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Deriving Shear Correction Factor for Thick Laminated Plates Using the Energy Equivalence Method

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Abstract: The cylindrical bending of thick laminated sandwich plates under static loading is studied based on the first order shear deformation theory (FSDT). FSDT generally requires a shear correction factor (SCF) to account for the deflection owing to the transverse shear. In this paper the SCF is derived using energy equivalence method. It is shown that depending on the mechanical and geometrical properties of the layers, the contribution of the transverse shear stress to the maximum deflection of the plate is variable and in some cases account for up to around 88% of the total deflection. The effects of non-dimensional parameters such as layers tensile and shear modulus ratio and layers thickness ratio on the SCF and on the maximum deflection are investigated. The analytical results are compared and verified with the finite element analysis.

keyword: Laminated Plates, Shear Correction Factor, Energy Equivalence Method, Cylindrical Bending, Transverse Shear

1 Introduction

Composite materials and structures are increasingly applied in various designs applications during the recent past decades, specially in high-tech sectors of industries such as aerospace and automotive. One of the main advantages of composite materials is their superiority in high strength to weight and stiffness to weight ratios. The rapid broadening areas of applications of composite materials require a continuous vigorous investigation on their mechanical behaviour leading to performance improvement of these materials.

In thick laminates, i.e. laminates with width-to-thickness ratio less than about 10, the analysis based on classical laminate theory (CLT) shows significant differences in the deflection and stress distribution of the laminates with the true mechanical behaviour. An improvement was introduced by the first order shear deformation theory (FSDT) proposed by Reissner [Reissner (1945)] based on stress approach and Mindlin [Mindlin (1951)] based on displacement approach. The displacement based FSDT is more widely used, though this leads to a uniform transverse shear strains through the plate thickness and this requires a shear correction factor to accommodate parabolic transverse shear stresses. Discussion about these approaches can be found elsewhere [Wang, et al (2001)]. The introduction of correction factors for the transverse shear moduli of the laminate in FSDT is an extension to Reissner [Reissner (1945)] and Mindlin [Mindlin (1951)] theories in the case of isotropic homogeneous plates. There has been an elaborate discussion about the shear correction factor in the literature for different test geometries (see [Whitney and Pagano (1970); Whitney (1972); Whitney (1973) and Kaneko (1975)] for an overview). The shear correction factor published by Stephen [Stephen (1980); Stephen and Hutchinson (2001)] was the first to incorporate a dependence on the aspect ratio of the cross-section. The same results were obtained by Hutchinson [Hutchinson (2001)] with a simple dynamic beam theory in a recent publication. Isaksson *et al.* [Isaksson , et al (2006)] derived shear correction factors from an equilibrium stress field for a corrugated board sandwich panel. Tanov and Tabiei [Tanov and Tabiei (2000)] used finite element analysis to obtain SCF. Puchegger *et al*. [Puchegger, et al (2003)] validated the SCF experimentally.

Most of these methods are based on equating certain global response of FSDT with its peer in elasticity theory. These global responses include transverse shear strain energy, natural frequency associated with the transverse shear vibration mode, and the velocity of propagation of a wave [Noor and Burton (1982)]. Many of the approaches to calculate the SCF can be found in [Cowper (1966); Chow (1971); Bert (1973); Dharmarajan and McCutchen (1973); Bank (1987)]. The most prevailing method, however, is equating transverse shear strain en-

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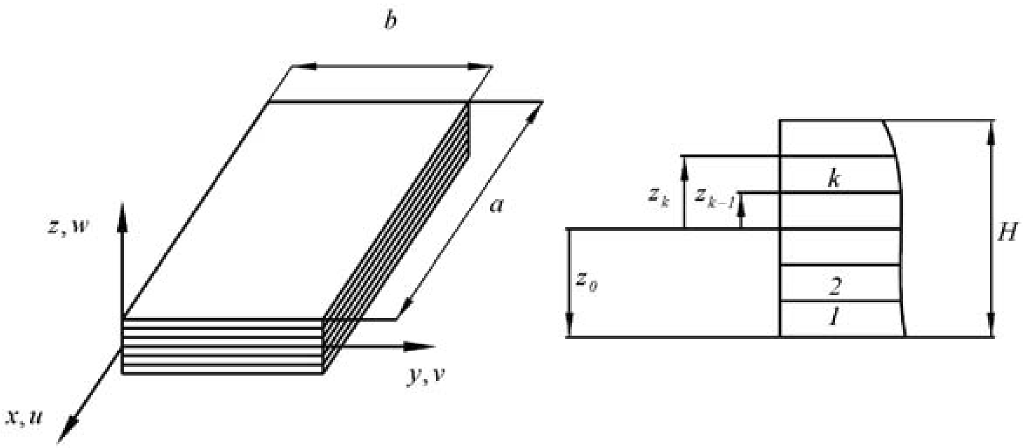

Figure 1 : Rectangular Laminated Plate

ergies obtained from FSDT to those obtained from three dimensional elasticity.

In the present study, the transverse strain energy equivalence method has been adopted to derive the SCF. The method was applied to derive the SCF for the two cases; an orthotropic single layer simply supported at its two opposite edges and a clamped-free laminated sandwich plate, assuming cylindrical bending. In each case the maximum deflection of the plate has been calculated and the contribution of transverse shear stress on the total deflection determined. Parametric analysis of the effects of layers tensile modulus ratio, layers transverse shear modulus ratio, and layers thickness ratio on the SCF and maximum deflection of the plate has been explored. The analytical results were verified using finite element analysis (FEA). It is shown that the analytical results correlate closely with those from the FEA.

2 Theoretical analysis

2.1 Displacement field

Consider a rectangular laminated plate of length a, width b and the total thickness H, composed of N orthotropic homogeneous layers, with a *Cartesian coordinate system* as illustrated in Fig. 1. The displacement field of the FSDT is given as

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\psi_x(x,y), \\ v(x,y,z) &= v_0(x,y) + z\psi_y(x,y), \\ w(x,y,z) &= w(x,y). \end{aligned} \quad (1)$$

where u, v and w are the displacements along x, y and z directions, u_0 and v_0 are the displacements of a point on the mid-plane and ψ_x and ψ_y are rotations about y and x axes, respectively.

