 SDHM, vol.2, no.3, pp.165-175, 2006

From Damage to Crack: A B.E. Approach

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Abstract: The formation of cracks and their propagation in brittle materials has been intensively studied in the last years. The main difficulty is related to the theoretical and numerical possibility to follow the development of regions of highly localised strains. The nonlinear phenomenon is physically different from the one which occurs in ductile materials: it starts with a narrow fracture process zone containing a large number of distributed microcracks which could lead to the formation of macrocracks and eventually to rupture. In the present paper, a simple nonlocal damage model is coupled to the crack analysis in order to describe the formation and propagation of a crack in quasibrittle materials. The numerical scheme is based on the dual boundary element method which allows for the discretisation of the boundary, of the crack and of the part of the domain in which the damage is supposed to occur.

keyword: Damage, Non-local, Crack, BEM.

1. Introduction

A great interest has been devoted in the last years to the description of non-linear phenomena accompanied by highly localised strains. Such an interest, mainly focused on brittle materials such as concrete, has triggered off the development of various techniques in the Finite Element (FE) context. The main idea is based on the possibility to enrich the standard finite element interpolations by strain or displacement discontinuities. The first attempt in such direction is due to [Ortiz and etc. (1987)], who proposed to enrich the strain field with only one weak discontinuity crossing each element, i.e. another weak discontinuity line in the neighboring element is necessary to model the entire localisation band. The idea was further improved by [Belytschko and etc. (1988)] who proposed to embed a localisation zone into the finite element, i.e. an element could contain a band of localised strain bounded by two parallel weak discontinuity lines. Finally a strong discontinuity was considered by [Dvorkin and etc. (1990)]. Afterwards, many similar techniques, enriching standard finite element interpolations by strain or displacement discontinuities, were proposed in the technical literature under different names and developed according to different types of enrichment in the element. A comparative study is given in [Jirásek (2000)] with special reference to the constant-strain triangle. Such an approach was proposed in order to overcome the stress locking which arises in the traditional smeared-crack models for concrete fracture. A practical implementation of a specific model with a strong (displacement) discontinuity embedded in a constant-strain triangular element is given in [Jirásek and Zimmermann (2001a)]. Such a formulation is tested on several typical fracture problems and the results are compared to those obtained with smeared crack models in [Jirásek and Zimmermann (2001b)]; stress locking is also shown to occur in consequence of an incorrect separation of nodes and a possible remedy, advocating a transition from a smeared to an embedded crack, is proposed. The smeared part is reformulated as non-local, which allows to have an additional improvement.

A different approach is based on the fictitious crack model. When an un-reinforced concrete structure is subjected to an increasing external load, the developing fracture zone behaves in a softening way, i.e. contact stresses decrease while the crack opening displacement increases. Such a formulation has been implemented in a number of finite element formulations, e.g. [Carpinteri (1989)] and [Carpinteri and Valente (1989)] for mode I cracking behaviour and [Bocca and etc. (1991)], [Gerstle and Xie (1992)] for mixed mode propagation. The dual boundary element method (DBEM) reported by [Portela and etc. (1992)] has shown to be very effective when applied to the fictitious crack model. In fact, in [Saleh and Aliabadi (1995)] the DBEM is shown to be computationally effective in simulating crack propagation especially when

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dealing with the nonlinear behaviour in concrete. With this method the crack propagation path is not to be known in advance: at each step of the crack extension the path is computed simultaneously, whereas the crack increment at each iteration has to be set in advance. In [Saleh and Aliabadi (1998)] the DBEM is applied to the analysis of crack growth in reinforced concrete. Both concrete and reinforcement are supposed to be connected, compatibly with the shear coefficient of the bond, and the reinforcement is assumed to behave as a elastic-perfectly plastic material.

All the approaches presented model the crack as a discontinuity immersed in a linearly elastic material, either as a displacement discontinuity incorporated into the finite element interpolation or as a crack line (surface in 3D) considered as a part of the boundary to discretise in the Boundary Element (BE) formulation.

