

Weight Functions for Structural Integrity Assessment: Method and Applications

Xue-Ren Wu¹

A review of the state-of-the-art is presented on the weight function method for fracture-mechanics-based structural integrity assessment with regard to crack-like defects. The weight function method provides a powerful tool for the determination of key parameters, such as stress intensity factors and crack opening displacements for cracked structural components. For two dimensional (2D) crack problems, weight functions were obtained in closed-form for both centre- and edge-crack configurations. For three dimensional (3D) cases, a combination of the closed-form 2D weight functions and the slice synthesis technique makes it possible for rapid determination of stress intensity factor at any point along the crack front. The versatility, efficiency and accuracy of the weight function method, especially for treating crack problems with complex loadings, were demonstrated with various examples.

1 Introduction

One of the most frequent causes of failure is the presence of crack-like defects, because they can lead to catastrophic structural failure well within the original design envelope without early warning. Fracture mechanics is a most valuable tool for examining the safety of cracked structures, for developing effective remedial measures and determining their remaining useful life. The key prerequisite for the application of linear elastic fracture mechanics (LEFM) is the knowledge of accurate fracture parameters, e.g. stress intensity factor (SIF) and crack opening displacement (COD) for cracked bodies subjected to the loading in consideration. Because of the singularity at the crack-tip and, the crack length as one additional variable, analysis of crack problems are much more difficult and time-consuming than uncracked cases. Various analysis methods for crack problems within the LEFM frame have been developed over the past decades. However, many of the methods are inefficient in handling complex load cases. The weight function method provides very powerful, reliable, easy-to-use and cost-effective means to overcome

¹ Beijing Institute of Aeronautical Materials, Beijing 100095, China. Email: xueren.wu@gmail.com

such difficulties. The present paper gives a state-of-the-art review of this versatile approach.

2 Weight Functions for 2D-Crack Problem Analyses

For 2D crack problems, it has been shown that SIF can be obtained by a quadrature of the product of the weight function and the crack line stress [Bueckner (1970), Wu and Carlsson (1983, 1991), Wu (1992), Fett and Munz (1997)]. The SIF is, in non-dimensional form:

$$K = f\sigma\sqrt{\pi aW}, \quad f = \int_0^a \frac{\sigma(x)}{\sigma} \frac{m(a,x)}{\sqrt{\pi a}} dx, \quad a = A/W, \quad x = X/W \quad (1)$$

where $m(a, x)$ is the weight function, $\sigma(x)$ is the crack line stress, a and x are normalized crack length and coordinate along the crack, respectively, W is the characteristic length of the crack body.

The essence of the weight function approach is the separation of the variables upon which SIF depends. Once determined, the weight function can be used unlimitedly for any load case, with good accuracy. Compared to the numerical methods such as FEM, the weight function method can drastically reduce computational effort, usually by several orders. The central issue of the method is the accurate determination of $m(a, x)$. One effective way is to employ the relationship between the weight function and the crack face displacement:

$$m(a, x) = \frac{E' \sqrt{W}}{K_r} \frac{\partial u_r(a, x)}{\partial a} \quad (2)$$

where $E' = E$ (plane stress), $E' = E/(1 - \nu^2)$ (plane strain), K_r and U_r are SIF and dimensionless crack face displacement ($u_r = U_r/W$) for a reference load case $\sigma_r(x)$, respectively.

To determine u_r , consider the general crack face loading - the polynomial type:

$$\sigma_r(x) = \sum_{m=0}^M S_m x^m \quad (3)$$

For centre crack(s), the crack face displacement u_r is assumed to take the form of

$$u_r(a, x) = \frac{\sigma a}{E'} \sqrt{1 - (x/a)^2} \sum_{j=1}^J F_j(a) [1 - (x/a)^2]^{j-1} \quad (4)$$

For edge crack(s), u_r is assumed as

$$u_r(a, x) = \frac{\sigma}{\sqrt{2}} \frac{a}{H} \cdot \sum_{j=1}^J F_j(a) \cdot \left(1 - \frac{x}{a}\right)^{j-\frac{1}{2}} \quad (5)$$

where the $F_j(a)$ -functions are determined by a number of conditions. The above crack opening displacement expressions lead to the following weight functions (see [Wu and Carlsson (1991)] for details),