The strain-displacement relations associated with Eq. 1 can be stated as

$$\begin{aligned} &\varepsilon_x = \varepsilon_x^0 + z\kappa_x, \quad \varepsilon_y = \varepsilon_y^0 + z\kappa_y, \quad \varepsilon_z = 0, \\ &\gamma_{yz} = \gamma_{yz}^0, \quad \gamma_{xz} = \gamma_{xz}^0, \\ &\gamma_{xy} = \gamma_{xy}^0 + z\kappa_{xy}. \end{aligned} \quad (2)$$

where

$$\begin{aligned} &\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \quad \kappa_x = \frac{\partial \psi_x}{\partial x}, \quad \varepsilon_y^0 = \frac{\partial v_0}{\partial y}, \quad \kappa_y = \frac{\partial \psi_y}{\partial y}, \\ &\gamma_{yz}^0 = \psi_y + \frac{\partial w}{\partial y}, \quad \gamma_{xz}^0 = \psi_x + \frac{\partial w}{\partial x}, \\ &\gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}, \quad \kappa_{xy} = \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}. \end{aligned} \quad (3)$$

Considering the cylindrical bending about y axis, i.e., $\frac{\partial}{\partial y} \equiv 0$, Eq. 3 is simplified to:

$$\begin{aligned} &\varepsilon_x^0 = \frac{du_0}{dx}, \quad \kappa_x = \frac{d\psi_x}{dx}, \quad \varepsilon_y^0 = \kappa_y = 0, \\ &\gamma_{yz}^0 = \psi_y, \quad \gamma_{xz}^0 = \psi_x + \frac{dw}{dx}, \\ &\gamma_{xy}^0 = \frac{dv_0}{dx}, \quad \kappa_{xy} = \frac{d\psi_y}{dx}. \end{aligned} \quad (4)$$

2.2 Constitutive relations

Assume that the orthotropic directions of each layer in the laminated plate are parallel to xyz directions. The stress-strain relationships for the kth layer are given by

$$\left\{\begin{array}{c}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{array}\right\}^{(k)}=\left[\begin{array}{ccc}Q_{11}&Q_{12}&0\\ Q_{12}&Q_{22}&0\\ 0&0&Q_{66}\end{array}\right]^{(k)}\left\{\begin{array}{c}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{array}\right\}^{(k)}, \quad (5)$$

and

$$\left\{\begin{array}{c}\sigma_{yz}\\ \sigma_{xz}\end{array}\right\}^{(k)}=\left[\begin{array}{cc}C_{44}&0\\ 0&C_{55}\end{array}\right]^{(k)}\left\{\begin{array}{c}\gamma_{yz}\\ \gamma_{xz}\end{array}\right\}^{(k)}. \quad (6)$$

where in Eq. 5, the reduced stiffness components, $Q_{ij}^{(k)}$, are

$$Q_{11}^{(k)}=\frac{E_x^{(k)}}{1-\nu_{xy}^{(k)}\nu_{yx}^{(k)}},\quad Q_{12}^{(k)}=\frac{\nu_{yx}^{(k)}E_x^{(k)}}{1-\nu_{xy}^{(k)}\nu_{yx}^{(k)}}=\frac{\nu_{xy}^{(k)}E_y^{(k)}}{1-\nu_{xy}^{(k)}\nu_{yx}^{(k)}},$$

$$Q_{22}^{(k)}=\frac{E_y^{(k)}}{1-\nu_{xy}^{(k)}\nu_{yx}^{(k)}},\quad Q_{66}^{(k)}=G_{xy}^{(k)} \quad (7)$$

Also in Eq. 6, the stiffness components $C_{ij}^{(k)}$, are given by

$$C_{44}^{(k)}=G_{yz}^{(k)},\quad C_{55}^{(k)}=G_{xz}^{(k)} \quad (8)$$

where $E_x^{(k)}$, $E_y^{(k)}$ are the tensile modulus, $G_{yz}^{(k)}$, $G_{xz}^{(k)}$ are the transverse shear modulus and $\nu_{xy}^{(k)}$, $\nu_{yx}^{(k)}$ are the Poisson's ratios of the kth layer.

The stress and moment resultants are

$$(N_x,N_y,N_{xy})=\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}(\sigma_x^{(k)},\sigma_y^{(k)},\sigma_{xy}^{(k)})dz,$$

$$(M_x,M_y,M_{xy})=\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}(\sigma_x^{(k)},\sigma_y^{(k)},\sigma_{xy}^{(k)})zdz,$$

$$(Q_y,Q_x)=\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}(\sigma_{yz}^{(k)},\sigma_{xz}^{(k)})dz. \quad (9)$$

Now, considering Eqs. 2, 5, 6 and 9, the laminates constitutive relations can be stated by

$$\left\{\begin{array}{c}N_x\\ N_y\\ N_{xy}\end{array}\right\}=\left[\begin{array}{ccc}A_{11}&A_{12}&0\\ A_{12}&A_{22}&0\\ 0&0&A_{66}\end{array}\right]\left\{\begin{array}{c}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{array}\right\}+\left[\begin{array}{ccc}B_{11}&B_{12}&0\\ B_{12}&B_{22}&0\\ 0&0&B_{66}\end{array}\right]\left\{\begin{array}{c}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{array}\right\},$$

$$\left\{\begin{array}{c}M_x\\ M_y\\ M_{xy}\end{array}\right\}=\left[\begin{array}{ccc}B_{11}&B_{12}&0\\ B_{12}&B_{22}&0\\ 0&0&B_{66}\end{array}\right]\left\{\begin{array}{c}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{array}\right\}+\left[\begin{array}{ccc}D_{11}&D_{12}&0\\ D_{12}&D_{22}&0\\ 0&0&D_{66}\end{array}\right]\left\{\begin{array}{c}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{array}\right\}$$

and

$$\left\{\begin{array}{c}Q_y\\ Q_x\end{array}\right\}=\left[\begin{array}{cc}\frac{A_{44}}{K_y}&0\\ 0&\frac{A_{55}}{K_x}\end{array}\right]\left\{\begin{array}{c}\gamma_{yz}^0\\ \gamma_{xz}^0\end{array}\right\} \quad (10)$$

where

$$(A_{ij},B_{ij},D_{ij})=\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}Q_{ij}^{(k)}(1,z,z^2)dz \quad (i,j=1,2,6),$$

$$A_{ij}=\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}C_{ij}^{(k)}dz \quad (i,j=4,5). \quad (11)$$

In Eq. 10, K_x and K_y are the SCFs.