It must be pointed out that such approaches do not cope properly with the existence of a narrow fracture process zone containing a large number of distributed microcracks at the fracture front. In the last twenty years, the continuum damage mechanics has been set up as a link between the classical continuum and the fracture mechanics. This approach tries to model the development, the growth and the coalescence of microdefects which could lead to the formation of macrocracks and eventually to rupture. One of the most commonly used damage models introduces an arbitrary variable tensor to model the growth and the diffusion of microcracks inside the solid. Such tensor is introduced in the constitutive equations in order to describe the region of the body in which a degradation of the material elastic properties due to the microcracking phenomenon occurs. The development of the material damage produces a strain softening behaviour. As pointed out already by Hadamard, the dynamic initial-boundary-value problem changes from hyperbolic to elliptic type and it becomes ill posed. The finite element numerical solution is non-objective with respect to the choice of the mesh and, upon the mesh refinement, it converges to a solution with a vanishing energy dissipation. To overcome such drawbacks, some regularisation techniques have been proposed. One of these is based on the formulation of a nonlocal continuum: the constitutive law at a point of a continuum involves weighted averages of a state variable over a certain neighbourhood of that point. A characteristic length is also introduced to control the spread of the nonlocal weight function. A lot of work has been done in the finite element context; see for example [Bažant and Jirásek (2002)] for a survey of progress of nonlocal integral formulations of plasticity and damage.

The very first coupling of BEM with nonlocal operators is given by [Garcia and etc. (1999)]. The main drawback is the assumption of constant damage in every internal cell and the collocation of the boundary integral equations in every internal node. It must be pointed out that boundary element formulations are very suited to represent high gradients and they can be recommended in the cases which exhibit stress or strain concentrations. The BEM, in fact, is able both to furnish a very precise evaluation of the stress tensor in any internal point and to reduce tremendously the number of the unknowns when the nonlinear zone is relatively small in comparison with the overall size of the finite domain. Only in 2002-2003 have [Lin and etc. (2002)] and [Sládek and etc. (2003)] incorporated a nonlocal strain softening localization limiter of integral type into the classical BEM formulation; nevertheless they implement a softening plasticity model rather than a damage model and they use a formulation which gives rise to some locking effects in FE and it is unable to deal with snap-back behaviours. Snap-back behaviours in physically nonlinear problems were faced by [Mallardo and Alessandri (2004)] who showed a general procedure to combine the arc-length technique with the BEM.

In the present paper the Authors intend to present a formulation which links the continuous damage approach with the formation and propagation of a macro-crack. The nonlocal damage theory is applied in order to evaluate the damage distribution over the solid, to regulate the starting point of the macro-crack and to lead its propagation. An automatic procedure is also used in order to formulate the weight function in terms of geodetic distance, i.e. the minimum length of the path not intersecting the boundary surface of the body. A numerical example is illustrated in order to show the efficiency of the procedure.

In Section 2, a brief description of the nonlocal damage model adopted is given; in Section 3 the DBEM formulation in damage mechanics is presented and in Section 4 some numerical results are provided.

2. The nonlocal damage model

In the present contribution the damage is assumed to be isotropic, i.e. d is scalar, and the formulation is confined to the case of small induced strains. The stress-strain relation can be written in the following way:

$$\sigma = (1-d)\mathbf{C}^{el} : \varepsilon \tag{1}$$

where C^{el} is the fourth order elastic moduli tensor, σ and ε are the stress and the strain tensor respectively

The model is set consistently with thermodynamic principles by the introduction of a kinematic internal variable α, a energy per unit of volume Y:

$$Y := \frac{1}{2}\varepsilon : \mathbf{C}^{el} : \varepsilon \tag{2}$$

and a force X of the type:

$$X := k \ln^n \frac{c}{1-\alpha} \tag{3}$$

where k, c and n are material parameters.

The existence of a damage activation function $g(Y,X)$ is now assumed. Under the hypothesis of generalised associative damage behaviour, the damage activation function can be written as:

$$g(Y,X) := Y - X \tag{4}$$

The flaw laws read:

$$g(Y,X) \le 0 \quad \dot{d} = \dot{\lambda} = \dot{\alpha} \quad \text{with } \dot{\lambda} \ge 0 \text{ and } \dot{\lambda} g = 0 \tag{5}$$

in any point of the body V.

At the generic iteration and at the points where the damage activation function is zero (points $\mathbf{x} \in V_d \subseteq V$), the response is elastic-damaging and the following relations must hold in V_d:

$$\dot{g} \le 0 \qquad \dot{\lambda} \ge 0 \qquad \dot{\lambda}\dot{g} = 0 \tag{6}$$

Expanding the damage activation function in its rate form leads to:

$$\dot{g}(Y,X) = \dot{Y} - \dot{X} \tag{7}$$

where:

$$\dot{Y} = \varepsilon : \mathbf{C}^{el} : \dot{\varepsilon} \tag{8}$$

and

$$\dot{X} = -\frac{kn}{1-\alpha}\dot{\alpha}\ln^{n-1}\left(\frac{c}{1-\alpha}\right) \tag{9}$$

The nonlocal version of the simple model described above is obtained by substitution of the strain energy Y with its nonlocal value. This choice has the advantage that for the linear damage model all the constitutive calculations can be carried out locally in a way which is formally equivalent to the local version, despite of the nonlocality introduced.

