature in a composite cylinder subjected to steady thermo-loads. Shupikov et al. [Shupikov, Smetankina and Svet (2007)] present an analytical solution of unsteady temperature filed in layered plate with complex planar shape. Kayhani et al. [Kayhani, Shariati, Nourozi et al. (2009)] proposed an exact solution of steady temperature filed in composite cylinders. Desgrosseilliers et al. [Desgrosseilliers, Groulx and White (2013)] present a model to predict the action under a constant temperature or heat flows in laminate films. Abdelal et al. [Abdelal, Abdalla and Gurdal (2010)] studied the thermal behavior of Variable Stiffness laminates with finite element method. Matysiak et al. [Matysiak and Perkowski (2011)] studied a regularly laminates with a cylindrical bore. They gave the temperature distribution and the distribution of heat flux on the basis of the Weber-Orr integral transforms. Tarn et al. [Tarn and Wang (2004)] studied heat-conducting property of composite and functionally graded materials cylinders, especially the end effect. Nemirovskii et al. [Nemirovskii and Yankovskii (2008)] analyzed unsteady thermal conductivity in the inhomogeneous layered shell and studied the asymptotic properties of thermal conductivity. Kulikov et al. [Kulikov and Plotnikova (2014)] used the method of sampling surfaces to analyze the exact three-dimensional heat conduction of laminated orthotropic and anisotropic shells. Delouei et al. [Delouei and Norouzi (2015)] present an exact solution of unsteady-state heat transfer in spherical laminates reinforced with composite. Kaminski [Kaminski (2003)] studied the homogenization about transient heat conduction problems in some composite materials. Mityushev et al. [Mityushev, Obnosov, Pesetskaya et al. (2008)] described a method of heat conduction in various type of composite materials. Delouei et al. [Delouei, Kayhani and Norouzi (2012)] studied the problem of transient heat transfer in the composite cylinders and gave an analytical solution. Kayhani et al. [Kayhani, Norouzi and Delouei (2012)] proposed an analytical solution for steady heat conduction in composite cylinders. Beck et al. [Beck, Wright, Haji-Sheikh et al. (2008)] gave an efficient procedure to solve the problem of poorly-convergence in heat conduction solutions. Ma et al. [Ma and Chang (2004)] proposed an analytic method to solve heat transfer problem in anisotropic laminated media. Savoia et al. [Savoia and Reddy (1995)] considered the polynomial and exponential temperature distributions through the thickness and presented the temperature analysis for multilayered plates subjected to thermal loads. Hsieh et al. [Hsieh and Ma (2002)] provided the analytical solution of heat transfer problem in a non-isotropic thin media with a heat source inside. Haji-Sheikh et al. [Haji-Sheikh, Beck and Agonafer (2003)] presented general solutions of temperature in laminates in a steady-state temperature environment. Norouzi et al. [Norouzi, Amiri and Seilsepour (2013)] studied spherical laminates reinforced with the fiber and solved the heat conduction problems under different temperature boundary condition. Liu et al. [Liu, Zhang, Cheng et al. (2016)] studied the thermal deformation of copper-steel wall in the process of smelting. Yang et al. [Yang and Liu (2017)] provided a general solution of heat transfer problems in composite cylinders in a temperature environment. Wang et al. [Wang, Miao and Zhu (2013)] proposed a new solution for thermal analysis of composites on the basis of hybrid boundary node method.

In this paper, exact analytical solutions of temperature in laminated shells are present from the heat conduction equation. The thin layer approach is introduced to simplify the temperature governing equation in the cylindrical coordinate system. Firstly, the general solutions of temperature satisfying the simplified three-dimensional governing equation in single-layered shell were deduced. Then, the temperature and heat flux relationships between the surfaces could be found by transferring matrixes. According to the continuities of temperature and heat flux in the adjacent interface of the laminates, the temperature and heat flux relationships of the surfaces were derived. With the temperature condition on the surfaces, the unknown coefficients in the general solution of temperature were obtained. There are two novelties in this paper: (1) This paper introduced the thinning layer approach to simplify the heat conduction equation. Then the effect is eliminated by the layer analysis and the exact solution can be obtained. (2) The presented method is appropriate for cylinders with any number of layers. And the solutions are derived only by operating a second order system of primary equations.

2 Solutions of temperature field in the open cylindrical shell

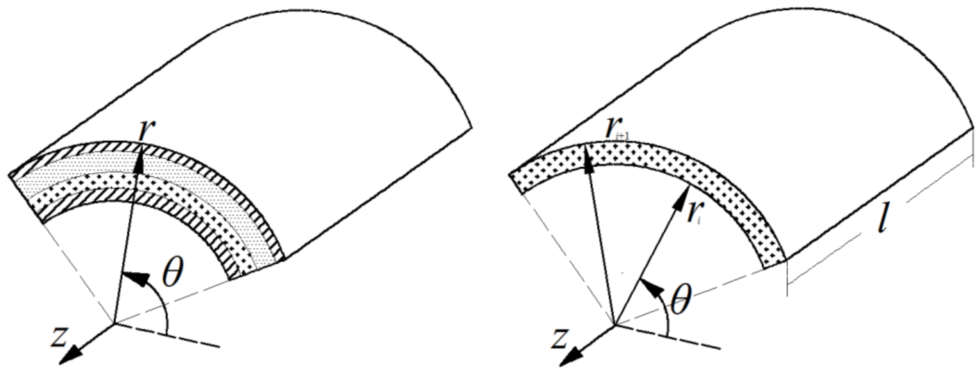
2.1 The general solution of temperature in a single-layer shell

An open cylindrical shell composed of H shelled laminations with radius r_j (j=0, 1, 2... H) and the angle θ^0 is considered, as shown in Fig. 1(a). The length of the shell is l. The known temperature value at the four edges of the open shell is same. We set the temperature as the baseline value of the temperature in the shell. The temperature of the outside surface in the laminated shell is $t_p(\theta, z)$, while the temperature of the internal surface is $t_1(\theta, z)$.