3 Deriving shear correction factors

The strain energy owing to the transverse shear component, σ_{xz}, can be obtained from

$$U_s=\frac{1}{2}\iiint_V\sigma_{xz}\gamma_{xz}dV=\frac{1}{2}\iiint_V\frac{\sigma_{xz}^2}{G_{xz}}dV \quad (12)$$

On the other hand, based on the FSDT, the strain energy owing to the transverse shear can be calculated from

$$U_s=\frac{1}{2}\iiint_v\sigma_{xz}\gamma_{xz}dv=\frac{1}{2}\int_0^a\int_0^b\int_{-\frac{H}{2}}^{\frac{H}{2}}\sigma_{xz}\gamma_{xz}dzdydx$$

This can be rewritten as

$$U_s=\frac{1}{2}\int_0^a\int_0^b(\int_{-\frac{H}{2}}^{\frac{H}{2}}\sigma_{xz}dz)\gamma_{xz}dydx \quad (13)$$

The transverse shear force can be obtained from

$$Q_x=\int_{-\frac{H}{2}}^{\frac{H}{2}}\sigma_{xz}dz$$

Also from Eq. 11 we have

$$\gamma_{yz}=\frac{K_x}{A_{55}}Q_x$$

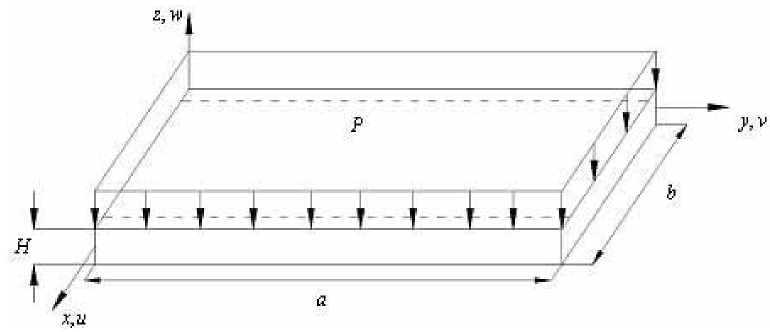

Figure 2 : Orthotropic rectangular plate simply supported at its two opposite edges

Substituting above into Eq. (13) results in

$$U_s = \frac{1}{2}\int_0^a \int_0^b K_x \frac{Q_x^2}{A_{55}}\, dy\, dx. \tag{14}$$

Equating Eq. 12 to Eq. 14, one can obtain the shear correction factor, K_x. Through a similar approach K_y can also be obtained.

4 Case studies

The above procedure will be applied to obtain SCF for the following two cases and the results will be compared with those from the FEA using ABAQUS software [ABAQUS User's Manual].

4.1 Case 1: An orthotropic single layer

Consider an orthotropic single layer rectangular plate, simply supported at its two opposite edges. The plate carries on the top surface at $z = H/2$, a uniformly distributed load of intensity P acting in the z-direction, see Fig. 2. Under the assumption $b >> a$, the plate deform under cylindrical bending about the y axis. Because of symmetrical conditions, the governing equilibrium equations owing to bending are decoupled from those of stretching. The governing equilibrium equations can be stated as

$$\frac{dM_x}{dx} - Q_x = 0,$$
$$\frac{dM_{xy}}{dx} - Q_y = 0,$$
$$\frac{dQ_x}{dx} - P = 0. \tag{15}$$

and the boundary conditions (B.C.'s) are

$$M_x = \psi_y = w = 0 \quad \text{at } x = 0 \text{ and } x = a. \tag{16}$$

Considering the constitutive relations, Eq. 15 can be stated in terms of the displacement components, if solved regarding B.C.'s Eq. 16, the results will be

$$w = -\frac{1}{24}\frac{P}{D_{11}}x^4 + \frac{1}{12}\frac{Pa}{D_{11}}x^3 + \frac{1}{2}\frac{K_xP}{A_{55}}x^2$$
$$-(\frac{1}{2}\frac{K_xPa}{A_{55}} + \frac{1}{24}\frac{a^3P}{D_{11}})x,$$
$$\psi_x = \frac{1}{6}\frac{P}{D_{11}}x^3 - \frac{1}{4}\frac{Pa}{D_{11}}x^2 + \frac{1}{24}\frac{a^3P}{D_{11}},$$
$$\psi_y \equiv 0. \tag{17}$$

Considering the elasticity equilibrium equations, one can write

$$\sigma_{xz} = -\int_{-\frac{H}{2}}^{z_0} \frac{\partial \sigma_x}{\partial x} dz. \tag{18}$$

Using Eq. 17 and the constitutive relations, σ_{xz} will be obtained from Eq. 18,

$$\sigma_{xz} = \frac{1}{128}\frac{Q_{11}(\frac{a}{2}-x)(z-\frac{H}{2})(z+\frac{H}{2})}{D_{11}}P. \tag{19}$$

From Eq. 19 and Eqs. 13, 14, 17 together with the constitutive relations, K_x will be obtained as

$$K_x = \frac{6}{5}, \tag{20}$$

Timoshenko obtained this correction factor by comparing his solution with a 2-D solution of the bending problem, and found a dependence on the Poisson's ratio ν [Timoshenko (1922)] as $K_{Timoshenko} = (6+5\nu)/(5+5\nu)$. Cowper derived a solution for the shear correction [Cowper (1966)], which differs only slightly from Timoshenko's solution $K_{Cowper} = (12+11\nu)/10(1+\nu)$. Both solution gave the same results as in Eq. 20 only if $\nu = 0$.