For centre cracks,

$$m(a, x) = \frac{1}{\sqrt{\pi a}} \sum_{i=1}^{J+1} \beta_i(a) \cdot \left[1 - \left(\frac{x}{a}\right)^2\right]^{i-\frac{3}{2}}; \quad (6)$$

$$\beta_i(a) = [a \cdot F'_{i-1}(a) - (2i-4) \cdot F_{i-1}(a) + (2i-1) \cdot F_i(a)] / f_r(a)$$

For edge cracks,

$$m(a, x) = \frac{1}{\sqrt{2\pi a}} \sum_{i=1}^{J+1} \beta_i(a) \cdot \left(1 - \frac{x}{a}\right)^{i-\frac{3}{2}} \quad (7)$$

$$\beta_i(a) = \left\{ a \cdot F'_{i-1}(a) + \frac{1}{2} [(2i-1) \cdot F_i(a) - (2i-5) \cdot F_{i-1}(a)] \right\} / f_r(a)$$

Inserting the weight function $m(a, x)$ into Eq. (1), the non-dimensional SIFs for arbitrarily loaded centre- or edge-crack(s) in a finite solid is readily calculated:

$$f = \frac{1}{\pi a} \int_0^a \frac{\sigma(x)}{\sigma} \sum_{i=1}^{J+1} \beta_i(a) \cdot \left[1 - \left(\frac{x}{a}\right)^2\right]^{i-\frac{3}{2}} dx, \quad \text{for centre crack} \quad (8)$$

$$f = \frac{1}{\sqrt{2\pi a}} \int_0^a \frac{\sigma(x)}{\sigma} \sum_{i=1}^{J+1} \beta_i(a) \cdot \left(1 - \frac{x}{a}\right)^{i-\frac{3}{2}} dx, \quad \text{for edge crack} \quad (9)$$

For many $\sigma(x)$ -distributions, the above integrations are often amenable to analytical treatment. It is advantageous to make use of the closed-form f -solutions for some basic loadings, as this will further minimize computation, and eliminate the possible error by numerical quadrature. A large variety of load cases have been considered [Wu and Carlsson (1991)], and solution accuracy has been thoroughly assessed. Three typical basic loadings are treated in the following.

(i) For a pair of point forces P (unit thickness) acting on the crack faces at x , the SIF becomes

$$K = \frac{P}{\sqrt{\pi a w}} G(a, x/a) \quad (10)$$

The function $G(a, x/a)$ is the Green's function (or influence function), which is related to weight function $m(a, x)$ by

$$G(a, x/a) = m(a, x) \cdot \sqrt{\pi a} \quad (11)$$

(ii) If the crack line stress distribution over the considered crack length is of power function type:

$$\sigma(x)/\sigma = |x^n|, \quad 0 \leq |x| \leq a \quad (12)$$

the corresponding non-dimensional SIF denoted by f_n becomes

$$f_n = \int_0^a x^n \cdot \frac{m(a,x)}{\sqrt{\pi a}} dx = \frac{a^n}{\pi} \sum_{i=1}^{J+1} \beta_i(a) \cdot T_{i,n} \quad \text{for centre crack} \quad (13)$$

$$f_n = \frac{2^{n+\frac{1}{2}} n! a^n}{\pi} \left[\sum_{i=1}^{J+1} \beta_i(a) \cdot \prod_{k=0}^n \frac{1}{2i-1+2k} \right] \quad \text{for edge crack} \quad (14)$$

The f_n -expressions is very useful, as most continuous crack line stress distributions can be represented by a single polynomial in $0 \leq |x| \leq a$:

$$\sigma(x)/\sigma = \sum_{n=0}^N S_n x^n, \quad 0 \leq |x| \leq a \quad (15)$$

The SIF is then simply obtained by superposition:

$$K = f\sigma\sqrt{\pi a W}, \quad f = \sum_{n=0}^N S_n f_n \quad (16)$$

(iii) For a linear stress segment acting at any part of the crack faces

$$\sigma(x)/\sigma = k|x| + b, \quad |x_1| \leq |x| \leq |x_2| \quad (17)$$

the SIF can be determined by:

$$K = f\sigma\sqrt{\pi a W}, \quad f = k f_l + b f_c \quad (18)$$

where f_l and f_c represent the contribution from the linear and constant part, respectively.

