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An Alternative 2D BEM for Fracture Mechanics in Orthotropic Materials

G. Davì[1] and A. Milazzo[1]

Abstract: An original and alternative single domain boundary element formulation and its numerical implementation are presented for the analysis of orthotropic two-dimensional cracked bodies. The problem is formulated employing the classical displacement boundary integral representation and an alternative integral equation deduced on the basis of the stress function theory. This integral equation written on the crack provides the relations needed to determine the problem solution in the framework of linear elastic fracture mechanics. Numerical examples are reported and discussed to demonstrate the accuracy of the proposed approach.

Keyword: Fracture mechanics, orthotropic materials, stress function, stress intensity factor.

1 Introduction

On a microscopic scale failure of composite materials involves different mechanisms like fiber fracture, matrix cracking, fiber debonding etc. that, together with anisotropy, affect their fracture behavior on a macroscopic scale. Such features need an accurate assessment in the framework of damage tolerance design and maintenance. The determination of stress intensity factors is the primary concern in the field of fracture mechanics since these parameters are related to fracture resistance and propagation. The solution of crack problems in composite materials generally requires the use of numerical techniques and actually both the finite element method and the boundary element method have been employed for fracture mechanics analyses in both isotropic and anisotropic materials. The boundary element method (BEM) has proved to be an efficient and powerful technique to deal with fracture mechanics problems where an accurate description of the high stress gradients at the crack tip and the numerical efficiency are essential. The application of BEM to the fracture mechanics of composites was proposed by Cruse and Svedlow (1971) and, starting from their pioneering work, different approaches have been proposed to analyze fracture mechanics problem in both isotropic and anisotropic media [Aliabadi (1997)]. Such approaches have been developed to overcome the impossibility of applying standard BEMs to deal with general crack problems, since the application of the displacement integral equation to coincident crack surfaces leads to a mathematical degeneration. The first approach is the Green function method, in which a special form of fundamental solution is introduced to avoid the modeling of crack surfaces; this approach is very accurate and advantageous for simple crack geometries for which the analytic expression of the Green function is known [Snyder and Cruse (1975), Chan and Cruse (1986)]. However, this Green function approach can be applied to any kind of geometry by numerically computing the corresponding Green function [Telles, Castor and Guimarães (1995); Castor and Telles (2000)].A general approach is the multidomain technique, which allows to model any crack problem. It is based on the artificial subdivision of the original domain into sub-regions whose boundaries contain the crack faces; the subregions are then joined along the fictitious boundaries or interfaces by means of the equilibrium of tractions and compatibility of displacements. This solution scheme gives rise to additional degrees of freedom and then the solving system has a greater order than that strictly required by the problem, with the consequent computational effort [Blandford, Ingraffea and Ligget (1981), Tan and Gao (1992), Sollero and Aliabadi (1993), Saez, Ariza and

[1] Dipartimento di Tecnologie e Infrastrutture Aeronautiche, Università di Palermo, Viale delle Scienze, 90128 Palermo – Italy. E-mail: davi@unipa.it

Dominguez (1997), Davì and Milazzo(2001)]. On the other hand, single-domain BEMs have been proposed and developed with the aim of analyzing general crack geometries without the computational effort of the multidomain technique. The fundamental single-domain approach, known as Dual Boundary Element Method, involves two different sets of boundary integral equations, one related to displacements and the other to stresses or tractions, which are simultaneously collocated on the crack boundaries. This method leads to a system of integral equations involving hypersingular integrals appearing in the traction equations, with consequent constraints on the kind of employed boundary elements [Gray, Martha and Ingraffea (1990), Portela, Aliabadi and Rooke (1992), Sollero and Aliabadi (1995), Chang and Mear (1995), Pan and Amadei (1996), Ammons and Vable (1996), Pan (1997), Garcia, Saez and Dominguez (2004)]. In this paper, an alternative single domain approach for fracture mechanics in orthotropic materials is presented. The paper is the extension of a previous authors' work [Davì and Milazzo (2006)] and is based on the employment of the stress function theory to obtain the integral equations needed for the solution of the crack problem. The approach preserves the computational advantages of single domain formulations without involving hypersingular kernels, so that the treatment of the resolving integral equations model requires no particular care. Numerical results are presented to show the accuracy and effectiveness of the proposed approach.