The maximum deflection can be found from Eq. 17,

$$w_{\max} = w\left(\frac{a}{2}\right) = -\frac{5}{32}\frac{Pa^4\left(1-v_{xy}v_{yx}\right)}{E_xH^3} - \frac{3}{20}\frac{Pa^2}{G_{xz}H}, \tag{21}$$

where the first term is the deflection due to the moment and the second term is the deflection due to the transverse shear. Let's define the dimensionless maximum deflection, $\hat{w}_{\max}$, as

$$\hat{w}_{\max} = \frac{E_xH^3}{12Pa^4\left(1-v_{xy}v_{yx}\right)}w_{\max}, \tag{22}$$

The selected layer material properties and loading are E_x=40 GPa, v_{xy}=0.3, v_{yz}=0.075, a=150 mm and P =1 MPa. For this case, the variation of $\hat{w}_{\max}$ versus E_x/G_{xz} and H/a is plotted in Fig. 3. It can be seen that by increasing E_x/G_{xz} at a constant H/a ratio, the contribution of deflection owing to shear will increase linearly, while by increasing H/a at a constant E_x/G_{xz} ratio, the contribution of deflection due to shear will increase parabolically. For example at E_x/G_{xz} = 100 and H/a = 0.1 about 49.55% of deflection was owing to the transverse shear. This problem was also solved with ABAQUS software using shell elements. In the FE analysis the S4R elements were used, a 4-node doubly curved thin and thick shell element with linear shape function and reduced integration. The B.C.'s were encastre (U1=U2=U3=UR1=UR2=UR3=0). The deformed shape of the plate is shown in Figure 4. The relative error between the analytical and the FEA deflection results is 8.4%.

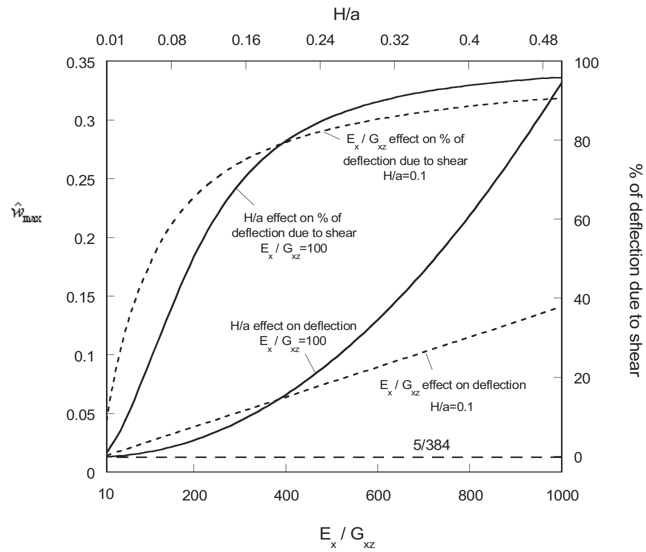

Figure 3 : Effect of E_x/G_{xz} and H/a on the deflection due to shear

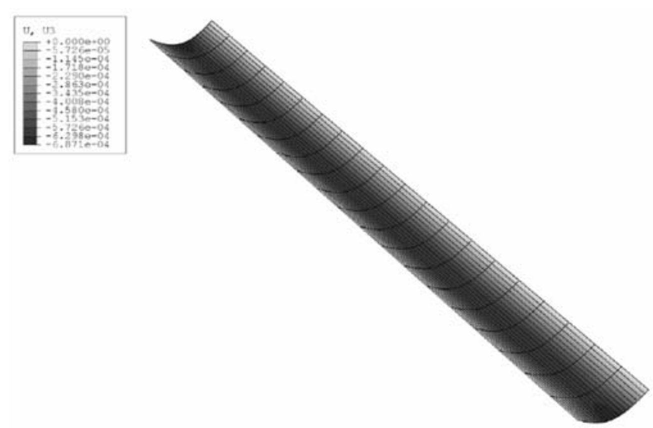

Figure 4 : Deformed shape of the orthotropic single layer rectangular plate from the FEA

4.2 Case 2: An orthotropic sandwich plate

The second case study is an orthotropic sandwich plate with clamped-free B.C.'s, subjected to a uniform load P on its free edge under the cylindrical bending about $y$axis (see Fig. 5) where $b >> a$. The governing equilibrium equations with the symmetrical conditions will become

$$\begin{aligned} &\frac{dM_x}{dx} - Q_x = 0, \\ &\frac{dM_{xy}}{dx} - Q_y = 0, \\ &\frac{dQ_x}{dx} = 0. \end{aligned} \tag{23}$$

The imposed B.C.'s are

$$\begin{aligned} &\psi_x = \psi_y = w = 0 \quad \text{at } x = 0, \\ &M_x = M_{xy} = 0, \quad Q_x = -P \quad \text{at } x = a. \end{aligned} \tag{24}$$

Solving the boundary value problem, Eq. 23 and Eq. 24, and considering the constitutive relations results in

$$\begin{aligned} &w = -\frac{1}{6}\frac{Px^2(3a-x)}{D_{11}} - \frac{K_xPx}{A_{55}} \\ &\psi_x = \frac{1}{2}\frac{Px(2a-x)}{D_{11}} \\ &\psi_y \equiv 0 \end{aligned} \tag{25}$$

