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# A Strain Energy Density Rate Approach to the BEM Analysis of Creep Fracture Problems

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**Abstract:** This paper explores the concept of strain energy density rate in relation to the crack initiation in fracture analysis problems arising in creeping cracked structural components. The analysis of the components is performed by using the boundary element methodology in association with the employment of singular boundary elements for the modeling of the crack tip region. The deformation of the material is assumed to be described by an elastic power law creep model. The strain energy density rate theory is applied to determine the direction of the crack initiation for a center cracked plate in tension which is subjected to Mode I loading conditions.

**keyword:** Fracture, Creep, Strain energy density rate, Boundary element method

## 1 Introduction

Cracks can degrade the integrity of structural components. This is a particular concern in the design of aircraft engines and steam turbines where the high temperature prevails and failure by creep components deformation is a concern. In these high level of temperatures the time-dependent creep fracture phenomenon can be considered as of multi-scale nature, particularly when physical size is scaled down to the dimensions of the material microstructure. For a dominant crack in metallic components that undergo creep deformation, the creation of macrocrack surface along the main crack (Mode I) path should be distinguished from the creation of microcrack surfaces off to the side of main crack where the creep enclaves are located. In this sense, creep fracture could be also considered as a multiscale process.

More than two decades ago, the strain energy density criterion was proposed by Sih (1973) as a fracture criterion in contrast to the conventional theory of G and K of the Griffith's energy release rate assumptions in elastic fracture mechanics. This provided an alternative approach to failure prediction for the same stress solution. The distinctions were empasized in the works of Sih (1991). The strain energy density criterion gained momentum and credibility in engineering. A review on the use of this criterion can be found in Gdoutos(1984) and Carpinteri (1986).

It is well known that for cases of realistic and practical problems in time-dependent fracture analysis of creeping cracked components the use of numerical solutions such as finite element method (FEM) and boundary element method (BEM) become imperative. For a review on the subject on can consult Beskos (1987).

In the search for an accurate, yet generalized, computational method for evaluating singular crack tip stress and strain fields, the singular element approach in conjunction with boundary element method (BEM) has been properly used in various fracture mechanics applications. Several researchers have contributed to this field: Blandford, Ingraffea & Ligget (1981) were the first who introduced the traction singular quarter-point boundary element approach in combination with a multi-domain formulation to the solution of both symmetrical and non-symmetrical crack problems. Thereafter, this approach has been extensively used in the application of the boundary element method to two- and three- dimensional crack problems. An extension of the quarter-point element technique was used by Hantschel, Busch, Kuna & Maschke (1990) who made an attempt to model crack tip fields arising in two-dimensional elastoplastic cracked panels by introducing some special singular boundary elements which took into account the HRR singularity field as presented in Hutchnison (1968) and Rice & Rosengren (1968) for locations near the crack tip.

In connection with the boundary element determination of near crack tip stress and strain fields in cracked structural components undergoing two-dimensional inelastic deformation one should mention the works of Professor Mukherjee and his co-workers for Mode I and II in Mukherjee & Morjaria (1981) and Morjaria & Mukher-

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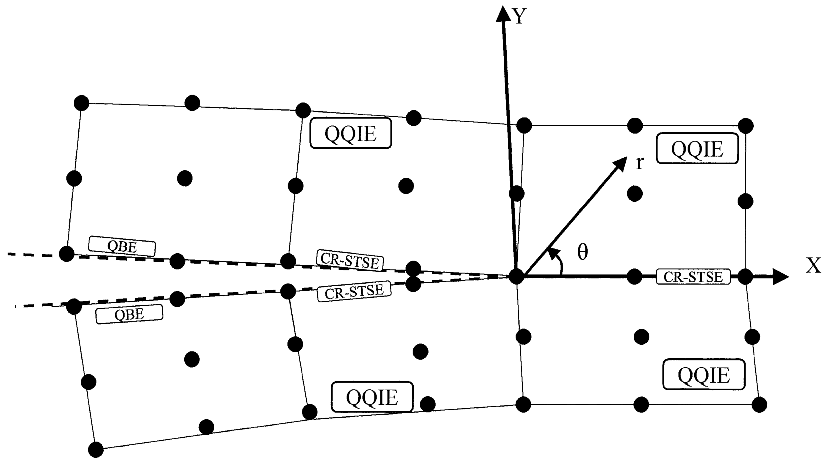


**Figure 1** : Geometry of the crack tip and CR-STSE element configuration

jee (1981) and Mode III in Morjaria & Mukherjee (1982). A more comprehensive review in BEM solutions of inelastic could be found in the review article of Aliabadi(1977).