2 Displacement boundary integral equation for orthotropic materials

Let us consider an elastic, orthotropic, two-dimensional body occupying the domain Ω with contour Γ and containing a crack whose representative boundary is denoted by Γ_f. The boundary integral representation of the displacement at the field point P_0 is given by the Somigliana identity which reads [Aliabadi (2002)]

$$\mathbf{c}\mathbf{u}(P_0) = \int_\Gamma [\mathbf{u}_j^T(P,P_0)\mathbf{p} - \mathbf{p}_j^T(P,P_0)\mathbf{u}]d\Gamma - \int_{\Gamma_f} \mathbf{p}_j^T(P,P_0)\Delta\mathbf{u}d\Gamma + \int_\Omega \mathbf{u}_j^T(P,P_0)\mathbf{f}d\Omega \quad (1)$$

where $\mathbf{u}$ is the displacement vector, $\mathbf{p}$ is the vector containing the boundary tractions, $\mathbf{f}$ are the body forces, $\Delta\mathbf{u}$ are the relative displacements along the crack, whereas $\mathbf{u}_j(P,P_0)$ and $\mathbf{p}_j(P,P_0)$ are the displacements and tractions of the problem fundamental solution associated with a concentrated body force $\mathbf{f}_j$ applied at the source point P_0. The anisotropic elasticity fundamental solution has been derived following different approaches which are based on the Lekhnitskii's stress functions [Lekhnitskii (1963), Sollero and Aliabadi (1993)] and the Stroh formalism [Stroh (1958), Stroh (1962)]. In the present paper an alternative notation introduced by the authors is employed [Davì and Milazzo (2001)] to infer the fundamental solution. This notation mixes the classical Lekhnitskii approach and the Stroh formalism producing a powerful matrix tool for the numerical implementation of the fundamental solution of 2D anisotropic elasticity and its extension to other problems. According to Davì and Milazzo [(2001)] the expressions of the 2D anisotropic fundamental solutions is obtained by introducing the complex variable $Z_k = [x_1(P) - x_1(P_0)] + \mu_k[x_2P) - x_2(P_0)]$ (k=1, 2) associated with the positive imaginary part eigenvalues μ_k of the following problem

$$\left[\mathbf{I}_1^T\mathbf{E}\mathbf{I}_1 + \mu\left(\mathbf{I}_1^T\mathbf{E}\mathbf{I}_2 + \mathbf{I}_2^T\mathbf{E}\mathbf{I}_1\right) + \mu^2\mathbf{I}_2^T\mathbf{E}\mathbf{I}_2\right]\mathbf{a} = \mathbf{0} \quad (2)$$

where $\mathbf{E}$ is the stiffness matrix whose elements are denoted by E_{ij}. The matrices $\mathbf{I}_m$ (m=1, 2) are obtained from the compatibility operator

$$\Xi^T = \begin{bmatrix} \partial/\partial x_1 & 0 & \partial/\partial x_2 \\ 0 & \partial/\partial x_2 & \partial/\partial x_1 \end{bmatrix} \quad (3)$$

by setting the derivatives with respect to x_m equal to one and replacing all the other terms with zeros. Denoting by $\mathbf{a}_k$ the eigenvectors related to the eigenvalues μ_k through Eq. (2), the fundamental solution displacements and tractions are written as

$$\mathbf{u}_j = \sum_{k=1}^{2} 2Re\left[\lambda_{kj}\mathbf{a}_k \ln Z_k\right] \quad (4)$$

$$\mathbf{t}_j = \sum_{k=1}^{2} 2Re\left[\lambda_{kj}\Xi_n^T\mathbf{E}\Xi_{\mu_k}\mathbf{a}_k\frac{1}{Z_k}\right] \quad (5)$$

where Ξ_n^T is the boundary traction operator obtained from the compatibility operator by replacing the derivatives with respect to x_1 and x_2 with the direction cosines α_1 and α_2 of the boundary outer normal, respectively. The matrix Ξ_{μ_k} is obtained from the compatibility operator by replacing the derivative with respect to x_1 with one and the derivative with respect to x_2 with μ_k. The vector $\lambda_j = [\lambda_{1j}\ \lambda_{2j}]^T$ is then computed by

$$\lambda_j = (\mathbf{B} + \tilde{\mathbf{B}}\tilde{\mathbf{A}}^{-1}\mathbf{A})^{-1}\mathbf{f}_j \tag{6}$$

where the tilde denotes the complex conjugate, $\mathbf{A}$ is the matrix of the eigenvectors $\mathbf{a}_k$ and the columns $\mathbf{b}_k$ of the matrix $\mathbf{B}$ are defined by

$$\mathbf{b}_k = \overline{\Xi}_k^T \mathbf{E} \Xi_{\mu_k} \mathbf{a}_k \tag{7}$$