For centre crack(s):

$$f_l = \frac{a}{\pi} \left[\sum_{i=1}^{J+1} \frac{\beta_i(a)}{2i-1} \left[1 - \left(\frac{x}{a} \right)^2 \right]^{i-\frac{1}{2}} \right]_{x_2}^{x_1}, \quad (19)$$

$$f_c = \frac{1}{\pi} \left[\sum_{i=1}^{J+1} \beta_i(a) Q_i(x/a) \right]_{x_1}^{x_2}$$

$$Q_1(x/a) = \sin^{-1}(x/a), \quad i = 1$$

$$Q_i(x/a) = \frac{1}{2i-2} \left\{ \frac{x}{a} \left[1 - \left(\frac{x}{a} \right)^2 \right]^{i-\frac{3}{2}} + (2i-3) \cdot Q_{i-1}(x/a) \right\} \quad i \geq 2$$

For edge crack(s):

$$f_l = \frac{\sqrt{2a}}{\pi} \left[\sum_{i=1}^{J+1} \frac{\beta_i(a) \cdot \left[2 + (2i-1) \cdot \frac{x}{a} \right]}{(2i-1)(2i+1)} \left(1 - \frac{x}{a} \right)^{i-\frac{1}{2}} \right]_{x_2}^{x_1}$$

$$f_c = \frac{\sqrt{2}}{\pi} \left[\sum_{i=1}^{J+1} \frac{1}{2i-1} \beta_i(a) \cdot \left(1 - \frac{x}{a} \right)^{i-\frac{1}{2}} \right]_{x_2}^{x_1} \tag{20}$$

The resultant SIF is readily obtained by summation of all the segment contributions over the entire crack length. The piecewise-linearization provides a most versatile way to calculate the SIFs for any crack line stress distribution. It is particularly useful for cases where a single polynomial is not adequate to fit the drastic stress variations over the *entire* crack length of interest.

A large number of SIFs for 2D crack problems were obtained by the present writer and co-workers using the 2D closed form weight functions [Wu and Carlsson (1991), Wu (1992a, 1992b)]. Examples of SIF-solutions are shown in Figs 1~6. The effectiveness and accuracy of this approach is clearly demonstrated. The weight function method was also extensively studied by Fett and Munz, useful information can be found in Reference [Fett and Munz (1997)].

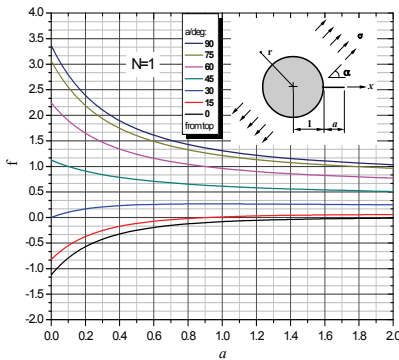


Figure 1: SIF for double radial cracks at a circular hole in infinite plate under inclined tension

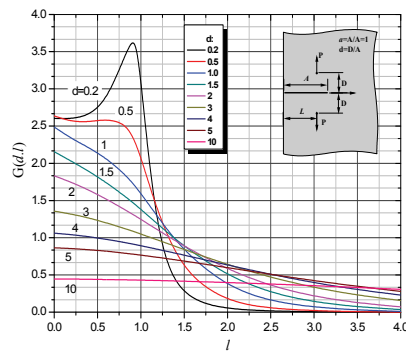


Figure 2: SIF for an edge crack in a semi-infinite plate loaded in the interior by a pair of forces P(unit thickness), $\nu = 1/3$

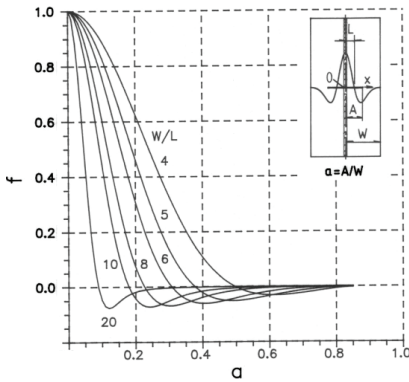


Figure 3: SIF for a centre crack in a butt-welded plate subjected to longitudinal welding residual stress