Now, the single-layer in the open cylindrical shell is considered individually (see Fig. 1(b)). In the cylindrical coordinate system r-θ-z, the three-dimensional equation of heat conduction is

$$\frac{\partial^2 T(r,\theta,z)}{\partial r^2}+\frac{1}{r}\frac{\partial T(r,\theta,z)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T(r,\theta,z)}{\partial \theta^2}+\frac{\partial^2 T(r,\theta,z)}{\partial z^2}=0 \tag{1}$$

Here, the thinning layer approach is introduced to simplify the equation. Each layer in the open shell is divided into H_j thinner layer. The laminated shell is divided into P ($P=\sum_{j=1}^{H} H_j$) layers. And the material properties of every layer in the new laminated open cylindrical shell are can be obtained easily. The heat conductivity is k_i (i=1, 2…P). In such case, the variable r in the governing equation can be approximately replaced by the center coordinate r of a layer. When the numbers of partitions increase, the accuracy of results increase. If the layer is thin enough, the solution required for accuracy can be obtained.



(a) multilayer shell (b) an isotropic shelled layer

Figure 1: Geometry of the open cylindrical laminated shell

The simplified governing equation of the ith (i=1, 2…P) layer is

$$\frac{\partial^2 T_i(r,\theta,z)}{\partial r^2}+\frac{1}{r_{i0}}\frac{\partial T_i(r,\theta,z)}{\partial r}+\frac{1}{r_{i0}^2}\frac{\partial^2 T_i(r,\theta,z)}{\partial \theta^2}+\frac{\partial^2 T_i(r,\theta,z)}{\partial z^2}=0 \tag{2}$$

From the temperature condition at the four edges of the shell

$$T_i\left(r,0,z\right)=T_i\left(r,\theta_0,z\right)=0, \qquad T_i\left(r,\theta,0\right)=T_i\left(r,\theta,l\right)=0 \tag{3}$$

According to condition of the temperature (3), the temperature distribution $T_i(r, \theta, z)$ can be derived

$$T_i(r,\theta,z)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}t_{mni}\left(r\right)\sin\left(\frac{m\pi\theta}{\theta_0}\right)\sin\left(\frac{n\pi z}{l}\right) \tag{4}$$

Eq. (4) is substituted into governing Eq. (2), differential equation of $t_{mn}(r)$ can be obtained.

$$t_{mn}^{"}\left(r\right)+\frac{1}{r_0}t_{mn}^{'}\left(r\right)-\left(\frac{m^2\pi^2}{\theta_0^2 r_0^2}+\frac{n^2\pi^2}{l^2}\right)t_{mn}\left(r\right)=0 \tag{5}$$

Finally, we can derive $T_i(r, \theta, z)$ according to Eqs. (4) and (5):

$$T_i(r,\theta,z)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(e^{\alpha_{mni}r}G_{mni}+e^{\beta_{mni}r}H_{mni}\right)\sin\left(\frac{m\pi\theta}{\theta_0}\right)\sin\left(\frac{n\pi z}{l}\right) \tag{6}$$

where,

$$\alpha_{mni}=\frac{\sqrt{\frac{4m^2\pi^2}{\theta_0^2}+\frac{4n^2\pi^2 r_{i0}^2}{l^2}+1}-1}{2r_{i0}}, \quad \beta_{mni}=\frac{-\sqrt{\frac{4m^2\pi^2}{\theta_0^2}+\frac{4n^2\pi^2 r_{i0}^2}{l^2}+1}-1}{2r_{i0}}$$

G_{mni} and H_{mni} are the undetermined constants. We can work out them with the temperature conditions on the surfaces of each layer.

2.2 Recursive formulae for temperature field

With the interlayer continuities of temperature condition in the interface of the laminated open cylindrical shell, we can get

$$T_{i+1}(r_i,\theta,z)=T_i(r_i,\theta,z),$$

$$k_{i+1}\left.\frac{\partial T_{i+1}(r,\theta,z)}{\partial r}\right|_{r=r_i}=k_i\left.\frac{\partial T_i(r,\theta,z)}{\partial r}\right|_{r=r_i}\quad (i=1,\,2,\,3\ldots P\text{-}1) \tag{7}$$

In order to facilitate the derivation, the temperature solution and the heat flux in the direction of r can be expressed as the matrix form from Eq. (6), i.e.

$$\begin{bmatrix} T_i(r,\theta,z) \\ k_i\dfrac{\partial T_i(r,\theta,z)}{\partial r}\end{bmatrix}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\phi_{mn}^i(r)\right]\sin\left(\frac{m\pi\theta}{\theta_0}\right)\sin\left(\frac{n\pi z}{l}\right) \tag{8}$$

where $\left[\phi_{mn}^i(r)\right]$ is a matrix function about variable r with the undetermined coefficients, which can be expressed as from temperature distribution (6),

$$\left[\phi_{mn}^i(r)\right]=\left[\varphi_{mn}^i(r)\right]\left[\Lambda_{mn}^i\right] \tag{9}$$

in which, $\left[\varphi_{mn}^i(r)\right]$ and $\left[\Lambda_{mn}^i\right]$ are

$$\left[\varphi_{mn}^i(r)\right]=\begin{bmatrix} e^{\alpha_{mni}r} & e^{\beta_{mni}r} \\ k_i\alpha_{mni}e^{\alpha_{mni}r} & k_i\beta_{mni}e^{\beta_{mni}r}\end{bmatrix},\quad \left[\Lambda_{mn}^i\right]=\begin{bmatrix} G_{mni} \\ H_{mni}\end{bmatrix} \tag{10}$$