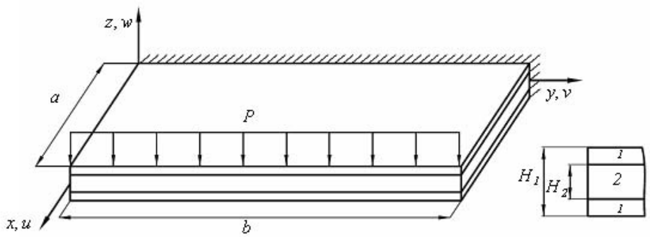

Figure 5 : Sandwich orthotropic plate with clamped-free B.C.'s

From Eq. 18, Eq. 25 and the constitutive relations, σ_{xz} becomes

$$\sigma_{xz} = \begin{cases} \frac{1}{2}\frac{Q_{11}^{(1)}P\left(z^2-\frac{H_1^2}{4}\right)}{D_{11}}, & -\frac{H_1}{2}<z<-\frac{H_2}{2} \\ \frac{1}{8}\frac{Q_{11}^{(1)}P\left(H_2^2-H_1^2\right)}{D_{11}}+\frac{1}{2}\frac{Q_{11}^{(2)}P\left(z^2-\frac{H_2^2}{4}\right)}{D_{11}}, & -\frac{H_2}{2}<z<\frac{H_2}{2} \\ \frac{1}{2}\frac{Q_{11}^{(1)}P\left(z^2-\frac{H_1^2}{4}\right)}{D_{11}}, & \frac{H_2}{2}<z<\frac{H_1}{2} \end{cases} \tag{26}$$

Using the same approach as in Case 1 and considering Eq. 7, one can obtain K_x from Eq. 26 as

$$K_x = \frac{3\left[G_{xz}^{(1)}(H_1-H_2)+G_{xz}^{(2)}H_2\right]}{2\left[E_x^{(1)}(H_1^3-H_2^3)+E_x^{(2)}H_2^3\right]^2} \times \left\{ \frac{E_x^{(2)}H_2^3\left[E_x^{(1)}H_1^2+(\frac{4}{5}E_x^{(2)}-E_x^{(1)})H_2^2\right]}{G_{xz}^{(2)}} + \frac{{E_x^{(1)}}^2(\frac{4}{5}H_1^5+\frac{1}{5}H_2^5-H_1^2H_2^3)}{G_{xz}^{(1)}} + 3{E_x^{(1)}}^2H_2(H_1^2-H_2^2) \times \frac{\left[\frac{1}{3}H_2^2+H_1^2(\frac{G_{xz}^{(1)}}{G_{xz}^{(2)}}-1)-H_2^2\frac{G_{xz}^{(1)}}{G_{xz}^{(2)}}(1-\frac{2E_x^{(2)}}{3E_x^{(1)}})\right]}{2G_{xz}^{(1)}} \right\} \tag{27}$$

The detail derivations of Eq. (27) are presented in Appendix A.

4.2.1 Isotropic sandwich plate

First consider the case that each layer has isotropic properties. The variation of K_x with respect to $E^{(1)}/E^{(2)}$ and H_2/H_1 is shown in Fig. 6.

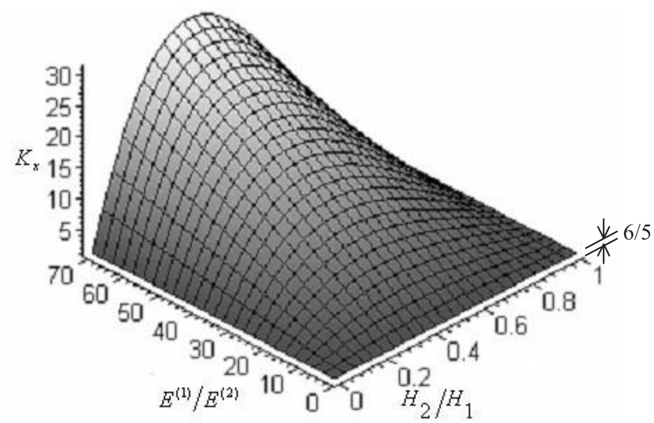

Figure 6 : Variation of K_x with $E^{(1)}/E^{(2)}$ and H_2/H_1 for an isotropic sandwich plate.

It can be seen from Figure 6 that for $H_2/H_1 = 0$ or 1 and for all $E_x^{(1)}/E_x^{(2)}$, the sandwich plate becomes a single layer and $K_x = 6/5 = 1.2$, as before in case 1. Also for $E_x^{(1)}/E_x^{(2)} = 1$ and for all H_2/H_1, the sandwich plate becomes a single layer and $K_x = 6/5 = 1.2$. At any other constant value of H_2/H_1, K_x increases by increasing $E_x^{(1)}/E_x^{(2)}$. At any $E_x^{(1)}/E_x^{(2)}$, the maximum K_x is at $H_2/H_1 = 0.5$, i.e. a sandwich plate made of half core material and half skin material.

4.2.2 Orthotropic sandwich plate

In this case it is assumed that all layers have orthotropic properties. For this case the variation of K_x versus $E_x^{(1)}/E_x^{(2)}$ and H_2/H_1 was investigated for a wide range of $1 \le G_{xz}^{(1)}/G_{xz}^{(2)} \le 600$. The results for $G_{xz}^{(1)}/G_{xz}^{(2)} = 1$, 2, 200 and 600 are shown in Fig. 7. The general pattern of K_x for $G_{xz}^{(1)}/G_{xz}^{(2)} > 10$ is similar to $G_{xz}^{(1)}/G_{xz}^{(2)} = 200$ case but with different K_x value.

It can be seen from Fig. 7 that for $H_2/H_1 = 0$ or 1, when $G_{xz}^{(1)}/G_{xz}^{(2)} = 1$, i.e. a single layer, $K_x = 6/5$, the same as case 1. Also, when $E_x^{(1)}/E_x^{(2)} = 1$, and for all values of H_2/H_1, $K_x = 6/5$.