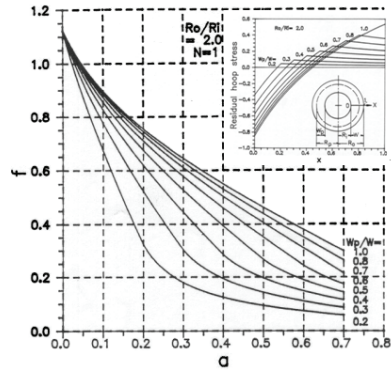


Figure 4: SIF for an internal radial crack in a hollow cylinder with autofrettage-induced residual stress

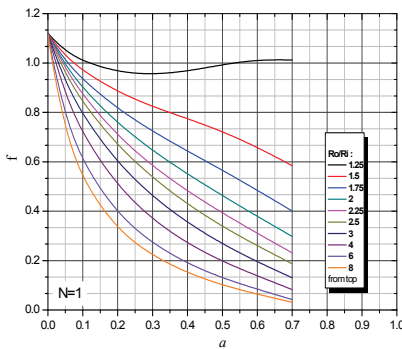


Figure 5: SIF for double radial internal cracks in circular hollow cylinder subjected to steady state thermal loading

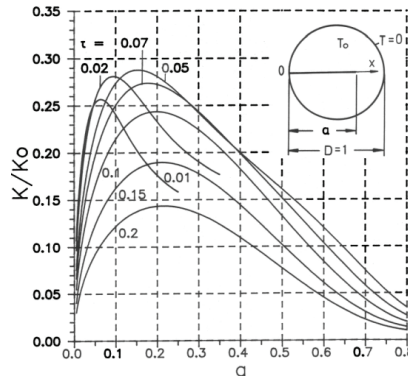


Figure 6: SIF \$K/K_0\$ for a single edge crack in a circular cylinder subjected to thermal shock loading

3 Weight Function Method for 3D-Crack Problems

Because of the SIF-variation along the crack front, 3D crack problems are much more difficult to analyze than the 2D cases. For 3D analysis, a weight function method was developed by Zhao and Wu (1989, 1990), by using a combination of the slice-synthesis procedure and the above 2D closed form weight function. In this approach, a 3D crack configuration is converted into an “equivalent” slice model which is able to simulate all the major 3D characteristics with the aid of the analytical solution for an embedded elliptical crack in an infinite body. The

3D WFM transforms the restraining effect of the un-cracked part of a 3D cracked body into a kind of elastic boundary condition on 2D slices. As shown in Fig.7, the cracked body is decomposed into two orthogonal slices (basic slice and spring slice) of infinitesimal thickness, being parallel with the major and minor axis, respectively. The restraining effect due to the un-cracked part is treated as a kind of elastic boundary condition on those boundaries of the 2D slices towards which the crack extends. The weight function for these slices are constructed through the two limiting cases, one for zero constraint, the other for fixed constraint, by using the general weight functions for 2D crack problems. The mechanical coupling between the two sets of slices is represented by the spring force $P(X, Y)$, which is determined by the crack opening displacement compatibility requirement:

$$V(x, y) = V(a_x, y) = V(c_y, x) \quad (23)$$

To calculate the crack opening displacements, 2D weight functions are again used. After having determined the spring forces, SIFs for the two sets of slices, one of which is subjected to spring forces alone and the other to both applied load and the negative of the spring forces, are calculated using the 2D weight functions. Finally these 2D-SIFs for each set of slices are used to compound the 3D-SIFs along the crack periphery according to the following general relationship between the 2D and 3D-SIFs:

$$k(\varphi) = \frac{1}{1 - \nu^2} \left\{ K_a^4(a_x) + \left[\frac{E}{E_s} K_c(c_y) \right]^4 \right\}^{1/4} (-1)^n \quad (24)$$

Details of this 3D approach and typical applications are referred to Zhao et al. (1989), Zhao and Wu (1990), Wu et al. (1998) and Zhao et al. (1995). Figure 8 shows the high accuracy level of the weight function solution as compared with Newman's various FEM-results for a semi-circular surface crack and quarter-circular corner crack at a semi-circular notch of single edge notch tension specimen [Wu et al. (1998)]. Bakuckas made a detailed study for the comparison of 3D SIFs for two symmetric corner cracks in a straight-shank hole, obtained by a variety of numerical methods and the above 3D weight function method. It was found that all the solutions were within a narrow band of 3% about the average solution, [Bakuckas (2001)].