In the present paper, the strain energy density rate concept is applied as a fracture criterion in association with the use of a previously developed, by the present authors, creep strain-traction singular element (CR-STSE) to determine the crack initiation involved in creeping cracked two-dimensional plates. A numerical example is presented for a shallow edge cracked plate (SENT). The creep constitutive model used in the numerical calculations is the Norton power law creep model (Nortan (1929)) but any other creep constitutive model having similar mathematical structure can be easily implemented in the proposed algorithm.

## 2 Asymptotic crack-tip fields in a creeping material

The material behavior in this paper is described by the elastic-nonlinear viscous constitutive relation according to the Norton power law relation (Nortran (1929))

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \dot{\varepsilon}_0 (\frac{\sigma}{\sigma_0})^m \tag{1}$$

where E is the elasticity modulus, $\sigma_0$ is a reference stress, $\dot{\varepsilon}_0$ is a reference creep strain rate and m is the creep exponent. Under the assumption of multiaxial stress states, the extension of equation (1) can be read as

$$\begin{aligned} \dot{\varepsilon}_{ij} &= \dot{\varepsilon}^e_{ij} + \varepsilon^n_{ij} \\ \dot{\varepsilon}^e_{ij} &= \frac{1+\nu}{E}\dot{S}_{ij} + \frac{1-2\nu}{3E}\dot{\sigma}_{kk}\delta_{ij} \\ \dot{\varepsilon}^n_{ij} &= \frac{3}{2}\dot{\varepsilon}_0 \left(\frac{\sigma_e}{\sigma_0}\right)^{m-1} \frac{S_{ij}}{\sigma_0} \end{aligned} \tag{2}$$

where $S_{ij}$ are the components of the deviatoric stress tensor and $S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$and $\sigma_e$ is the Misses effective stress defined by $\sigma_e = ((3/2)S_{ij}S_{ij})^{1/2}$. From the inspection of (1) and (2) it could be noted that if there is a singular crack tip field at time t=0 the elastic singularity fields prevail at the crack tip. In subsequent time step and at distances sufficient close to the crack tip the creep strain part of the total strain rate is much larger than the elastic strain rates and it seems to control the crack tip fields (m¿1). Thus, the constitutive equations (1) and (2) become power law creep relationships. Using the Hoff analogy (Hoff(1954)) to contrast the power-law creep relation with the power-law hardening relation, Riedel & Rice (1988) and Ohji, Ogura & Kubo (1979) presented the HRR-type singularity fields for power-law creep material described by the equations

$$\begin{aligned} \sigma_{ij} &= \sigma_0 \left(\frac{C(t)}{\dot{\varepsilon}_0\sigma_0 I_n r}\right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(\theta) \\ \dot{\varepsilon}_{ij} &= \dot{\varepsilon}_0 \left(\frac{C(t)}{\dot{\varepsilon}_0\sigma_0 I_n r}\right)^{\frac{n}{n+1}} \tilde{\varepsilon}_{ij}(\theta) \\ \dot{u}_i &= \dot{\varepsilon}_0 r \left(\frac{C(t)}{\dot{\varepsilon}_0\sigma_0 I_n r}\right)^{\frac{n}{n+1}} \tilde{u}_{ij}(\theta) \end{aligned} \tag{3}$$

where the radial distance r from the crack tip and the angle $\theta$ in relation to the x axis are shown in Figure 1. The dimensionless constants $I_n$ and the $\theta$-variation functions of the suitably normalized functions $\tilde{\sigma}_{ij}$, $\tilde{\varepsilon}_{ij}$ and $\tilde{u}_{ij}$ depend on the creep exponent m and have been tabulated in Shih(1983).