In Eq. (7) the matrix $\overline{\Xi}_k^T$ is again obtained from the compatibility operator by replacing the derivatives with respect to x_1 with the quantity $-\pi\left(1+\sqrt{-1}\mu_k\right)/\left(1+\mu_k^2\right)$ and the derivative with respect to x_2 with $\pi\left(\sqrt{-1}-\mu_k\right)/\left(1+\mu_k^2\right)$.

By using two independent fundamental solutions (j=1, 2), associated with concentrated body forces $\mathbf{f}_j$ directed along the coordinate axes, the boundary integral representation of the displacement components at the point P_0 in terms of the displacements and tractions on the boundary is obtained

$$\mathbf{cu}(P_0) = \int_\Gamma [\mathbf{U}^*(P,P_0)\mathbf{p} - \mathbf{P}^*(P,P_0)\mathbf{u}]d\Gamma - \int_{\Gamma_f} \mathbf{P}^*(P,P_0)\Delta\mathbf{u}d\Gamma + \int_\Omega \mathbf{U}^*(P,P_0)\mathbf{f}d\Omega \tag{8}$$

where $\mathbf{U}^*$ and $\mathbf{P}^*$ are the kernels of the fundamental solutions [Aliabadi (2002)]. For points P_0 belonging to the boundary, provided that singular integrals are computed in the sense of their principal value and that $\mathbf{c}$ is suitably computed [Davì and Milazzo (2001), Aliabadi (2002)], Eq. (8) becomes the boundary integral equation which, taking the prescribed boundary conditions into account, models the solution of the elastic problem in terms of displacements and tractions on the boundary [Aliabadi (2002)]. It straightforwardly appears that Eq. (8), in its numerical application to cracked bodies, when collocated on the two crack faces, originates a mathematical degeneration in the problem formulation. In particular the resolving system presents more unknowns than equations and additional equations are required to solve the problem [Aliabadi (1997)].

In the present work, such additional equations are retrieved by an alternative boundary integral representation based on the use of the stress function theory and described in the next section. The new boundary integral equation, collocated on the crack representative boundary, determines the problem solution.

3 Alternative boundary integral equation

For homogeneous, orthotropic two-dimensional body the stress field can be derived from a single function, the stress function $\Phi = \Phi(x_1, x_2)$, so that the equilibrium equations are trivially fulfilled. Assuming that the body force field $\mathbf{f}$ is conservative there exists a potential function $\Psi = \Psi(x_1, x_2)$ such that

$$\mathbf{f} = \begin{bmatrix} \frac{\partial\Psi}{\partial x_1} & \frac{\partial\Psi}{\partial x_2} & 0 \end{bmatrix}^T \tag{9}$$

and one has [Chou and Pagano (1992)]

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{12} \end{bmatrix}^T = \mathbf{C}\Phi - \tilde{\mathbf{I}}\Psi \tag{10}$$

where the involved operators are defined as

$$\mathbf{C}^T = \begin{bmatrix} \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial x \partial y} \end{bmatrix} \tag{11}$$

$$\tilde{\mathbf{I}}^T = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \tag{12}$$

Once the stress function is introduced, the displacement field can be obtained by integrating the strain-displacement relations and for orthotropic bodies one obtains the following displacement field decomposition

$$\mathbf{u} = \mathbf{v} + \Lambda\mathbf{S}\Phi \tag{13}$$

where $\mathbf{S} = [\partial/\partial x_1\ \partial/\partial x_2]^T$ is the gradient operator and

$$\Lambda = \begin{bmatrix} E_{11}\mu_1^2 + E_{12} & 0 \\ 0 & \frac{E_{22}}{\mu_1^2} + E_{12} \end{bmatrix} \tag{14}$$