Taking r=r_i and r=r_{i+1} in Eq. (9) respectively, the unknown coefficients $\left[\Lambda_{mn}^i\right]$ in Eq. (10) can be eliminated. Therefore, one has

$$\left[\phi_{mn}^{i+1}(r_{i+1})\right]=\left[\varphi_{mn}^{i+1}(r_{i+1})\right]\left[\varphi_{mn}^{i+1}(r_i)\right]^{-1}\left[\phi_{mn}^{i+1}(r_i)\right] \tag{11}$$

The interlayer continuity in Eq. (7) is expressed as a matrix equation,

$$\left[\phi_{mn}^{i+1}(r_i)\right]=\left[\phi_{mn}^i(r_i)\right] \tag{12}$$

The relationship of the undetermined coefficients between the qth layer (q=2…P) and the first layer can be recursively worked out with the interlayer continuity.

$$\begin{bmatrix} G_{mnq} \\ H_{mnq}\end{bmatrix}=\left[\varphi_{mn}^q(r_q)\right]^{-1}\left\{\prod_{j=q}^{1}\left[\varphi_{mn}^j(r_j)\right]\left[\varphi_{mn}^j(r_{j-1})\right]^{-1}\right\}\left[\varphi_{mn}^1(r_0)\right]\begin{bmatrix} G_{mn1} \\ H_{mn1}\end{bmatrix} \tag{13}$$

2.3 Undetermined coefficients in the analytical solution

Assume that the temperature distributions of the external surface and the internal surface of the layered shell are $t_p(\theta,z)$ and $t_1(\theta,z)$ respectively, i.e.

$$T_1(r_0,\theta,z)=t_1(\theta,z),\qquad T_p(r_p,\theta,z)=t_p(\theta,z) \tag{14}$$

Substitute Eq. (6) into the temperature condition Eq. (14) and multiply Eq. (14) by $\sin(\frac{i\pi\theta}{\theta_0})\sin(\frac{j\pi z}{l})$. Finally, the equation is integrated over z and θ at two sides. We can get

$$\int_0^l\int_0^{\theta_0} T_1(r_0,\theta,z)\sin(\frac{i\pi\theta}{\theta_0})\sin(\frac{j\pi z}{l})d\theta dz = \int_0^l\int_0^{\theta_0} t_1(\theta,z)\sin(\frac{i\pi\theta}{\theta_0})\sin(\frac{j\pi z}{l})d\theta dz\ ,$$

$$\int_0^l\int_0^{\theta_0} T_p(r_p,\theta,z)\sin(\frac{i\pi\theta}{\theta_0})\sin(\frac{j\pi z}{l})d\theta dz = \int_0^l\int_0^{\theta_0} t_p(\theta,z)\sin(\frac{i\pi\theta}{\theta_0})\sin(\frac{j\pi z}{l})d\theta dz \tag{15}$$

According to temperature analytic formula (6), Eq. (15) can be written as

$$e^{\alpha_{mn1}r_0}G_{mn1} + e^{\beta_{mn1}r_0}H_{mn1} = \frac{4}{\theta_0 l}\int_0^l\int_0^{\theta_0} t_1(\theta,z)\sin(\frac{m\pi\theta}{\theta_0})\sin(\frac{n\pi z}{l})d\theta dz,$$

$$e^{\alpha_{mnp}r_p}G_{mnp} + e^{\beta_{mnp}r_p}H_{mnp} = \frac{4}{\theta_0 l}\int_0^l\int_0^{\theta_0} t_p(\theta,z)\sin(\frac{m\pi\theta}{\theta_0})\sin(\frac{n\pi z}{l})d\theta dz \tag{16}$$

Simultaneously solving Eq. (13) (taking q=P) and Eq. (16), G_{mn1}, H_{mn1}, G_{mnp} and H_{mnp} can be uniquely determined. The undetermined coefficients G_{mnq} and H_{mnq} of the qth layer (q=2…P) can be calculated by Eq. (13). Then, the temperature distribution in the laminated open shell is obtained by substituting the coefficients into the analytical solution (6).

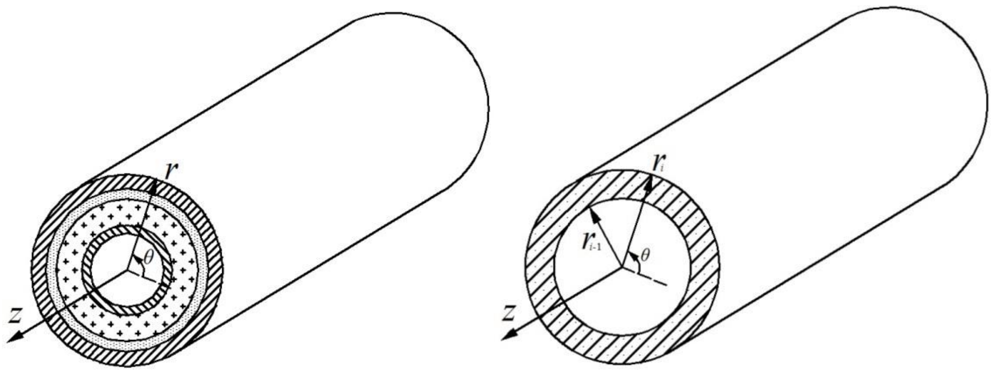
3 Solutions of temperature field in the cylindrical shell

3.1 The general solution of temperature in a single-layer shell

Consider a cylinder of radius r_j (j=0, 1, 2... H) and the angle θ^0, as shown in Fig. 2(a). The cylinder is composed of H layers. The length of the shell is l. The known temperature value at the two edges of the shell is same. We set the temperature as the baseline value of the temperature in the shell. The temperature of the outside surface in the laminated shell is $t_p(\theta, z)$, while the temperature of the internal surface is $t_1(\theta, z)$.