## 3 Derivation of boundary integral equations

The Navier equation for the displacement rates of a structural component undergoing plane strain deformation and under the presence of non-elastic strains can be written as

$$\dot{u}_{i,kk}+\frac{1+\nu}{1-\nu}\dot{u}_{k,ki}=-\frac{\dot{F}_i}{G}+2\dot{\varepsilon}^n_{ij,j}+\frac{2\nu}{1-\nu}\dot{\varepsilon}^n_{kk,i} \tag{4}$$

where $F_i$ is the prescribed body force per unit volume, G, $\nu$ and $\alpha$ are the shear modulus, Poisson's ratio and coefficient of linear thermal expansion, respectively, $u_i$ is the displacement vector. Suitable traction and displacement rate boundary conditions must be prescribed. The integral representation of the solution of a point P on the boundary of the body (with $\dot{F}_i=0$) has the following initial strain form

$$\begin{aligned}(\delta_{ij}-C)\,\dot{u}_i(P) &= \int_S [U_{ij}(P,Q)\,\dot{\tau}_j(Q) \\ &\quad - T_{ij}(P,Q)\,\dot{u}_j(Q)]\,dS_q \\ &\quad + \int_V \Sigma_{jki}(P,Q)\,\dot{\varepsilon}^n_{jk}(q)\,dV_q\end{aligned} \tag{5}$$

where $\delta_{ij}$ is Kronecker delta, P,Q are boundary points, q is an interior point, $\Gamma$ and $\Omega$ are the boundary and the surface of the body, respectively. The kernels $U_{ij}$,$T_{ij}$,$\Sigma_{jki}$and $\Sigma_{jki}$ are known singular solutions due to a point load in an infinite elastic solid in plane strain (Mukherjee (1977)). The traction and displacement rates are denoted by $\dot{\tau}$ and $\dot{\tau}$, respectively. The coefficients $C_{ij}$ are known functions of the included angle at the boundary corner at P, the angle between the bisector of the corner angle and the x-axis. Equation (5) is a system of integral equations for the unknown traction and displacements rates in terms of their prescribed values on the boundary, and the non-elastic strain rates. The unknown quantities only appear on the boundary of the body and the surface integrals are known at any time through the constitutive equations.

The stress rates can be obtained by direct differentiation of equation (5) resulting in

$$\begin{aligned}\dot{\sigma}_{ij}(p) &= \int_\Gamma \left[\overline{U}_{ijk}(p,Q)\,\dot{\tau}_k(Q) - \overline{T}_{ijk}(p,Q)\,\dot{u}_k(Q)\right] d\Gamma_q \\ &\quad - 2G\dot{\varepsilon}^n_{ij}(p) - 3K\alpha\dot{T}(p)\delta_{ij} + \int_\Omega [\overline{\Sigma}_{ijkl}(p,q)\,\dot{\varepsilon}^n_{kl}(q) \\ &\quad + \overline{\overline{\Sigma}}_{ijkl}(p,q)\delta_{kl}\alpha\dot{T}(q)]d\Omega_q \qquad (i,j,k,l=1,2)\end{aligned} \tag{6}$$

where G and K are the shear and bulk modulus, respectively; $\Sigma_{jki}$and $\Sigma_{jki}$ are inelastic and temperature effect kernel functions, respectively, which are also defined in the work of Mukherjee(1977).

## 4 Singular element implementation and solution procedure

The integral equations (5) and (6) are expressed in this paper by discretizing the boundary and the interior into a number of standard three-noded quadratic boundary elements and nine-noded quadratic quadrilateral interior surface elements, respectively, provided that they are not adjacent to the crack tip.

By following the procedure developed in Providakis & Kourtakis (2002) to produce a special element which presents the HRR-type singularity of equations (3) at the crack tip (Figure 1), one can obtain the following new set of shape functions $N^u_a$ which depend upon the creep exponent m

$$\begin{aligned}N^u_1 &= 2^{\frac{1}{1+m}}[(\frac{r}{l})^{1+\frac{1}{1+m}}-(\frac{r}{l})^{\frac{1}{1+m}}]+(\frac{r}{l}) \\ N^u_2 &= 2^{1+\frac{1}{1+m}}[(\frac{r}{l})^{\frac{1}{1+m}}-(\frac{r}{l})^{1+\frac{1}{1+m}}] \\ N^u_3 &= 2^{\frac{1}{1+m}}[(\frac{r}{l})^{1+\frac{1}{1+m}}-(\frac{r}{l})^{\frac{1}{1+m}}]-(\frac{r}{l})+1\end{aligned} \tag{7}$$

where l is the length of the new special quadratic element, the distance r=l-x and the ratio can be defined in terms of the intrinsic coordinate $\zeta$ as (r/l)=(1-$\zeta$)/2. By taking the derivatives of the new shape functions (7) one can observe that these derivatives display a $r^{-m/(m+1)}$ singularity near the crack tip which is the actual situation for the strain rate singularities according to (3).