If $Y(\mathbf{x})$ is the local value at a point $\mathbf{x} \in V$, the corresponding nonlocal value is defined by:

$$\overline{Y}(\mathbf{x}) = \int_V W(\mathbf{x},\mathbf{y}) Y(\mathbf{y}) dV(\mathbf{y}) \tag{10}$$

The weight function $W(\mathbf{x},\mathbf{y})$ is given by:

$$W(\mathbf{x},\mathbf{y}) = \frac{W_0(\mathbf{x},\mathbf{y})}{\overline{W}(\mathbf{x})} \tag{11}$$

where $W_0(\mathbf{x},\mathbf{y})$ is a monotonically decreasing nonnegative function of the distance r here chosen as the Gauss distribution function, i.e.:

$$W_0(\mathbf{x},\mathbf{y}) = e^{\frac{-\|\mathbf{x}-\mathbf{y}\|^2}{2l^2}} \tag{12}$$

$\overline{W}(\mathbf{x})$ is introduced in order to ensure that uniform damage fields are not modified by the spatial average. This is guaranteed by posing:

$$\overline{W}(\mathbf{x}) = \int_V W_0(\mathbf{x},\mathbf{y}) dV(\mathbf{y}) \tag{13}$$

The distance r must be intended as the geodetic distance, i.e. the minimum distance between the points $\mathbf{x}$ and $\mathbf{y}$ entirely contained in the domain and never crossing the entire boundary, cracks included. This means that the distance r must take into account possible nonconvex sides of both the domain and internal cracks.

The geodetic distance between the central nodes of every couple of domain cells is evaluated just once in the entire nonlinear analysis. Such computation results to be useful in the evaluation of the nonlocal value $\overline{Y}(\mathbf{x})$. The integration (10) is performed by evaluating the radius r between the cell node and the Gaussian point as sum of two terms: one is the geodetic distance (previously evaluated and stored) between $\mathbf{x}$($\mathbf{x} \equiv$A in figure 1) and the

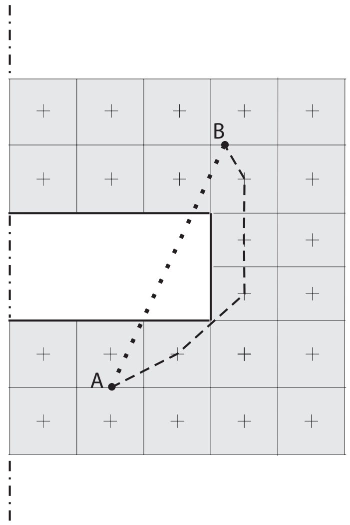

Figure 1 : Comparison between the geodetic distance (dashed line) and the minimum distance (dotted line) connecting two generic points A and B

central node of the domain cell surrounding the Gaussian point and proximate to **x**, the latter is the distance between such a central node and the Gaussian point (≡B in figure 1).

The current results are already able to show the reliability of the adopted technique which however need be further investigated.

The parameter l in Eq. (12) plays the role of an internal length controlling the nonlocal spatial spread of the damage.

On the assumptions made, the nonlocal damage activation function becomes:

$$g(\overline{Y},X)=\overline{Y}-X \tag{14}$$

whereas the flaw laws and the damage evolutive laws mantain the same local expressions.

In the iterative procedure of the incremental DBEM problem the distribution of the damage parameter d will provide both the level over which the crack occurs and its direction of propagation.

3. The elastic-damaging DBEM

The boundary integral equations governing the problem given by the continuum damage relations introduced in the previous section can be obtained as in the classical plasticity.