4 Crack Opening Displacements

Another important application of the weight function method is the evaluation of crack opening displacements (COD), or profiles, for cracks subjected to arbitrary

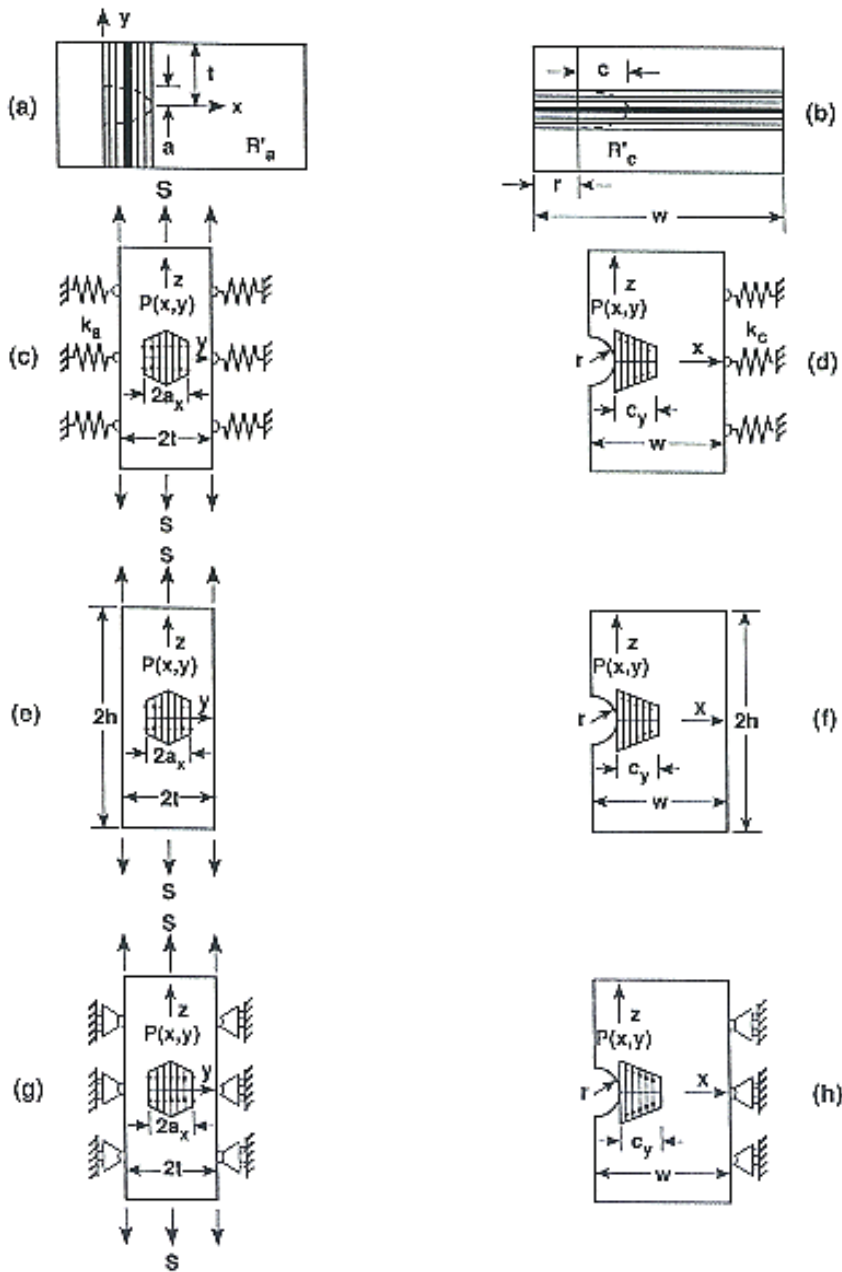


Figure 7: Slice model for a surface crack at a notch: basic slices, (a), (c) and (g); spring slices, (b), (d), (f) and (h)