In the displacement decomposition given by Eq. (13) $\mathbf{v}$ is a vector, whose components $v_1(x_1, x_2)$

and $v_2(x_1,x_2)$ are functions that satisfy the orthotropic Laplace equation

$$\frac{\partial^2 v_i}{\partial x_1^2} - \mu_1^2 \frac{E_{11}}{E_{22}} \frac{\partial^2 v_i}{\partial x_2^2} = 0 \quad i = 1,2 \tag{15}$$

Additionally, the functions v_1 and v_2 are conjugate according to the following relations

$$\frac{\partial v_1}{\partial x_1} = -\mu_1^2 \frac{E_{11}}{E_{22}} \frac{\partial v_2}{\partial x_2} \tag{16}$$

$$\frac{\partial v_1}{\partial x_2} = -\frac{\partial v_2}{\partial x_1} \tag{17}$$

The boundary tractions $\mathbf{p}$ can be also expressed in terms of stress function and one obtains

$$\mathbf{p} = \frac{\partial}{\partial s}\mathbf{H}\mathbf{S}\Phi - \mathbf{n}\Psi \tag{18}$$

where $\partial/\partial s$ indicates the tangent derivative, $\mathbf{n}$ is the boundary unit normal vector, whereas the matrix $\mathbf{H}$ is defined by

$$\mathbf{H} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{19}$$

Taking into account the displacement decomposition, the integral equation (8) becomes

$$\begin{aligned} &\mathbf{c}\mathbf{v}(P_0) + \mathbf{c}\Lambda\mathbf{S}\Phi(P_0) \\ &= \int_\Gamma [\mathbf{U}^*(P,P_0)\mathbf{p} - \mathbf{P}^*(P,P_0)\mathbf{u}]d\Gamma \\ &\quad - \int_{\Gamma_f} \mathbf{P}^*(P,P_0)\Delta\mathbf{u}d\Gamma + \int_\Omega \mathbf{U}^*(P,P_0)\mathbf{f}d\Omega \end{aligned} \tag{20}$$

Due to the potential nature of the components of $\mathbf{v}$ applying the Green theorem one has

$$\mathbf{c}\mathbf{v}(P_0) = \int_\Gamma \left[\varphi^*(P,P_0)\frac{\partial \mathbf{v}}{\partial \tilde{n}} - \frac{\partial \varphi^*(P,P_0)}{\partial \tilde{n}}\mathbf{v}\right]d\Gamma \tag{21}$$

where φ^* is the orthotropic potential fundamental solution

$$\varphi^* = \sqrt{-\frac{E_{22}}{\mu_1^2 E_{11}}}\ln\frac{1}{r(P,P_0)} \tag{22}$$

where $r(P,P_0)$ is the reduced distance between the domain point P and the source point P_0

$$r(P,P_0) = \sqrt{[x_1(P) - x_1(P_0)]^2 - \frac{E_{22}}{\mu_1^2 E_{11}}[x_2(P) - x_2(P_0)]^2} \tag{23}$$

and $\partial/\partial\tilde{n}$ denotes the derivative along the boundary co-normal defined by

$$\partial/\partial\tilde{n} = \alpha_1\partial/\partial x_1 - \alpha_2\left(\mu_1^2 E_{11}/E_{22}\right)\partial/\partial x_2 \tag{24}$$

By using the conjugate relations between v_1 and v_2, Eqs (16) and (17), and taking the displacement field decomposition and the expression of the boundary traction in terms of the stress function into account, Eq. (21) becomes

$$\begin{aligned} \mathbf{c}\mathbf{v}(P_0) &= \int_\Gamma [\mathbf{u}^*(P,P_0)\mathbf{p} - \mathbf{p}^*(P,P_0)\mathbf{u}]d\Gamma \\ &+ \int_\Gamma \Lambda\mathbf{S}\Phi\frac{\partial \varphi^*(P,P_0)}{\partial \tilde{n}}d\Gamma + \int_\Gamma \mathbf{u}^*(P,P_0)\mathbf{n}\Psi d\Gamma \end{aligned} \tag{25}$$

where

$$\mathbf{u}^*(P,P_0) = \begin{bmatrix} \Lambda_{22}\mu_1^2\frac{E_{11}}{E_{22}}\varphi^*(P,P_0) & 0 \\ 0 & -\Lambda_{11}\varphi^*(P,P_0) \end{bmatrix} \tag{26}$$

$$\mathbf{p}^*(P,P_0) = \begin{bmatrix} \frac{\partial\varphi^*(P,P_0)}{\partial\tilde{n}} & \mu_1^2\frac{E_{11}}{E_{22}}\frac{\partial\varphi^*(P,P_0)}{\partial s} \\ \frac{\partial\varphi^*(P,P_0)}{\partial s} & \frac{\partial\varphi^*(P,P_0)}{\partial\tilde{n}} \end{bmatrix} \tag{27}$$