Now, the single-layer in the open cylindrical shell is considered individually (see Fig. 2(b)). In the cylindrical coordinate system r-θ-z, the three-dimensional equation of heat conduction is the same with Eq. (1).

Here, the thinning layer approach is also introduced to simplify the equation. Each layer in the cylinder is divided into H_j thinner layer. The laminated shell is divided into P ($P=\sum_{j=1}^{H}H_j$) layers. And the material properties of every layer in the new laminated cylinder are can be obtained easily. The heat conductivity is k_i (i=1, 2…P). In such case, the variable r in the governing equation can be approximately replaced by the center coordinate r of a layer. When the numbers of partitions increase, the accuracy of results increase. If the layer is thin enough, the solution required for accuracy can be obtained. The simplified heat conduction equation is the same as Eq. (2).



(a) multi-layer cylindrical shell (b) an isotropic layer

Figure 2: Geometry of multi-layer cylindrical shell

From the temperature condition at the two edges of the shell

$$T_i\left(r,\theta,0\right)=T_i\left(r,\theta,l\right)=0 \tag{17}$$

According to condition of the temperature (17), the temperature distribution can be given in the complete Fourier series form of

$$T_i\left(r,\theta,z\right)=\sum_{n=1}^{\infty}t_n^i\left(r\right)\sin\left(\frac{n\pi z}{l}\right)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}^i\left(r\right)\cos\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}b_{mn}^i\left(r\right)\sin\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right) \tag{18}$$

Eq. (18) is substituted into Eq. (2), differential equations of $t_n^i\left(r\right)$, $a_{mn}^i\left(r\right)$ and $b_{mn}^i\left(r\right)$ can be obtained. Finally, we can derive $T_i(r, \theta, z)$ according to Eq. (18).

$$\begin{aligned}T_i\left(r,\theta,z\right)=&\sum_{n=1}^{\infty}\left(e^{\alpha_n^i r}H_{ni}^1+e^{\beta_n^i r}H_{ni}^2\right)\sin\left(\frac{n\pi z}{l}\right)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(e^{\alpha_{mn}^i r}H_{mni}^3+e^{\beta_{mn}^i r}H_{mni}^4\right)\cos\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right)\\&+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(e^{\alpha_{mn}^i r}H_{mni}^5+e^{\beta_{mn}^i r}H_{mni}^6\right)\sin\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right)\end{aligned} \tag{19}$$

where,

$$\alpha_n^i=\frac{\sqrt{\frac{4n^2\pi^2r_{i0}^2}{l^2}+1}-1}{2r_{i0}}, \qquad \beta_n^i=\frac{-\sqrt{\frac{4n^2\pi^2r_{i0}^2}{l^2}+1}-1}{2r_{i0}},$$

$$\alpha_{mn}^i=\frac{\sqrt{4m^2+\frac{4n^2\pi^2r_{i0}^2}{l^2}+1}-1}{2r_{i0}}, \quad \beta_{mn}^i=\frac{-\sqrt{4m^2+\frac{4n^2\pi^2r_{i0}^2}{l^2}+1}-1}{2r_{i0}}$$

And H_{ni}^{1}, H_{ni}^{2}, H_{mni}^{3}, H_{mni}^{4}, H_{mni}^{5} and H_{mni}^{6} are the undetermined constants. We can work out them with the temperature conditions on the surfaces of each layer.

3.2 Recursive formulae for temperature field

With the interlayer continuities of temperature condition in the interface of the laminated cylindrical shell, we can get

$$T_{i+1}(r_i,\theta,z)=T_i(r_i,\theta,z),$$

$$k_{i+1}\left.\frac{\partial T_{i+1}(r,\theta,z)}{\partial r}\right|_{r=r_i}=k_i\left.\frac{\partial T_i(r,\theta,z)}{\partial r}\right|_{r=r_i}\qquad (i=1,2,...,P-1) \tag{20}$$

In order to facilitate the derivation, the temperature solution and the heat flux in the direction of r can be expressed as the matrix form from Eq. (19), i.e.

$$\begin{bmatrix} T_i(r,\theta,z) \\ k_i\dfrac{\partial T_i(r,\theta,z)}{\partial r}\end{bmatrix}=\left[\psi_{mn}^{i}(r,\theta,z)\right]$$

$$=\sum_{n=1}^{\infty}\left[\phi_n^i(r)\right]\sin\left(\frac{n\pi z}{l}\right)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\phi_{mn1}^{i}(r)\right]\cos(m\theta)\sin\left(\frac{n\pi z}{l}\right)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\phi_{mn2}^{i}(r)\right]\sin(m\theta)\sin\left(\frac{n\pi z}{l}\right) \tag{21}$$

where $\left[\phi_n^i(r)\right]$, $\left[\phi_{mn1}^i(r)\right]$ and $\left[\phi_{mn2}^i(r)\right]$ are the matrix functions about variable r with undetermined coefficients, which can be described as from temperature distribution (6),

$$\left[\phi_n^i(r)\right]=\left[\varphi_n^i(r)\right]\left[\Lambda_n^i\right],$$

$$\left[\phi_{mn1}^i(r)\right]=\left[\varphi_{mn1}^i(r)\right]\left[\Lambda_{mn1}^i\right],$$

$$\left[\phi_{mn2}^i(r)\right]=\left[\varphi_{mn2}^i(r)\right]\left[\Lambda_{mn2}^i\right] \tag{22}$$