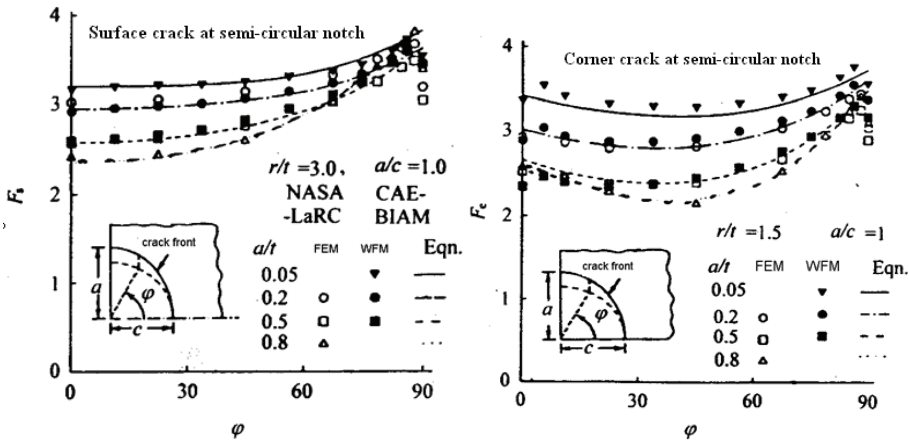


Figure 8: (a) Comparison of SIFs by 3D weight function method and FEM, surface crack at semi-circular notch; (b) Comparison of SIFs by 3D weight function method and FEM, corner crack at semi-circular notch

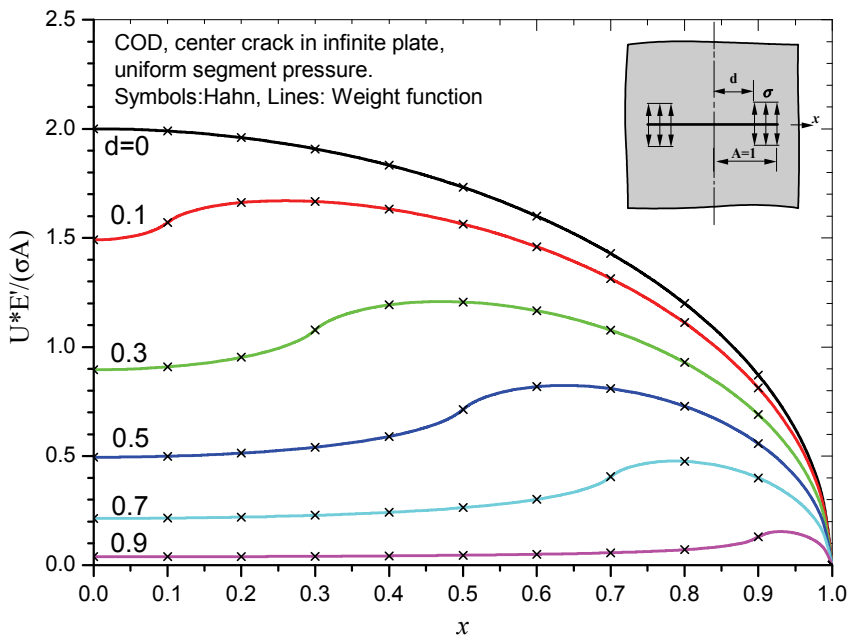


Figure 9: COD for a centre crack in an infinite plate, segment loading in the crack wake

loading. Such knowledge is very useful for crack-tip plastic zone analysis, crack-closure-based fatigue crack growth life prediction, modeling of various material toughening mechanisms, bridging stress computation, residual stress influence on crack tip parameters, and experimental determination of crack lengths using compliance methods, etc.

A general way to determine COD for arbitrary crack line loadings is to use the relationship between the weight function $m(a, x)$ and the COD, $u(a, x)$, via Eq. 2:

$$u(a, x) = \frac{\sigma}{E'} \int_{a_0}^a [f(s)\sqrt{\pi s}] \cdot m(s, x) ds \quad (25)$$

where $f(s)$ is the non-dimensional SIF associated with the load case for which the COD is desired. This $f(s)$, if not available, can be evaluated with the weight function method for any crack line loading $\sigma(x)$. In explicit form, Eq. 21 becomes

$$u(a, x) = \frac{\sigma}{E'} \int_{a_0}^a [f(s) \sum_{i=1}^I \beta_i(s) \cdot [1 - (\frac{x}{s})^2]^{i-\frac{3}{2}}] ds, \text{ for centre crack} \quad (26)$$

$$u(a, x) = \frac{\sigma}{E'\sqrt{2}} \int_{a_0}^a [f(s) \sum_{i=1}^I \beta_i(s) \cdot (1 - \frac{x}{s})^{i-\frac{3}{2}}] ds, \text{ for edge crack} \quad (27)$$