In fact, the stress tensor rate $\dot{\sigma}_{ij}$ can be written as:

$$\dot{\sigma}_{ij}=\dot{\sigma}^{el}_{ij}-\dot{\sigma}^{d}_{ij}=\dot{\sigma}^{el}_{ij}-d\dot{\sigma}^{el}_{ij}=C^{el}_{ijlm}\dot{\varepsilon}_{lm}-dC^{el}_{ijlm}\dot{\varepsilon}_{lm} \tag{15}$$

where $\dot{\sigma}^{el}_{ij}$ is named elastic stress tensor rate and $\dot{\sigma}^{d}_{ij}$ is named damaged stress tensor rate. Analogously, the traction rate can be expressed as:

$$\dot{t}_i=\dot{\sigma}_{ij}n_j=\dot{t}^{el}_i-\dot{t}^{d}_i=\dot{t}^{el}_i-d\dot{t}^{el}_i \tag{16}$$

Furthermore, by manipulating the governing differential equilibrium equation in a way which is classical in the BE context, an integral relation, giving the displacement at any internal point $\mathbf{X}\in\overset{\circ}{\Omega}$ (Ω is the body under analysis with boundary Γ), can be obtained:

$$\dot{u}_i(\mathbf{X})=\int_\Gamma u^*_{ij}(\mathbf{X},\mathbf{x})\dot{t}_j(\mathbf{x})d\Gamma(\mathbf{x})-\int_\Gamma t^*_{ij}(\mathbf{X},\mathbf{x})\dot{u}_j(\mathbf{x})d\Gamma(\mathbf{x}) + \int_\Omega \varepsilon^*_{ijk}(\mathbf{X},\mathbf{x})\dot{\sigma}^{d}_{jk}(\mathbf{x})d\Omega(\mathbf{x}) \tag{17}$$

where $u^*_{ij}(\mathbf{X},\mathbf{x})$ and $t^*_{ij}(\mathbf{X},\mathbf{x})$ are respectively displacement and traction in the direction given by the unit vector $\mathbf{e_j}$ (direction j) at point $\mathbf{x}$ corresponding to a unit point force acting in the direction i applied at point $\mathbf{X}$ of an infinite elastic medium. The term $\varepsilon^*_{ijk}(\mathbf{X},\mathbf{x})$ represents the strain ε^*_{jk} at any point $\mathbf{x}$ due to a unit point load acting in the i direction and applied at the point $\mathbf{X}$ of an infinite elastic medium.

The stress state at any point can be obtained by applying the generalized Hooke's law to the elastic part of the total strain rate tensor:

$$\dot{\sigma}_{ij}(\mathbf{X})=\int_\Gamma U^*_{ijk}(\mathbf{X},\mathbf{x})\dot{t}_k(\mathbf{x})d\Gamma(\mathbf{x}) - \int_\Gamma T^*_{ijk}(\mathbf{X},\mathbf{x})\dot{u}_k(\mathbf{x})d\Gamma(\mathbf{x}) + \int_\Omega \Sigma^*_{ijkl}(\mathbf{X},\mathbf{x})\dot{\sigma}^{d}_{kl}(\mathbf{x})d\Omega(\mathbf{x})+g_{ij}(\dot{\sigma}^{d}_{kl}) \tag{18}$$

where U^*_{ijk}, T^*_{ijk}, Σ^*_{ijkl} and g_{ij} are related to the Kelvin's fundamental solution and can be found in any BE book (see for instance [Aliabadi (2002)]).

The calculation of stresses at any point of Ω is of fundamental importance for the stepwise solution of nonlinear material problems; for this reason the expression of stress, rather than strain, tensor rate has been preferred. It is worthy to underline that the expression (18) is the mathematically correct value of the stress tensor rate related to the displacement-traction rate distribution $\dot{u}_j$, $\dot{t}_j$

on Γ, and not an approximate evaluation as in the case of FE or Finite Difference (FD) approaches. Of course, the stress value will be a consequence of the assumed shape functions on the boundary and on the part of the domain where the nonlinear term is different from zero.

The integral representation of the displacement rate for a boundary point ξ can be obtained by a suitable limiting process ($\mathbf{X} \rightarrow \xi$) in which the point $\mathbf{X}$ is an internal point surrounded by part of a spherical (or circular in 2D) surface of radius ε, i.e. by the application of the concept of Cauchy principal value of an integral. The arising boundary integral equation is given by:

$$c_{ij}(\xi)\dot{u}_j(\xi) + \int_\Gamma t^*_{ij}(\xi,\mathbf{x})\dot{u}_j(\mathbf{x})d\Gamma(\mathbf{x}) = \int_\Gamma u^*_{ij}(\xi,\mathbf{x})\dot{t}_j(\mathbf{x})d\Gamma(\mathbf{x}) + \int_\Omega \varepsilon^*_{ijk}(\xi,\mathbf{x})\dot{\sigma}^d_{jk}(\mathbf{x})d\Omega(\mathbf{x}) \quad (19)$$

where the integral on the left hand side is to be interpreted in the sense of Cauchy principal value and the coefficient $c_{ij}(\xi)$ is equal to 0.5 if the tangent plane at ξ is continuous. Eq. (19) furnishes an integral equation involving boundary variable only, i.e. displacement and traction vector rates.