Finally, by using Eq. (25), the integral equation (20) is written as

$$\begin{aligned} \mathbf{c}\Lambda\mathbf{S}\Phi(P_0) = &\int_\Gamma [\mathbf{U}^*(P,P_0)\mathbf{p} - \mathbf{P}^*(P,P_0)\mathbf{u}]d\Gamma \\ &- \int_{\Gamma_f} \mathbf{P}^*(P,P_0)\Delta\mathbf{u}d\Gamma \\ &+ \int_\Gamma [\mathbf{p}^*(P,P_0)\mathbf{u} - \mathbf{u}^*(P,P_0)\mathbf{p}]d\Gamma \\ &+ \int_{\Gamma_f} \mathbf{p}^*(P,P_0)\Delta\mathbf{u}d\Gamma \\ &- \int_\Gamma \Lambda\mathbf{S}\Phi\frac{\partial\varphi^*(P,P_0)}{\partial\tilde{n}}d\Gamma \\ &+ \int_\Gamma \mathbf{u}^*(P,P_0)\mathbf{n}\Psi d\Gamma \\ &+ \int_\Omega \mathbf{U}^*(P,P_0)\mathbf{f}d\Omega \end{aligned} \tag{28}$$

This equation can be regarded as a boundary integral representation of the stress function gradient. Recalling that the components of the resultant of the tractions applied between the point $\overline{P}$ and a generic point P_A are defined as

$$\mathbf{R} = \int_{\overline{P}}^{P_A} \mathbf{p} d\Gamma \tag{29}$$

by integration of the Eq. (18) one obtains

$$\mathbf{S}\Phi = \mathbf{H}^{-1}\mathbf{R} + \mathbf{k} \tag{30}$$

where $\mathbf{k}$ is a vector whose components are arbitrary constants. For a source point P_0 belonging to the crack line the integral equation (28) becomes

$$\begin{aligned}\mathbf{c}\Lambda\mathbf{k} = &\int_{\Gamma} [\mathbf{U}^*(P,P_0)\,\mathbf{p} - \mathbf{P}^*(P,P_0)\,\mathbf{u}]d\Gamma \\ &- \int_{\Gamma_f} \mathbf{P}^*(P,P_0)\,\Delta\mathbf{u} d\Gamma \\ &+ \int_{\Gamma} [\mathbf{p}^*(P,P_0)\,\mathbf{u} - \mathbf{u}^*(P,P_0)\,\mathbf{p}]d\Gamma \\ &+ \int_{\Gamma_f} \mathbf{p}^*(P,P_0)\,\Delta\mathbf{u} d\Gamma \\ &- \int_{\Gamma} \Lambda\mathbf{H}^{-1}\mathbf{R}\frac{\partial\varphi^*(P,P_0)}{\partial\tilde{n}} d\Gamma\end{aligned} \tag{31}$$

This equation allows the problem solution through the boundary element method.

4 Numerical scheme

The boundary integral formulation proposed is numerically solved by the boundary element method [Aliabadi (2002)]. In the following, for the sake of simplicity, the numerical scheme employed is described for the case of zero body forces. The body boundary Γ is discretized into M boundary elements and over each of these elements $\Gamma_{\langle k\rangle}$ the displacements $\mathbf{u}$ and the tractions $\mathbf{p}$, are expressed in terms of their respective nodal values $\delta_{\langle k\rangle}$ and $\mathbf{t}_{\langle k\rangle}$

$$\mathbf{u} = \mathbf{N}_u \delta_{\langle k\rangle} \quad \text{on } \Gamma_{\langle k\rangle} \tag{32}$$

$$\mathbf{p} = \mathbf{N}_p \mathbf{t}_{\langle k\rangle} \quad \text{on } \Gamma_{\langle k\rangle} \tag{33}$$

where $\mathbf{N}_u$ and $\mathbf{N}_p$ are shape function matrices. The crack representative boundary Γ_f is discretized into m boundary elements and over each of these elements $\Gamma_{f\langle k\rangle}$ the crack relative displacements $\Delta\mathbf{u}$ are expressed in terms of their respective nodal values $\Delta\delta_{\langle k\rangle}$ by means of the shape function matrix $\mathbf{N}$

$$\Delta\mathbf{u} = \mathbf{N}\Delta\delta_{\langle k\rangle} \quad \text{on } \Gamma_{f\langle k\rangle} \tag{34}$$