The general case of SCF for a sandwich plate made of orthotropic layers happens when $G_{xz}^{(1)}/G_{xz}^{(2)} > 10$. In these cases as shown in Fig. 7, for the values of $E_x^{(1)}/E_x^{(2)}$ >about 5 and at any specific H_2/H_1, there is not any significant changes in K_x by increasing $E_x^{(1)}/E_x^{(2)}$. For the values of $E_x^{(1)}/E_x^{(2)}$ less than about 10, however, K_x increases rapidly by decreasing $E_x^{(1)}/E_x^{(2)}$. The mag-

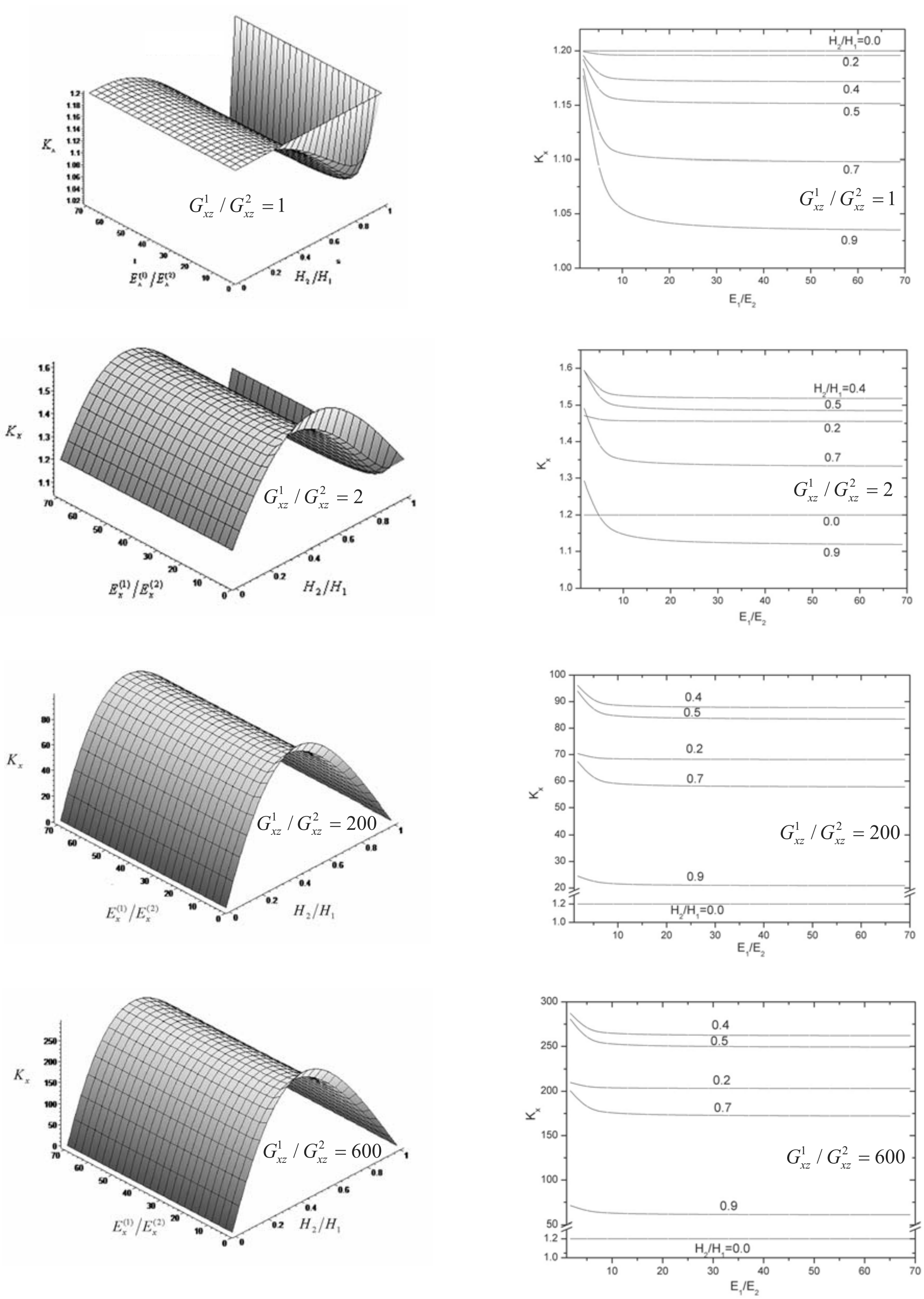

Figure 7 : Variation of K_x with $E_x^{(1)}/E_x^{(2)}$ and H_2/H_1 for an orthotropic sandwich plate for $G_{xz}^{(1)}/G_{xz}^{(2)} = 1, \quad 2, \quad 200$ and 600.

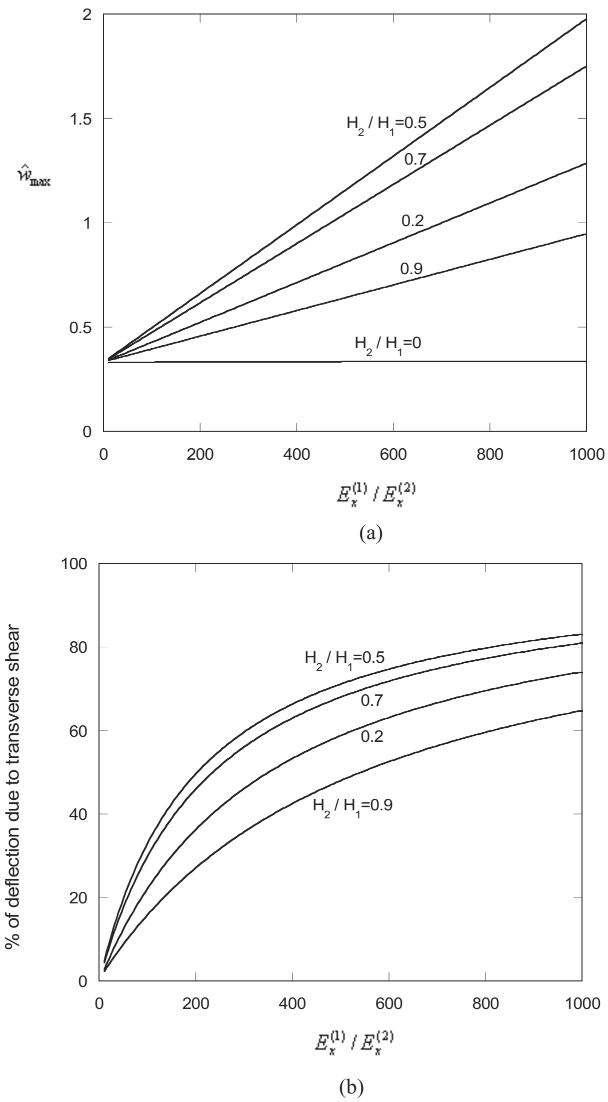

Figure 8 : Effect of H_2/H_1 on the deflection of orthotropic sandwich plate: (a) Variation of $\hat{w}_{\max}$ versus $E_x^{(1)}/E_x^{(2)}$, (b) Contribution of transverse shear to the total deflection. $G_{xz}^{(1)}/G_{xz}^{(2)} = 600$.