Relation (18) was deduced for points located within the body; therefore it cannot be used to determine the boundary stress rates. The limit of such expression, when the load point $\mathbf{X}$ approaches the boundary, would contain a hypersingular term. Its evaluation could be avoided by expressing the stress at any boundary point in terms of both boundary tractions and displacements and strain components along the tangent direction to the boundary evaluated by means of the shape functions. Alternatively, discontinuous domain cells could be used in proximity of the discretised boundary.

The integral relations (19) and (18) are not sufficient to solve the problem when a crack is present. Both for bilateral and unilateral crack, the unknowns at every point of the crack lines are the double of the corresponding available equations. The first approach in the BE context, solving such a drawback, proposed the multi-region formulation, in which the crack line belongs to the boundary of two different subregions. A more novel and efficient approach (DBEM) is based on the procedure proposed by [Portela and etc. (1992)] in which a traction equation at every crack node is added in order to obtain a square system of equations. The dual equations are given respectively by the displacement equation at the node ξ' of the crack as belonging to one line and by the traction equation at the node ξ'' of the crack as belonging to the opposite face. The displacement boundary integral equation at a crack node is slightly different (see for instance [Alessandri and Mallardo (1999)] for details) from the classical one. Provided that the tangent line at the crack node is continuous, it is given by the following relation:

$$\frac{1}{2}u_i(\xi') + \frac{1}{2}u_i(\xi'') + \int_\Gamma t^*_{ij}(\xi',\mathbf{x})\dot{u}_j(\mathbf{x})d\Gamma(\mathbf{x}) = \int_\Gamma u^*_{ij}(\xi',\mathbf{x})\dot{t}_j(\mathbf{x})d\Gamma(\mathbf{x}) + \int_\Omega \varepsilon^*_{ijk}(\xi',\mathbf{x})\dot{\sigma}^d_{jk}(\mathbf{x})d\Omega(\mathbf{x}) \quad (20)$$

In order to give the expression of the traction equation it is necessary to write the integral expression of the stress tensor rate on a smooth boundary point:

$$\frac{1}{2}\dot{\sigma}_{ij}(\xi) + \oint_\Gamma T^*_{ijk}(\xi,\mathbf{x})\dot{u}_k(\mathbf{x})d\Gamma(\mathbf{x}) = \int_\Gamma U^*_{ijk}(\xi,\mathbf{x})\dot{t}_k(\mathbf{x})d\Gamma(\mathbf{x}) + \int_\Omega \Sigma^*_{ijkl}(\xi,\mathbf{x})\dot{\sigma}^d_{kl}(\mathbf{x})d\Omega(\mathbf{x}) + \frac{1}{2}g_{ij}(\dot{\sigma}^d_{kl}) \quad (21)$$

and on a smooth crack point:

$$\frac{1}{2}\dot{\sigma}_{ij}(\xi'') + \frac{1}{2}\dot{\sigma}_{ij}(\xi') + \oint_\Gamma T^*_{ijk}(\xi'',\mathbf{x})\dot{u}_k(\mathbf{x})d\Gamma(\mathbf{x}) = \int_\Gamma U^*_{ijk}(\xi'',\mathbf{x})\dot{t}_k(\mathbf{x})d\Gamma(\mathbf{x}) + \int_\Omega \Sigma^*_{ijkl}(\xi'',\mathbf{x})\dot{\sigma}^d_{kl}(\mathbf{x})d\Omega(\mathbf{x}) + \frac{1}{2}g_{ij}(\dot{\sigma}^d_{kl}) \quad (22)$$

where $\oint$ denotes the Hadamard principal value integral. Both previous equations stem from a careful limiting process carried out on the Eq. (18) when the internal point $\mathbf{X}$ approaches the external boundary and the crack surfaces. The traction components, $\dot{t}_i$, can be obtained by the relation $\dot{t}_i = \dot{\sigma}_{ij}n_j$, and on a smooth crack point they are given by:

$$\frac{1}{2}\dot{t}_i(\xi'') - \frac{1}{2}\dot{t}_i(\xi') + n_j(\xi'') \oint_\Gamma T^*_{ijk}(\xi'',\mathbf{x})\dot{u}_k(\mathbf{x})d\Gamma(\mathbf{x}) = n_j(\xi'') \int_\Gamma U^*_{ijk}(\xi'',\mathbf{x})\dot{t}_k(\mathbf{x})d\Gamma(\mathbf{x}) + n_j(\xi'') \int_\Omega \Sigma^*_{ijkl}(\xi'',\mathbf{x})\dot{\sigma}^d_{kl}(\mathbf{x})d\Omega(\mathbf{x}) + \frac{1}{2}n_j(\xi'')g_{ij}(\dot{\sigma}^d_{kl}) \quad (23)$$

The boundary Γ is divided into N_Γ continuous three noded (quadratic) elements whereas the crack Γ_c is divided into N_{Γ_c} discontinuous flat elements. The elements on the crack are chosen to be flat in order to evaluate the singular integrals analytically. The displacement and traction rates, as well as the geometry, can be approximated in each boundary element Γ_l by products of the shape function $M_\Gamma^n(\eta)$ and the nodal values, where η represents the local coordinate. The domain discretisation needs to be performed only for that region $\Omega_d \subseteq \Omega$ susceptible of damage, whereas for other numerical techniques the whole domain must be discretised. This is especially attractive in damage mechanics for brittle materials since the area expected to be damaged tends to the macro-crack shape. The domain Ω_d is divided into N_Ω quadratic triangular or quadrilateral cells (with shape functions $M_\Omega^n(\eta,\zeta)$).

Before the crack initiation, the discretised form of the Eq. (19), along with the expression (18) of the stress tensor rate at any internal point, is adopted in the iterative scheme which is necessary to satisfy the rate constitutive equations. The classical collocation approach furnishes the unknown displacement and traction rates on the boundary and the distribution of the damage d at every cell node. Once the damage in one or some points passes a fixed threshold close to the unity, i.e. a macro-crack arises, a crack line is inserted and some new unknowns are introduced into the formulation. Such unknowns are given only by the two displacement rates at every crack node, provided that the crack is supposed to behave bilaterally. In order to obtain these new unknowns, it is necessary to add the discretised form of the Eqs. (20), (23) to the previous system of equations. In both cases, all the equations can be collected in a matrix form as:

$$A\dot{\mathbf{x}} = \dot{\mathbf{f}} + Q\dot{\sigma}^d \tag{24}$$

$$\dot{\sigma} = -A'\dot{\mathbf{x}} + \dot{\mathbf{f}}' + (Q' + E')\dot{\sigma}^d \tag{25}$$

where the Dirichlet/Neumann conditions on the boundary Γ are included, the bold letters mean vectors and the capital letters mean matrices. The unknowns, displacement/traction rate on the external boundary, plus (possible) displacement rates on the crack faces are collected in $\dot{\mathbf{x}}$ whereas A, A', Q, Q' and E' involve boundary and domain integrals of the fundamental solutions. The weakly singular integrals are evaluated either by logarithmic quadrature rule or by Gaussian quadrature after performing a suitable coordinate transformation. The numerical value of the strongly singular integrals is obtained by using the rigid body technique in the displacement integral equation, the procedure proposed by [Guiggiani and Gigante (1990)] in the other cases. The hypersingular terms are evaluated analytically (see for example [Alessandri and Mallardo (1999)] for details) provided that the crack line is geometrically straight. The nearly singular terms are evaluated by dividing in subelements.

In a finite time step Δt, the evolution problem, governed both by the Eqs. (24), (25) and by the constitutive relations, presents finite increment unknowns $\Delta\mathbf{x}$, $\Delta\sigma$ rather than derivatives $\dot{\mathbf{x}}$, $\dot{\sigma}$ with respect to the time parameter. The proposed formulation may furnish snap-back branches in the equilibrium path, i.e. it may be necessary to introduce an arclength constrain in the general system of equations. This is achieved by following the scheme proposed in [Mallardo and Alessandri (2004)].

The constitutive relations are integrated by the Euler backward scheme, i.e. the increment of damage d is obtained by imposing the consistency condition at the end of the step. Due to the linearity of the adopted damage function $1-d$ and to the choice of Y as a nonlocal variable, the expression of the damage increment results to be formally equivalent to the local formuation. The nonlocal value of the variable Y is obtained by using the same domain discretisation adopted for the evaluation of the nonlinear term occurring in the integral equations.