The discretized version of eqn. (8) for a collocation point P_i on the boundary is therefore given by

$$\mathbf{c}_i^*\mathbf{u}(P_i) + \sum_{k=1}^{M} \mathbf{H}_{ik}\delta_{\langle k\rangle} + \sum_{k=1}^{M} \mathbf{G}_{ik}\mathbf{t}_{\langle k\rangle} + \sum_{k=1}^{m} \mathbf{Q}_{ik}\Delta\delta_{\langle k\rangle} = \mathbf{0} \tag{35}$$

where

$$\mathbf{H}_{ik} = \int_{\Gamma_{\langle k\rangle}} \mathbf{P}^*(P,P_i)\,\mathbf{N}_u(P) d\Gamma_{\langle k\rangle} \tag{36}$$

$$\mathbf{G}_{ik} = -\int_{\Gamma_{\langle k\rangle}} \mathbf{U}^*(P,P_i)\,\mathbf{N}_p(P) d\Gamma_{\langle k\rangle} \tag{37}$$

$$\mathbf{Q}_{ik} = \int_{\Gamma_{f\langle k\rangle}} \mathbf{P}^*(P,P_i)\,\mathbf{N}(P) d\Gamma_{f\langle k\rangle} \tag{38}$$

To obtain the discretized form of eqn. (31) one observes that the boundary traction resultant at the point P belonging to the boundary element $\Gamma_{\langle j\rangle}$ can be written as

$$\begin{aligned}\mathbf{R}(P) = &\sum_{r=1}^{j-1} \int_{\Gamma_{\langle r\rangle}} \mathbf{N}_p(P)\, d\Gamma_{\langle r\rangle}\mathbf{t}_{\langle r\rangle} \\ &+ \int_{P_{\langle j\rangle}}^{P} \mathbf{N}_p(P)\, d\Gamma_{\langle j\rangle}\mathbf{t}_{\langle j\rangle} \\ = &\sum_{r=1}^{j-1} \mathbf{W}_r\mathbf{t}_{\langle r\rangle} + \Theta_j(P)\,\mathbf{t}_{\langle j\rangle}\end{aligned} \tag{39}$$

where $P_{\langle j\rangle}$ is the first node of the j-th boundary element. On the other hand, the discretized version of eqn. (31) for a collocation point P_i on the crack representative boundary is written as

$$\mathbf{c}\Lambda\mathbf{k} + \sum_{k=1}^{M} \tilde{\mathbf{H}}_{ik}\delta_{\langle k\rangle} + \sum_{k=1}^{m} \tilde{\mathbf{Q}}_{ik}\Delta\delta_{\langle k\rangle} + \sum_{k=1}^{M} \left(\tilde{\mathbf{G}}_{ik} + \tilde{\mathbf{Z}}_{ik}\right)\mathbf{t}_{\langle k\rangle} = \mathbf{0} \tag{40}$$

where

$$\tilde{\mathbf{H}}_{ik} = \int_{\Gamma_{\langle k \rangle}} \left[\mathbf{P}^* (P, P_i) - \mathbf{p}^* (P, P_i) \right] \mathbf{N}_u (P) d\Gamma_{\langle k \rangle} \tag{41}$$

$$\tilde{\mathbf{G}}_{ik} = -\int_{\Gamma_{\langle k \rangle}} \left[\mathbf{U}^* (P, P_i) - \mathbf{u}^* (P, P_i) \right] \mathbf{N}_p (P) d\Gamma_{\langle k \rangle} \tag{42}$$

$$\tilde{\mathbf{Q}}_{ik} = \int_{\Gamma_{f\langle k \rangle}} \left[\mathbf{P}^* (P, P_i) - \mathbf{p}^* (P, P_i) \right] \mathbf{N} (P) d\Gamma_{f\langle k \rangle} \tag{43}$$

$$\tilde{\mathbf{Z}}_{ik} = \Lambda \mathbf{H}^{-1} \sum_{j=k+1}^{M} \int_{\Gamma_{\langle j \rangle}} \frac{\partial \varphi^* (P, P_0)}{\partial \tilde{n}} \mathbf{W}_j d\Gamma_{\langle j \rangle} + \Lambda \mathbf{H}^{-1} \int_{\Gamma_{\langle k \rangle}} \frac{\partial \varphi^* (P, P_0)}{\partial \tilde{n}} \Theta_k d\Gamma_{\langle k \rangle} \tag{44}$$