In the above equation, $\left[\varphi_n^i(r)\right]$, $\left[\varphi_{mn1}^i(r)\right]$, $\left[\varphi_{mn2}^i(r)\right]$ and $\left[\Lambda_n^i\right]$, $\left[\Lambda_{mn1}^i\right]$, $\left[\Lambda_{mn2}^i\right]$ are, respectively,

$$\left[\varphi_n^i(r)\right]=\begin{bmatrix} e^{\alpha_n^i r} & e^{\beta_n^i r} \\ k_i\alpha_n^i e^{\alpha_n^i r} & k_i\beta_n^i e^{\beta_n^i r}\end{bmatrix},\ \left[\Lambda_n^i\right]=\begin{bmatrix} H_{ni}^1 \\ H_{ni}^2\end{bmatrix},$$

$$\left[\varphi_{mn1}^i(r)\right]=\begin{bmatrix} e^{\alpha_{mn}^i r} & e^{\beta_{mn}^i r} \\ k_i\alpha_{mn}^i e^{\alpha_{mn}^i r} & k_i\beta_{mn}^i e^{\beta_{mn}^i r}\end{bmatrix},\ \left[\Lambda_{mn1}^i\right]=\begin{bmatrix} H_{mni}^3 \\ H_{mni}^4\end{bmatrix}$$

$$\left[\varphi_{mn2}^i(r)\right]=\begin{bmatrix} e^{\alpha_{mn}^i r} & e^{\beta_{mn}^i r} \\ k_i\alpha_{mn}^i e^{\alpha_{mn}^i r} & k_i\beta_{mn}^i e^{\beta_{mn}^i r}\end{bmatrix},\ \left[\Lambda_{mn2}^i\right]=\begin{bmatrix} H_{mni}^5 \\ H_{mni}^6\end{bmatrix} \tag{23}$$

Taking, respectively, $r=r_i$ and $r=r_{i-1}$, the unknown coefficients $\left[\Lambda_n^i\right]$, $\left[\Lambda_{mn1}^i\right]$ and $\left[\Lambda_{mn2}^i\right]$ can be eliminated. Therefore, one has

$$\left[\phi_n^i(r_i)\right]=\left[\varphi_n^i(r_i)\right]\left[\varphi_n^i(r_{i-1})\right]^{-1}\left[\phi_n^i(r_{i-1})\right],$$

$$\left[\phi_{mn1}^i(r_i)\right]=\left[\varphi_{mn1}^i(r_i)\right]\left[\varphi_{mn1}^i(r_{i-1})\right]^{-1}\left[\phi_{mn1}^i(r_{i-1})\right],$$

$$\left[\phi_{mn2}^i(r_i)\right]=\left[\varphi_{mn2}^i(r_i)\right]\left[\varphi_{mn2}^i(r_{i-1})\right]^{-1}\left[\phi_{mn2}^i(r_{i-1})\right] \tag{24}$$

The interlayer continuity in Eq. (20) is expressed as a matrix equation,

$$\left[\psi_{mn}^i\left(r_i,\theta,z\right)\right]=\left[\psi_{mn}^{i+1}\left(r_i,\theta,z\right)\right] \tag{25}$$

The relationship of the undetermined coefficients between the qth layer (q=2…P) and the first layer can be recursively worked out with the interlayer continuity.

$$\left[\phi_n^q(r_q)\right]=\left\{\prod_{i=q}^{1}\left[\varphi_n^i(r_i)\right]\left[\varphi_n^i(r_{i-1})\right]^{-1}\right\}\left[\phi_n^1(r_0)\right],$$

$$\left[\phi_{mn1}^q(r_q)\right]=\left\{\prod_{i=q}^{1}\left[\varphi_{mn1}^i(r_i)\right]\left[\varphi_{mn1}^i(r_{i-1})\right]^{-1}\right\}\left[\phi_{mn1}^1(r_0)\right],$$

$$\left[\phi_{mn2}^q(r_q)\right]=\left\{\prod_{i=q}^{1}\left[\varphi_{mn2}^i(r_i)\right]\left[\varphi_{mn2}^i(r_{i-1})\right]^{-1}\right\}\left[\phi_{mn2}^1(r_0)\right] \tag{26}$$

3.3 Undetermined coefficients in the analytical solution

Assume that the temperature distributions of the external surface and the internal surface of the layered shell are $t_p(\theta, z)$ and $t_1(\theta, z)$ respectively, i.e.

$$T_1(r_0,\theta,z)=t_1(\theta,z),\quad T_p(r_p,\theta,z)=t_p(\theta,z) \tag{27}$$

The temperature loads $t_p(\theta, z)$ and $t_1(\theta, z)$ can be expanded into the Fourier series, respectively:

$$\begin{aligned}t_1\left(\theta,z\right)=&\sum_{n=1}^{\infty}\left[\frac{1}{\pi l}\int_{-\pi}^{\pi}\int_0^l t_1\left(\theta,z\right)\sin\left(\frac{n\pi z}{l}\right)dzd\theta\right]\sin\left(\frac{n\pi z}{l}\right)\\&+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{2}{\pi l}\int_{-\pi}^{\pi}\int_0^l t_1\left(\theta,z\right)\cos\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right)dzd\theta\right]\cos\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right)\\&+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{2}{\pi l}\int_{-\pi}^{\pi}\int_0^l t_1\left(\theta,z\right)\sin\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right)dzd\theta\right]\sin\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right),\end{aligned}$$

$$t_p\left(\theta,z\right)=\sum_{n=1}^{\infty}\left[\frac{1}{\pi l}\int_{-\pi}^{\pi}\int_0^l t_p\left(\theta,z\right)\sin\left(\frac{n\pi z}{l}\right)dzd\theta\right]\sin\left(\frac{n\pi z}{l}\right)$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{2}{\pi l}\int_{-\pi}^{\pi}\int_{0}^{l}t_p\left(\theta,z\right)\cos\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right)dzd\theta\right]\cos\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right)$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{2}{\pi l}\int_{-\pi}^{\pi}\int_{0}^{l}t_p\left(\theta,z\right)\sin\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right)dzd\theta\right]\sin\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right) \tag{28}$$