4. The crack modelling strategy

A criterion, which is similar to Beltrami's one, can be assessed for the proposed damage model. All the material elements ahead of the crack tip, where the damage parameter d is higher than a critical value $\overline{d}$, fail. It can be postulated that the crack increment Δa is a function of the position of the point in which d is maximum, i.e.

$$\Delta a = \beta \mid \mathbf{x}_{crack\,tip} - \mathbf{x}_{d_{max}} \mid \tag{26}$$

When dealing with mode I behaviours, it is simpler to determine in advance the direction of the crack growth. In the example presented in this paper, for instance, the crack trajectory is collinear with the initial crack and thus well known. On the other hand, in the mixed mode problems it can be postulated that the growth of the crack

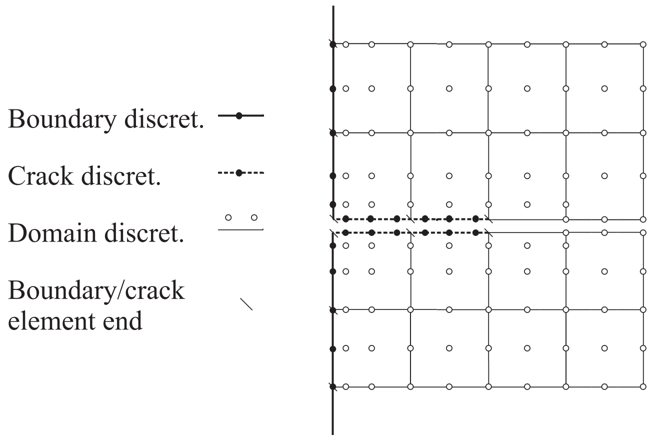

Figure 2 : Boundary, crack and domain discretisation

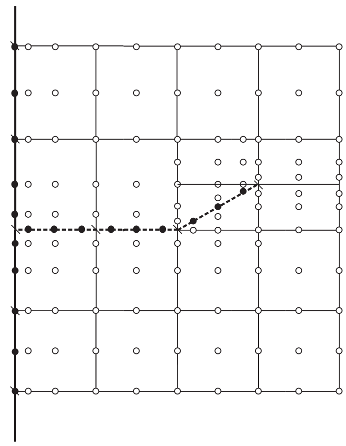

Figure 3 : Adaption of the discretisation

occurs in the direction given by the line connecting the crack tip and the point in which the maximum value of the damage d occurs. As a matter of fact, it is worthy to observe that in the general situation, it may be necessary to implement a correction technique, similar to the one proposed by [Portela and etc. (1993)], in order to obtain the direction of the actual crack-extension increment.

Considering the evolution damage-crack problem from a discrete incremental standpoint for a finite time step $\Delta t = t_{s+1} - t_s$, the system of equations (24-25) can be written as:

$$A\Delta\mathbf{x}_s = \mathbf{f}\Delta\lambda_s + Q\Delta\sigma_s^d \tag{27}$$

$$\Delta\sigma_s = -A'\Delta\mathbf{x}_s + \mathbf{f}'\Delta\lambda_s + \overline{Q}\Delta\sigma_s^d \tag{28}$$

where $\Delta\lambda_s$ represents the s^{th} load factor increment.

In an incremental crack extension analysis, each new crack-extension increment is modelled with new discontinuous boundary elements. The remeshing of the boundary is not required in the DBEM. The new discontinuous boundary elements generate new equations and update those already existing with new unknowns, i.e. new rows and new columns are generated in the matrix of the system of equations (27). The new set of integral equations are obtained by collocating both (21) and (23) at the opposite crack nodes. If the crack has a point on the boundary Γ, the boundary elements converging into this point must be transformed into discontinuous elements, i.e. edge discontinuous elements are employed in this region to avoid a common node at the intersection.

The part of the domain where the damage is supposed to occur is modelled with continuous quadratic triangular/quadrilateral cells. The cells, which have at least one side coincident either with the external boundary Γ or with the crack line, are discontinuous.

A general frame showing the discretisation at the generic intermediate step after the crack initiation is drawn in figure 2.

The general modelling strategy developed in the present paper can be summarised as follows:

1. The external boundary and the part of the domain where the damage is supposed to occur are modelled with either continuous or discontinuous quadratic elements.

2. The system of equations (27-28) furnishes the solution in terms of both displacement and traction increments on the boundary Γ and damage d in the domain.

3. The first crack element is positioned in terms of the first damage distribution in which at least one damaged point exceeding the critical value is present.