The numerical solution of the problem is achieved according to the following scheme. The eq. (35) is collocated at the nodes on the external boundary. The Eq. (39) is collocated on the crack representative boundary at a number of source points equal to the number of unknown nodal crack relative displacements plus one. By so doing a linear algebraic system is obtained whose solution, taking into account the boundary conditions given in terms of prescribed displacements and tractions on the external boundary and zero relative displacement at the crack tips, provides the unknown displacements and tractions on the external boundary, the unknown crack relative displacements and the constant **k**.

5 Applications

To demonstrate the accuracy and efficiency of the proposed approach, some numerical results are presented for classical fracture mechanics problems widely analyzed in the literature [Sollero and Aliabadi (1995), Garcia, Saez and Dominguez (2004), Bowie and Freeze (1972), Sollero (1994); Aliabadi and Sollero (1998)].

The first application deals with the computation of the stress intensity factors for a square plate $h/w = 1$ with a central horizontal crack of length $2a$. The plate is loaded with a uniform traction at the two opposite sides parallel to the crack (see Figure 1). Different material properties have been considered in such a way that the shear modulus $G_{12} = 6GPa$ and the Poisson's ratio $\nu_{12} = 0.03$ have fixed values, whereas different values are set for the Young's moduli according to the relations $E_1 = G_{12}(\zeta + 2\nu_{12} + 1)$ and $E_2 = E_1\zeta$. The discretization employed consists of 60 linear boundary elements and the influence coefficients are computed by using 6-point Gauss quadrature. The crack collocation points corresponds to the midpoint between the crack element nodes. The stress intensity factors (SIF) are computed by the extrapolation method of the relative crack displacements [Pan (1997)], which are directly available from the solution. Results for the normalized mode I SIF $K_I/\sigma\sqrt{\pi a}$ are presented in Table 1 for the parameter ζ varying from 0.1 to 4.5 and a crack length $a = 0.2w$. In Table 1 the comparison of the present results with those available in the literature [Bowie and Freeze (1972), Garcia, Saez and Dominguez (2004), Sollero (1994)] is presented. Analyses have also been performed for a crack length $a = 0.5w$ and the comparison of the results with those of the references [Bowie and Freeze (1972), Sollero and Aliabadi (1995), Garcia, Saez and Dominguez (2004)] is shown in Table 2.

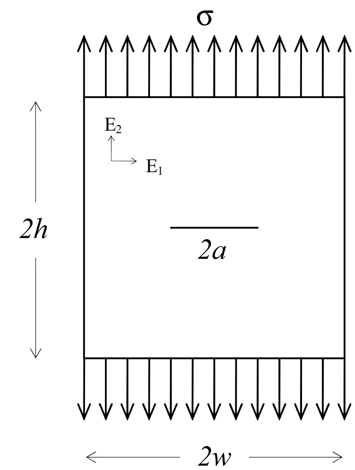

Figure 1: Square plate with a central horizontal crack.

Table 1: Normalized mode I SIF for an horizontal central crack $a/w = 0.2$ in a square plate

$\zeta = E_1/E_2$	Present	Bowie and Freeze (1972)	Sollero (1994)	Garcia, Saez and Dominguez (2006)
0.1	1.16	1.16	1.16	1.15
0.3	1.10	1.10	1.10	1.10
0.5	1.08	1.08	1.08	1.08
0.7	1.07	1.07	1.07	1.07
0.9	1.06	1.06	1.06	1.06
1.1	1.05	1.05	1.05	1.05
1.5	1.05	1.05	1.05	1.05
2.5	1.04	1.04	1.04	1.04
3.5	1.03	1.03	1.03	1.03
4.5	1.03	1.03	1.03	1.03