Substituting the solution of temperature (19) into Eq. (28) gives

$$e^{\alpha_n^1 r_0}H_{n1}^1+e^{\beta_n^1 r_0}H_{n1}^2=\frac{1}{\pi l}\int_{-\pi}^{\pi}\int_{0}^{l}t_1\left(\theta,z\right)\sin\left(\frac{n\pi z}{l}\right)dzd\theta,$$

$$e^{\alpha_{mn}^1 r_0}H_{mn1}^3+e^{\beta_{mn}^1 r_0}H_{mn1}^4=\frac{2}{\pi l}\int_{-\pi}^{\pi}\int_{0}^{l}t_1\left(\theta,z\right)\cos\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right)dzd\theta,$$

$$e^{\alpha_{mn}^1 r_0}H_{mn1}^5+e^{\beta_{mn}^1 r_0}H_{mn1}^6=\frac{2}{\pi l}\int_{-\pi}^{\pi}\int_{0}^{l}t_1\left(\theta,z\right)\sin\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right)dzd\theta,$$

$$e^{\alpha_n^p r_p}H_{np}^1+e^{\beta_n^p r_p}H_{np}^2=\frac{1}{\pi l}\int_{-\pi}^{\pi}\int_{0}^{l}t_p\left(\theta,z\right)\sin\left(\frac{n\pi z}{l}\right)dzd\theta,$$

$$e^{\alpha_{mn}^p r_p}H_{mnp}^3+e^{\beta_{mn}^p r_p}H_{mnp}^4=\frac{2}{\pi l}\int_{-\pi}^{\pi}\int_{0}^{l}t_p\left(\theta,z\right)\cos\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right)dzd\theta,$$

$$e^{\alpha_{mn}^p r_p}H_{mnp}^5+e^{\beta_{mn}^p r_p}H_{mnp}^6=\frac{2}{\pi l}\int_{-\pi}^{\pi}\int_{0}^{l}t_p\left(\theta,z\right)\sin\left(m\theta\right)\sin\left(\frac{n\pi z}{l}\right)dzd\theta \tag{29}$$

Simultaneously solving Eq. (26) (taking q=P) and Eq. (29), H_{n1}^1, H_{n1}^2, H_{mn1}^3, H_{mn1}^4, H_{mn1}^5, H_{mn1}^6 and H_{np}^1, H_{np}^2, H_{mnp}^3, H_{mnp}^4, H_{mnp}^5, H_{mnp}^6 can be uniquely determined. Taking H_{n1}^1, H_{n1}^2, H_{mn1}^3, H_{mn1}^4, H_{mn1}^5 and H_{mn1}^6 back to Eq. (26), H_{nq}^1, H_{nq}^2, H_{mnq}^3, H_{mnq}^4, H_{mnq}^5 and H_{mnq}^6 of the qth layer (q=2…P) can be calculated by Eq. (26). Then, the temperature distribution in the laminated cylinders is obtained by substituting the coefficients into the analytical solution (19).

4 Convergence and comparison studies, layer analysis

Some numerical examples for the convergence and comparison studies of temperatures are performed to confirm the validity and accuracy of the proposal method. Consider a three-layer laminated open cylindrical shell with the parameters r_0=1.7 m, r_1=2.1 m, r_2=2.5 m, r_3=2.9 m, θ_0=7π/8 rad, l=20 m. The materials of the open shells are that surface layer is steel and the middle is concrete, i.e k_1=50 W/(m·ºC), k_2=2 W/(m·ºC), k_3=50 W/(m·ºC). The shell has prescribed temperatures on the surfaces: $t_p(\theta, z)$=100°C and $t_1(\theta, z)$=0°C.

The series of number terms m and n are truncated up to N in Eq. (6), the approximate solution for temperature field can be obtained. We have focused on five series of number terms about the temperature solution. Tab. 1 shows the temperature values in the positions: z=12 m, θ=1.2, r=1.80 m; r=2.00 m; r=2.20 m; r=2.40 m; r=2.60 m; r=2.80 m, respectively. We can find that the convergence of the results is excellent from Tab. 1.

And the solutions for N=24 and N=30 are nearly same. So, we can fix the number of terms at N=24 in the coming calculation.

Table 1: Convergence of temperature field

N	r=1.80 m	r=2.00 m	r=2.20 m	r=2.40 m	r=2.60 m	r=2.80 m
6	1.15	2.88	25.4	65.6	90.8	95.6
12	1.19	3.11	27.5	69.8	92.5	96.8
18	1.21	3.38	28.1	72.6	96.4	98.2
24	1.21	3.44	29.0	74.8	97.0	98.4
30	1.21	3.44	29.0	74.8	97.0	98.4

Table 2: The influence of dividing method about temperature

Position	P	r=1.80 m	r=2.00 m	r=2.20 m	r=2.40 m	r=2.60 m	r=2.8 m
θ=1.2, z=12	6	1.15	3.33	27.9	73.8	96.3	97.5
	9	1.18	3.33	28.3	73.9	96.4	97.9
	12	1.19	3.35	28.5	74.5	96.5	98.2
	15	1.21	3.44	29.0	74.8	97.0	98.4
	18	1.21	3.44	29.0	74.8	97.0	98.4

Table 3: Comparative researches of the temperature solutions at z=12 m, θ=1.2 rad

Position	Method	Temperature	Position	Method	Temperature
r=1.80 m	Present	1.21	r=2.40 m	Present	74.8
	FE	1.19		FE	74.5
r=2.00 m	Present	3.44	r=2.60 m	Present	97.0
	FE	3.44		FE	96.5
r=2.20 m	Present	29.0	r=2.80 m	Present	98.4
	FE	28.8		FE	98.0

To eliminate effects from the dividing method, the three-layer shell has been divided into 6 layers, 9 layers, 12 layers, 15 layers and 18 layers. Tab. 2 shows the temperature solution in six positions: z=12 m, θ=1.2 rad, r=1.80 m; r=2.00 m; r=2.20 m; r=2.40 m; r=2.60 m; r=2.80 m, respectively. We can see form Tab. 2 that the solutions for P=18 have the same three figures as those for P=15. So, we can find the error can be eliminated by the layer analysis. In order to confirm the correctness of the results, ANSYS software is used to simulate the shell in the temperature environment with the element Shell57 and the data are obtained. Tab. 3 gives the comparison studies of the temperatures at six points: z=12 m, θ=1.2 rad, r=1.80 m; r=2.00 m; r=2.20 m; r=2.40 m; r=2.60 m; r=2.80 m, respectively. We can find the temperature results are very close to the data from the software in Tab. 3.