Since in boundary element methodology displacement and tractions are independently represented the above derived singular element for the simulation of crack tip behavior of displacement rates, fails to model the expected from equations (3) crack tip behavior of tractions which displays an order of –1/(m+1) singularity. Thus, for the proper simulation of the traction rate singularity different shape functions are derived by the use of the derivatives of the shape functions (7) and finally modified to the following separate forms $N_a^t$ in terms of creep exponent m

$$
\begin{aligned}
N_1^t &= 2^{\frac{m}{1+m}}[(\frac{l}{r})^{\frac{1}{1+m}} - (\frac{r}{l})^{\frac{m}{1+m}}] - 2 + 2(\frac{r}{l}) \\
N_2^t &= 2^{\frac{m}{1+m}}[(\frac{l}{r})^{\frac{1}{1+m}} - (\frac{r}{l})^{\frac{m}{1+m}}] \\
N_3^t &= 2^{\frac{m}{1+m}}[-(\frac{l}{r})^{\frac{1}{1+m}} + (\frac{r}{l})^{\frac{m}{1+m}}] + 1
\end{aligned} \tag{8}
$$

where now r=x and the ratio (r/l)=(1+ζ)/2. A simultaneous simulation of displacement and traction rate fields, by the use of the shape functions (7) and (8), respectively, yields to the proposed, in the present BEM approach, creep strain-traction singular element (CR-STSE) (Figure 1). Then, by applying a boundary nodal point collocation procedure to the discretized versions of equations (5) and (6) one can obtain the following system of equations in matrix form

$$[A]\{\dot{u}\} = [B]\{\dot{\tau}\} + \{\dot{b}\} \tag{9}$$

$$[A]\{\dot{u}\} = [B]\{\dot{\tau}\} + \{\dot{b}\} \tag{10}$$

However, the vector $\{\dot{b}\}$ is known at any time through the constitutive equations and the stress rates of equation (6) while the vector $\{\dot{b}\}$ could be easily computed through the known values of temperature profile for the whole structural component. Half of the total number of components of $\{\dot{u}\}$ and $\{\dot{\tau}\}$ are prescribed through the boundary condition while the other half are unknowns.

Then, the initial distribution of the nonelastic strain has to be prescribed. Thus, the only existed strains at time step t=0 are elastic and then, the thermal and initial stresses and displacements can be obtained from the solution of the corresponding elastic problem. By the use of equations (9) and (10) the displacement and stress rates can be obtained at time step t=0 while the rates of change of the nonelastic strains can be computed from constitutive equations. Thus, the initial rates of all the relevant variables are now known and their values at a new time Δt can be obtained by integrating forward in time. The rates are then obtained at time Δt and so on, and finally the time histories of all the variables can be computed. Another important task in this approach is the choice of a suitable time integration scheme. For the purposes of the present paper, an Euler type algorithm with automatic time-step control is employed.

## 5 The strain energy rate

For power law creep materials the strain energy density rate (SEDR) can be analytically determined as

$$[A]\{\dot{u}\} = [B]\{\dot{\tau}\} + \{\dot{b}\} \tag{11}$$

The SEDR was estimated according to the boundary element procedure developed previously by solving in time the system of equations (9) and (10) and then by using the analytic equation (11) for each time step.

## 6 Numerical results

Consider a central cracked plate (CCP) specimen with height (h) and width (w) = 40.6 cm x 20.3 cm made by a power law creeping material (superalloy Inconel 800H at 650$^o$C) with properties E=153.7 GPa, $\sigma_0$=417.04 MPa, ν=0.33, creep exponent m=5 and the parameter

$$B = \frac{\dot{\varepsilon}_0}{\sigma_0^m} = 2.1x10^{-32}x6894.73 \quad (Pa)^{-5}/h \tag{12}$$

The specimen contains a central crack of depth a= 0.125w. The specimen is subjected to a remote uniform load of 129.2 MPa which is suddenly applied. The symmetry of the specimen was used and thus a quarter of the plate was analyzed.

The computer software established according to the BEM methodology presented in this paper can provide sufficient data related to stress and strain distribution history. Based on these data and after using equation (11) the strain energy density rate distribution could be predicted at each time step. Typical diagrams of contours of strain energy density rate distributions for a quadrant of the specimen and for two different time steps (t=0.0 and 5.01 hours) are shown in Figures 2 and 3, respectively.