For a segment of uniform pressure acting in the immediate wake of the crack tip,

$$u(a, d/a, x/a) = \frac{\sigma}{E'\pi} \int_{a_0}^a [1 - (\frac{x}{s})^2]^{-\frac{1}{2}} \cdot \sum_{i=1}^I \beta_i(s) \cdot Q_i(\frac{d}{s}) \cdot \sum_{i=1}^I \beta_i(s) \cdot [1 - (\frac{x}{s})^2]^{i-1} ds \quad (28)$$

$$Q_1(\frac{d}{s}) = \cos^{-1}(\frac{d}{s}), \quad i = 1$$

$$Q_i(\frac{d}{s}) = \frac{1}{2i-2} \left\{ (2i-3) \cdot Q_{i-1}(\frac{d}{s}) - (\frac{d}{s}) \cdot [1 - (\frac{d}{s})^2]^{i-\frac{3}{2}} \right\}, \quad i \geq 2$$

for centre crack, and

$$u(a, d/a, x/a) = \frac{\sigma}{E'\pi} \int_{a_0}^a \left[\frac{1 - \frac{d}{s}}{1 - \frac{x}{s}} \right]^{-\frac{1}{2}} \cdot \sum_{i=1}^I \frac{\beta_i(s)}{2i-1} \cdot (1 - \frac{d}{s})^{i-1} \cdot \sum_{i=1}^I \beta_i(s) \cdot (1 - \frac{x}{s})^{i-1} ds \quad \text{for edge crack} \quad (29)$$

A large number of cases for COD were obtained in Wu and Carlsson (1991). One example for a partially loaded centre crack is given in Fig. 9, which shows good agreement of the COD between the results from the above weight function

method and the exact solution. Using the weight function approach, Liu and Wu extended the crack-closure model to study fatigue crack closure behavior for various cracked geometries, and developed analytical approximate COD-expressions for edge cracks subjected to a segment uniform crack face pressure [Liu and Wu (1997)], which can facilitate fatigue crack growth analysis.

5 Conclusions

The weight function method is a very powerful method for the evaluation of fracture mechanics parameters, such as stress intensity factors and crack opening displacements, especially for cracks in complex stress fields, and when a large number of load cases are considered. The distinct advantages of the method are versatility, high efficiency, easy-to-use and good accuracy. The method is well established, and provides a very efficient and reliable tool for engineering structural integrity assessment where the presence of crack-like defects is a major concern.

6 References

- Bakuckas, J.G.** (2001): *Engng Fract. Mech*, 68, pp. 1095-1106.
- Bueckner, H.F.** (1970): *Z. Angew. Math. Mech.*, 50, pp. 529-545.
- Fett, T.; Munz, D.** (1997): *Stress Intensity Factors and Weight Functions*, Computational Mechanics Publications, Southampton, UK.
- Liu, J.Z.; Wu, X.R.** (1997): *Engng Fract. Mech*, 58, pp. 107-119.
- Wu, X.R.** (1992): *Engineering Analysis with Boundary Elements*, 9, pp. 307-322.
- Wu, X.R.** (1992): in *Thermal Shock and Thermal Fatigue Behavior of Advanced Ceramics*, Eds. Schneider and Petzow, Kluwer Academic Publishers, pp. 119-142.
- Wu, X.R.; Carlsson, J.** (1983): *J. Mech. Phys. Solids*, 31, pp. 485-497.
- Wu, X.R.; Carlsson, J.** (1991): *Weight Functions and Stress Intensity Factor Solutions*, Pergamon Press.
- Wu, X.R.; Newman, J.C.; Zhao, W.; Swain M.H.; Ding, C.F.; Philips, E.P.** (1998): *Fatigue and Fracture of Engng. Mat. & Struct.* 21, pp. 1289-1306.
- Zhao, W.; Wu, X.R.; Yan, M.G.** (1989): *Engng Fract. Mech.* 34, pp. 593-624.
- Zhao, W.; Wu, X.R.** (1990): *Fatigue and Fract. Engng Mater. Struct.* 13, pp. 347-360.
- Zhao, W.; Newman, J.C.; Sutton, M.A.; Wu, X.R.; Shivakumar, K.N.** (1995): *NASA Technical Memorandum 110144*.