5 Numerical examples

In this part, several numerical examples are studied to investigated the effects to temperature distribution from surface temperatures, sizes of the shells, numbers of layers and the material properties.

5.1 Effect of different surface temperatures

In this section, the effect of surface temperature t_0 on the distributions of temperature in the layered cylinders is discussed. The example is a three-layered cylindrical shell. The radii are r_0=1.6 m, r_1=2.0 m, r_2=2.4 m, r_3=2.8 m, respectively. The length of the shell is l=20 m. The materials of the cylindrical shells are that surface layer is steel and the middle is concrete. The shell has prescribed temperatures on the surfaces: $t_1(\theta, z)$=0°C and $t_p(\theta, z)$ is as follows,

$$t_p(\theta,z)=\begin{cases} t_0 & 0\le\theta\le\pi \\ 0 & -\pi<\theta<0 \end{cases} \tag{30}$$

where t_0=50°C, 100°C, 150°C, respectively.

The distribution of temperature at θ=1.1 rad, z=8 m is plotted in Fig. 3 for different thermal-loads. We can find the temperature solutions in the surface layer are almost the same, while the temperature variation is considerable in the concrete layer. That is because the heat conductivity of the steel is much larger than the concrete. The temperature distribution along the θ direction at r=2.1 m, z=8 m is given in Fig. 4. We can find the temperature of each point in the cylinders increases when the thermo-load increases.

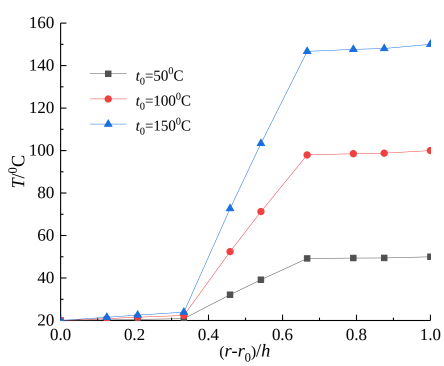


Figure 3: The temperature distribution along the thickness at θ=1.1 rad, z=8 m for different boundary temperatures

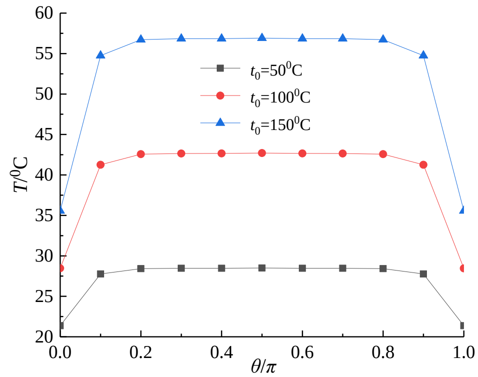


Figure 4: The temperature in the radial direction at r=2.1 *m*, z=8 *m* for different boundary temperatures

5.2 Effect of thickness and radius

In this part, the example is a three-layered open cylindrical shell. The materials of the open cylindrical shells are that surface layer is steel and the middle is concrete. The thickness of the open cylindrical shell is $h_1=h_2=h_3=0.4$ *m* (i.e. $h=r_3-r_0=1.2$ *m*). The open angle θ_0 of the shell is $\pi/2$ rad. The inner radiuses of the open cylindrical shell are: $r_0=1.0$ *m*, 2.0 *m*, 6.0 *m* (i.e. $h/r_0=1.2$, 0.6, 0.2), respectively. The shell has prescribed temperatures on the surfaces: $t_1(\theta, z)=0$°C and $t_p(\theta, z)=100$°C.

The distribution of temperature along the thickness direction at z=7 *m*, $\theta=\pi/5$ rad are plotted in Fig. 5. We can find that temperatures along the thickness direction are almost the same to different h/r_0. The distribution of temperature along the radial direction at r-r_0=0.7 *m*, z=7 *m* is shown in Fig. 6. It is seen that the figure is symmetrical in the radial direction. We can find from Fig. 6 that the temperature variation near to the edges is more remarkable than that within the interior.

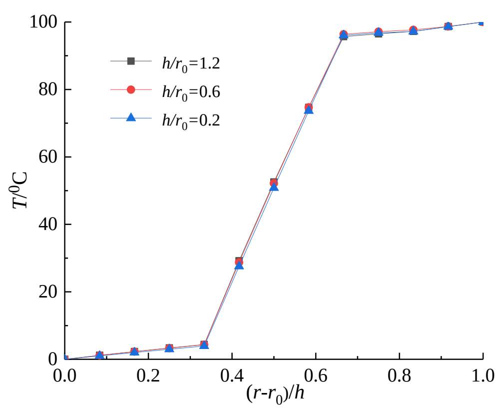


Figure 5: The temperature in the thickness direction at z=7 *m*, $\theta=\pi/5$ rad to different h/r_0