Depicted in Figure 4 are the angular variations of the strain energy density rate for different time steps. It could be noted from the inspection of this figure that the strain energy density rate decreases with increasing time. This trend was expected since the applied load remains fixed in time and the crack is assumed to be stationary. It could be also noted that all curves posses a minimum at angle $\theta_o$=0.

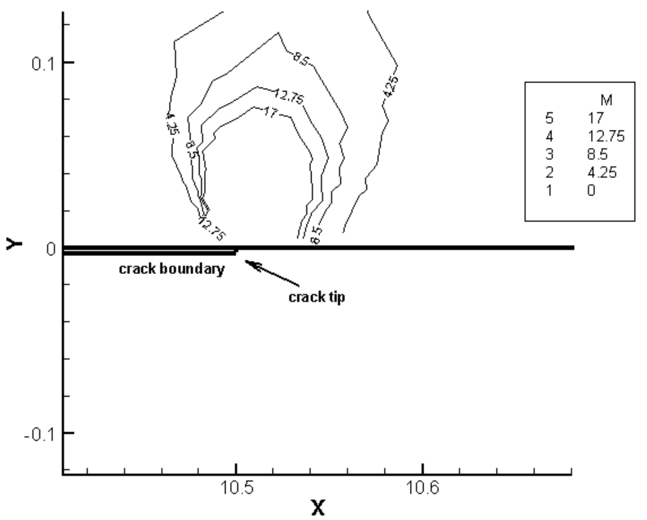


**Figure 2** : Contours of strain energy density rate for time step t=0.0 hours

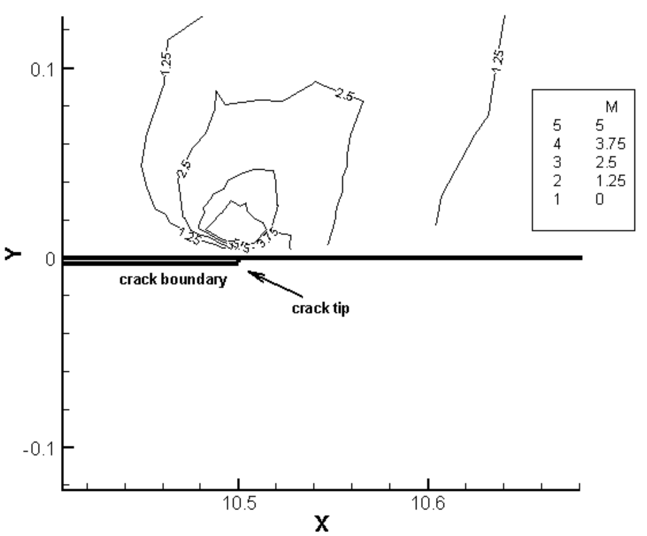


**Figure 3** : Contours of strain energy density rate for time step t=5.01 hours

Taking into account the strain energy density criterion this indicates that crack would initiate at $\theta_o$=0 and along the axis of symmetry of load symmetry under present crack mode I.

## 7 Conclusions

In this paper a new singular boundary element approach based on the implementation of a special singular boundary element is performed for the estimation of the strain

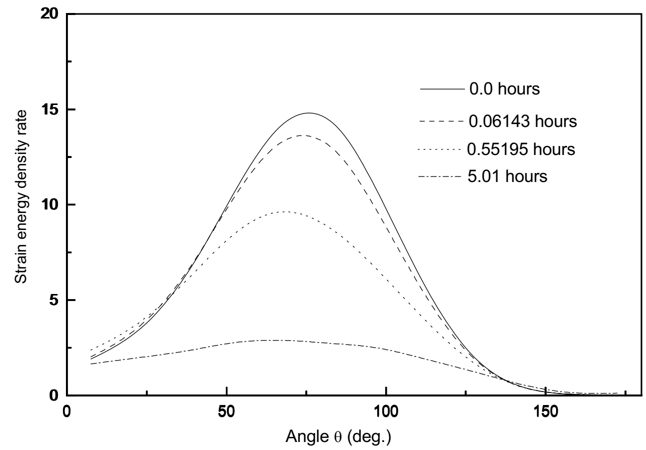


**Figure 4** : Angular variation of strain energy rate for different time steps.

energy density rate distribution close to crack tip fields arising in creeping structural components undergoing Mode I deformation under the effect of remote loading condition.

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