nitude of this variation depends on the value of H_2/H_1 at any specific $G_{xz}^{(1)}/G_{xz}^{(2)}$ value.

For this case the dimensionless maximum deflection, $\hat{w}_{max}$, is defined as

$$\hat{w}_{\max} = \frac{E_x^{(2)}}{12(1-\nu_{xy}\nu_{yx})P} \times \left[\frac{E_x^{(1)}}{E_x^{(2)}} - \left(\frac{E_x^{(1)}}{E_x^{(2)}} - 1\right)\left(\frac{H_2}{H_1}\right)^3\right]\left(\frac{H_1}{a}\right)^3 w_{\max} \quad (28)$$

From Eq. 25 and Eq. 27 and the constitutive relations, $\hat{w}_{\max}$ will be given by

$$\hat{w}_{\max} = \frac{1}{3} + \frac{\frac{E_x^{(2)}}{G_{xz}^{(2)}}\left(\frac{H_1}{a}\right)^2}{80\left[\frac{E_x^{(1)}}{E_x^{(2)}} - \left(\frac{E_x^{(1)}}{E_x^{(2)}} - 1\right)\left(\frac{H_2}{H_1}\right)^3\right](1-\nu_{xy}\nu_{yx})\frac{G_{xz}^{(1)}}{G_{xz}^{(2)}}} \times \left\{\left(\frac{E_x^{(1)}}{E_x^{(2)}}\right)^2\left(\frac{H_2}{H_1}-1\right)^2\left[3\left(\frac{H_2}{H_1}\right)^3\left(5\frac{G_{xz}^{(1)}}{G_{xz}^{(2)}}-1\right) + 6\left(\frac{H_2}{H_1}\right)^2\left(5\frac{G_{xz}^{(1)}}{G_{xz}^{(2)}}-1\right) + \frac{H_2}{H_1}\left(15\frac{G_{xz}^{(1)}}{G_{xz}^{(2)}}+1\right) + 8\right] - 20\frac{E_x^{(1)}}{E_x^{(2)}}\frac{G_{xz}^{(1)}}{G_{xz}^{(2)}}\left(\frac{H_2}{H_1}\right)^3\left(\frac{H_2}{H_1}-1\right)\left(\frac{H_2}{H_1}+1\right) + 8\frac{G_{xz}^{(1)}}{G_{xz}^{(2)}}\left(\frac{H_2}{H_1}\right)^5\right\} \quad (29)$$

In Fig. 8 the variation of $\hat{w}_{\max}$ is plotted versus $E_x^{(1)}/E_x^{(2)}$ for different values of H_2/H_1 when $G_{xz}^{(1)}/G_{xz}^{(2)} = 600$. Fig. 8a shows that $\hat{w}_{\max}$ varies linearly with $E_x^{(1)}/E_x^{(2)}$ and the maximum contribution to the deflection due to transverse shear occurs when $H_2/H_1 = 0.4-0.5$. The maximum contribution to total deflection due to the transverse shear is shown in Fig. 8b and it can be as high as 80%.

In Fig. 9, the variation of $\hat{w}_{\max}$ is plotted versus $G_{xz}^{(1)}/G_{xz}^{(2)}$ for different values of H_2/H_1 when $E_x^{(1)}/E_x^{(2)} = 1000$. Fig. 9a shows that for $G_{xz}^{(1)}/G_{xz}^{(2)}$ >about 10, for each specific sandwich plate, i.e. H_2/H_1 =constant, the deflection due to transverse shear for all $G_{xz}^{(1)}/G_{xz}^{(2)}$ remains constant. The maximum contribution to the deflection happens when $H_2/H_1 = 0.4$. Fig. 9b shows that the maximum contribution to total deflection due to the transverse shear can be as high as 85%.

In summary, the contribution of the transverse shear to the total deflection of sandwich thick plate increases by increasing $E_x^{(1)}/E_x^{(2)}$ and decreases by increasing $G_{xz}^{(1)}/G_{xz}^{(2)}$. However, for $G_{xz}^{(1)}/G_{xz}^{(2)} > 10$, the effect of shear modulus ratio is negligible. The maximum deflection due to shear for any $E_x^{(1)}/E_x^{(2)}$ and $G_{xz}^{(1)}/G_{xz}^{(2)}$ happens when $H_2/H_1 = 0.4-0.5$.