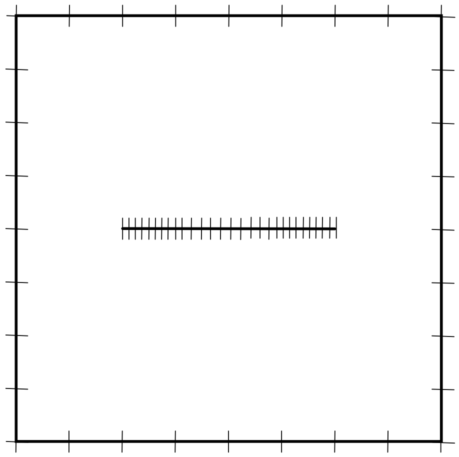

Figure 2: Mesh for the square plate with a central horizontal crack. (a/w=0.5)

The second example deals with an edge horizontal crack in a rectangular plate with $h/w = 2$ and $a/w = 0.5$ as shown in Figure 3. The plate consists of a graphite-epoxy material with orthotropic properties as $E_1 = 114.8GPa$, $E_2 = 11.7GPa$, $G_{12} = 9.66GPa$ and $\nu_{12} = 0.21$. The discretization employed consists of 50 linear boundary elements (see Figure 4) with the crack collocation points corresponding to the mid-point between the crack element nodes. The results obtained for the normalized mode I SIF are given in Table 3 where a comparison with literature values is presented [Aliabadi and Sollero (1998)]. The present results show an excellent agreement with those obtained by other methods and confirm the soundness of the method for cracks in orthotropic materials.

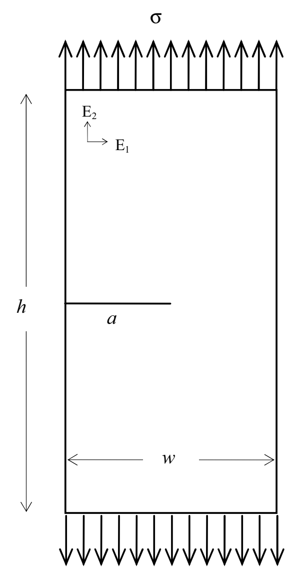

Figure 3: Rectangular plate with an edge horizontal crack.

6 Conclusions

An alternative single domain boundary element method for two-dimensional orthotropic elastic solids has been presented. The method rests on the use of additional integral equations deduced by using the stress function. These integral equations collocated on the crack representative boundary provide the relations needed to determine the solution in terms of external bound-

Table 2: Normalized mode I SIF for an horizontal central crack $a/w = 0.5$ in a square plate

$\zeta = E_1/E_2$	Present	Bowie and Freeze (1972)	Sollero and Aliabadi (1995)	Garcia, Saez and Dominguez (2006)
0.1	1.85	1.85	1.85	1.85
0.3	1.57	1.57	1.57	1.57
0.5	1.46	1.46	1.46	1.46
0.7	1.39	1.39	1.39	1.39
0.9	1.35	1.35	1.35	1.35
1.1	1.32	1.32	1.32	1.32
1.5	1.28	1.28	1.28	1.28
2.5	1.24	1.24	1.24	1.24
3.5	1.22	1.22	1.22	1.22
4.5	1.21	1.20	1.20	1.20

Table 3: Normalized mode I SIF for an horizontal edge crack in a rectangular orthotropic plate plate

	Present	Aliabadi and Sollero (1998)	Asadpoure, Mohammadi and Vafai (2006)*
$K_I/\sigma\sqrt{\pi a}$	2.94	2.96	2.8

* Extrapolated from graph.

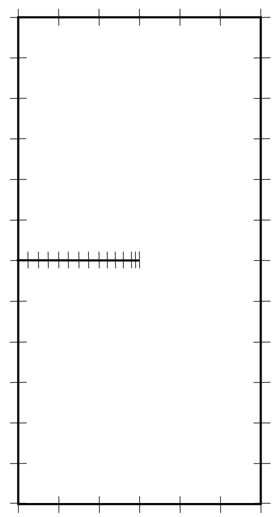

Figure 4: Mesh for edge horizontal crack.

ary displacements and tractions and crack relative displacements. No hypersingular integrals are involved in the formulation with the resulting simplification in numerical implementation. The numerical results obtained show the accuracy and efficiency of the proposed approach to determine the stress intensity factors pertaining to cracks in orthotropic materials

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