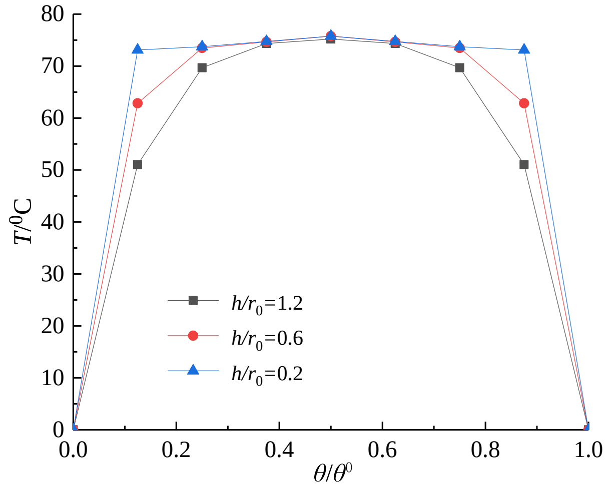


Figure 6: The temperature in the radial direction at r-r_0=0.7 m, z=7 m to different h/r_0

5.3 Effect of shell composition

In this part, the numerical example is three different layered open cylindrical shells. Here, there are three different materials: the wood with k_1=0.1 $W/(m·°C)$; the steel with k_2=50 $W/(m·°C)$; the concrete with k_3=2 $W/(m·°C)$. All the shells have the same inner radius r_0=2.0 m, the same thickness h (i.e. r_3-r_0)=1.2 m and the same angle θ_0=π/2. The length of the shell is l=20 m. The shell has prescribed temperatures on the surfaces: $t_1(\theta, z)$=0°C and $t_p(\theta, z)$=100°C.

The three-layered open cylindrical shells are two-layer shell, three-layer shell and four layer shell, respectively. The material properties of two-layer shell with the radii r_1=2.6 m, r_2=3.2 m are: The inner layer is steel and the outer layer is wood. The material properties of three-layer shell with the radii r_1=2.4 m, r_2=2.8 m, r_3=3.2 m are: The inner layer is concrete, the outer layer is wood and the middle is steel. The material properties of three-layer shell with the radii r_1=2.3 m, r_2=2.6 m, r_3=2.9 m, r_4=3.2 m are: The surface layer is wood, the second layer is concrete and the third layer is steel.

The temperature distribution along the thickness direction at θ=0.9 rad, z=9 m are plotted in Fig. 7. We can find that the temperature distributions are different although the exterior dimensions of the three shells are the same. The distribution of temperature in the radial direction at r=2.7 m, z=12 m is presented in Fig. 8. It can be found from Fig. 8 that the composition of the shell has a strong impact on the temperature variation. The temperature of two-layer shell is much higher than the other two shells. That is because the coefficient of heat conduction of wood is the lowest one among the three materials.

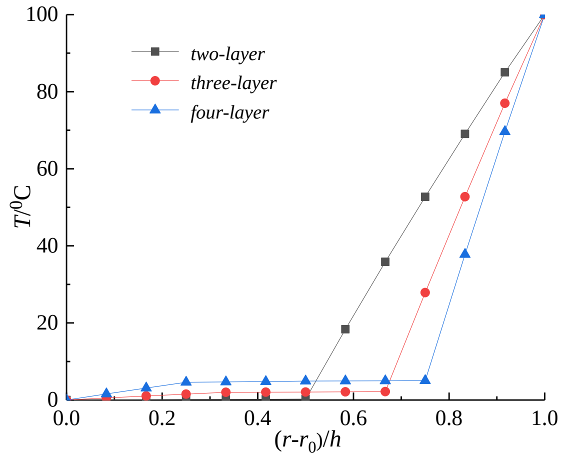


Figure 7: The temperature in the thickness direction at θ=0.9 rad, z=9 m for the open shell made up of different materials

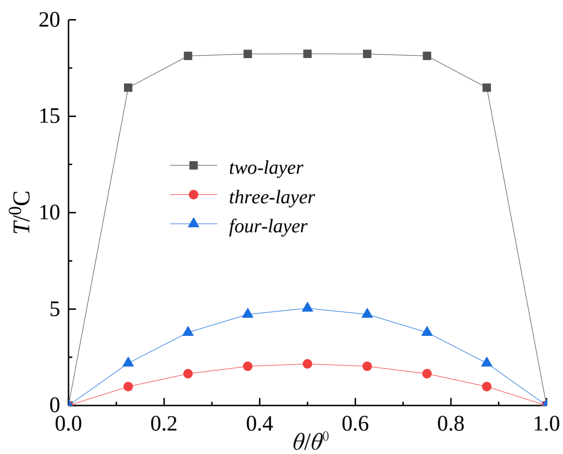


Figure 8: The temperature in the radial direction at r=2.7 m, z=12 m for the open shell made up of different materials

6 Conclusions

The three-dimensional temperature field within a laminated shell, including the open cylindrical shell and the cylindrical shell, has been investigated on the basis of the exact thermal theory. An exact solution is present to get the three-dimensional temperature field. We introduce a thinning layer approach to simplify the calculation of differential equations. Firstly, the layered shell is divided into N thinner layers. The governing equation was simplified by replacing the variable r by r_0 in the center line of every thin layer. The general solutions of temperature satisfying the simplified three-dimensional governing equation in single-layered shell are deduced in the cylindrical coordinate system. Then, the temperature and heat flux relationships between the surfaces can be found by transferring matrixes. According to the continuities of temperature and heat flux in the interface of the laminates, the temperature and heat flux relationships of the surfaces are derived. With the temperature condition on the surfaces, the unknown coefficients in the general solution of temperature are obtained. Finally, the effects from the thinning layer approach are eliminated by analyzing different numbers of thin layers.

The ANSYS software is applied to simulate the structure in the thermal environment. The validity and accuracy are proved from the comparison of results. And several numerical examples are studied to investigated the temperature effects from surface temperatures, geometric size of the shells and composition of layers.

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