This problem was also solved with ABAQUS software using shell elements. In the FE analysis as in the pre-

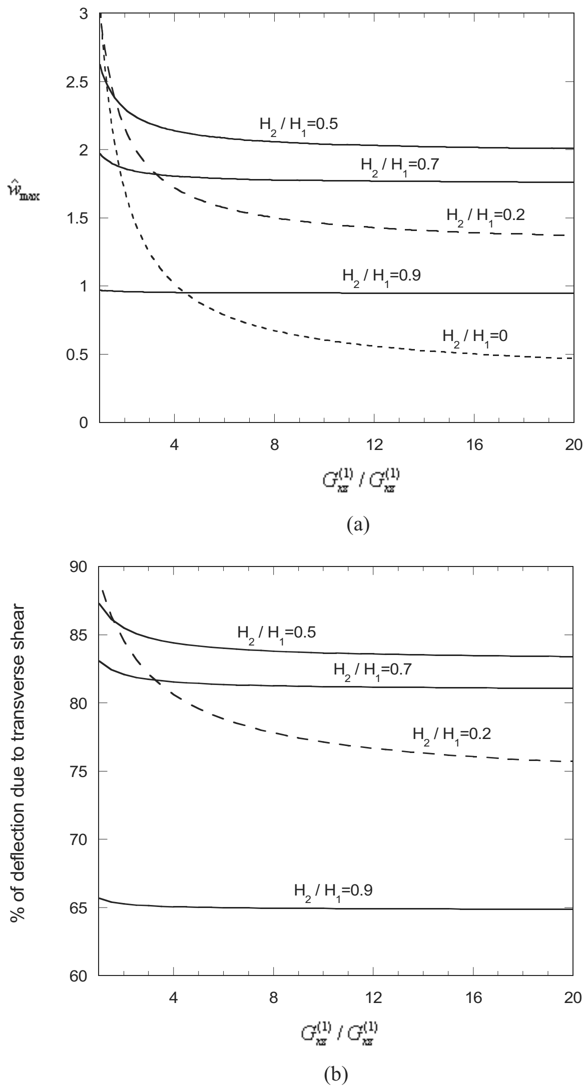

Figure 9 : Effect of H_2/H_1on the deflection of orthotropic sandwich plate: (a) Variation of $\hat{w}_{\max}$ versus $G_{xz}^{(1)}/G_{xz}^{(2)}$, (b) Contribution of transverse shear to the total deflection. $E_x^{(1)}/E_x^{(2)} = 1000$.

vious case the S4R elements were used, a 4-node doubly curved thin and thick shell element, with linear shape function and reduced integration. The B.C.'s were pinned (U1=U2=U3=0). The FEA maximum deflection was 6.662 mm and the analytical one was 6.636 mm, an error of about 0.4%.

5 Concluding remarks

In this paper, shear correction factor for thick laminated sandwich plates was derived using energy equivalence method. In this method, the transverse shear strain energy obtained from FSDT was equated to that obtained from three-dimensional elasticity. The method was tested for two case studies. In the first case, the shear correction factor for a single layer orthotropic plate simply supported at its two opposite edges was shown to be the same as that obtained by Cowper [Cowper (1966)]. The second case was a general sandwich plate clamped at its edge made from orthotropic materials. For this case, it was shown that the SCF is a function of $E_x^{(1)}/E_x^{(2)}$, $G_{xz}^{(1)}/G_{xz}^{(2)}$ and H_2/H_1. The effect of these non-dimensional parameters on the shear correction factor and the maximum deflection of the plate were studied. The conditions under which the dominant contributing factor to the total plate deflection is transverse shear were demonstrated.

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Appendix A: Derivation of Eq. 27

From Eq. 12 we have

$$U_s = \frac{1}{2}\iiint_V \sigma_{xz}\gamma_{xz}dV = \frac{1}{2}\iiint_V \frac{\sigma_{xz}^2}{G_{xz}}dV = \frac{ab}{2}\int_{-\frac{H_1}{2}}^{-\frac{H_2}{2}} \frac{\sigma_{xz}^2}{G_{xz}^{(1)}}dz + \frac{ab}{2}\int_{-\frac{H_2}{2}}^{\frac{H_2}{2}} \frac{\sigma_{xz}^2}{G_{xz}^{(2)}}dz + \frac{ab}{2}\int_{\frac{H_2}{2}}^{\frac{H_1}{2}} \frac{\sigma_{xz}^2}{G_{xz}^{(1)}}dz. \quad (A.1)$$

Now From Eq. 26 we may write

$$\int_{-\frac{H_1}{2}}^{-\frac{H_2}{2}} \frac{\sigma_{xz}^2}{G_{xz}^{(1)}}dz = \frac{1}{1920} \times \frac{{Q_{11}^{(1)}}^2 P^2(8H_1^2+9H_1H_2+3H_2^2)(H_1-H_2)^3}{D_{11}^2 G_{xz}^{(1)}}, \quad (A.2)$$

$$\int_{-\frac{H_2}{2}}^{\frac{H_2}{2}} \frac{\sigma_{xz}^2}{G_{xz}^{(2)}}dz = \frac{1}{960} \times \frac{P^2H_2(8{Q_{11}^{(2)}}^2H_2^4 - 20Q_{11}^{(1)}Q_{11}^{(2)}H_2^4 + 20Q_{11}^{(1)}Q_{11}^{(2)}H_1^2H_2^2}{D_{11}^2 G_{xz}^{(2)}} + \frac{15{Q_{11}^{(1)}}^2H_2^4 - 30{Q_{11}^{(1)}}^2H_1^2H_2^2 + 15{Q_{11}^{(1)}}^2H_1^4)}{D_{11}^2 G_{xz}^{(2)}}, \quad (A.3)$$

$$\int_{\frac{H_2}{2}}^{\frac{H_1}{2}} \frac{\sigma_{xz}^2}{G_{xz}^{(1)}}dz = \frac{1}{1920} \times \frac{{Q_{11}^{(1)}}^2 P^2(8H_1^2+9H_1H_2+3H_2^2)(H_1-H_2)^3}{D_{11}^2 G_{xz}^{(1)}}. \quad (A.4)$$

From Eqs. 8, 11 we conclude

$$A_{55} = G_{xz}^{(1)}(H_1-H_2) + G_{xz}^{(2)}H_2. \quad (A.5)$$

Also we know that

$$Q_x = P. \quad (A.6)$$

Substituting from Eqs. A.5, A.6 into Eq. 14 we will get

$$U_s = \frac{ab}{2}\frac{K_xP^2}{G_{xz}^{(1)}(H_1-H_2)+G_{xz}^{(2)}H_2}. \quad (A.7)$$

Next we substitute from Eqs. A.2, A.3, A.4 into Eq. A.1 and set the result equal to Eq. A.7 and then solve the equation for K_x to obtain Eq. 27.