4. A new arclength step is proceeded. A new damage distribution is obtained numerically and, therefore, the crack extends for a quantity Δa in the direction of the maximum damage increment. The domain discretisation around the new crack elements must be transformed.

5. The analysis proceeds further until the rupture occurs.

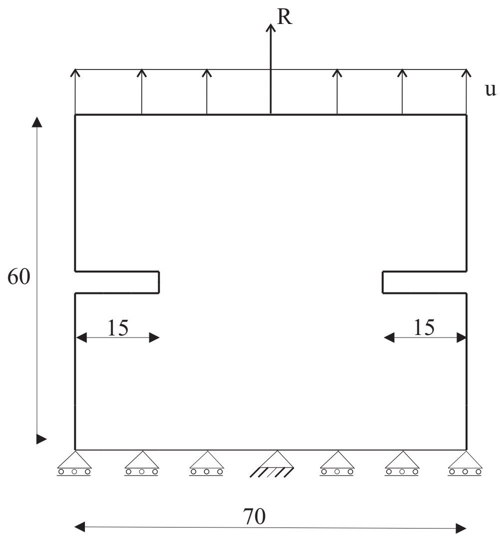

Figure 4 : Two-dimensional specimen subjected to direct tension test. Geometry and load condition.

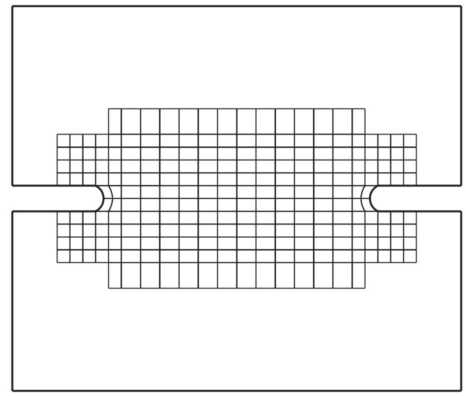

Figure 5 : Domain discretisation adopted for the slab.

For each step applying the proposed damage crack model a re-meshing of the part of the domain close to the crack tip needs to be performed (see for instance figure 3). This is due to the new crack elements which penetrate into a domain cell.

Another fundamental requirement of re-meshing arises from the geometric modification of the element. Once a new mesh has been created, the history-dependent damage should be mapped from the old mesh to the new one. The mapping involves one step: to interpolate the damage parameter from the nodal (cell) points of the old mesh to the nodal (cell) points of the new mesh by using the shape functions. The procedure is much simpler than the one developed in the FE context where three steps are necessary: first transfer all the damage of the old mesh from the Gaussian points to the element nodes, then interpolate the state variable from the nodal points of the old mesh to the nodal points of the new mesh, finally redetermine the value in the Gaussian integration points of the new mesh.

An important question arises concerning the crack line which is inserted according to the damage distribution. On the basis of the proposed model, the crack extends when the damage around the crack tip reaches a critical value $\overline{d}$. If $\overline{d}$ is set equal to the unity, the stress transfer is fully correct, i.e. the crack extends towards points in which the stress tensor is zero. On the contrary, this does not happen if the critical value of the damage is set less than one, i.e. lines on which the stress tensor is different from zero are substituted by traction free crack lines. Such a phenomenon may be source of numerical errors and needs to be analysed carefully. The drawback could be avoided if the transition between damage and traction-free crack is led by using the fictitious crack model. The material in the fracture zone is partly destroyed ($d = \overline{d} < 1$), but it is still able to transfer stresses. Both a closing normal force and a frictional force act on both crack lines. The intensity of these forces has an initial value which depends on the previous value of the damage d and, for instance linearly, on the relative displacement on the crack.

5. Numerical examples

A numerical example is presented in order to show the efficiency of the procedure. The example refers to the slab in figure 4 where a plane stress behaviour is assumed and measures are given in millimeters. The Young modulus and the Poisson coefficient are respectively $E = 36000MPa$ and $\nu = 0.15$. Such an example is a useful benchmark, often adopted in the FE context, which is able to show clearly the onset of the localisation of the strain in correspondence of the crack initiation. An increasing displacement is applied on the top edge of the slab, the corresponding distributed reaction with resultant R is not uniform.

The analysis of the localisation and the effect of the nonlocal approach, as a remedy to avoid the corresponding mesh-dependency of the numerical results, have been deeply investigated by the Authors. It can be demonstrated that the nonlocal damage model here presented

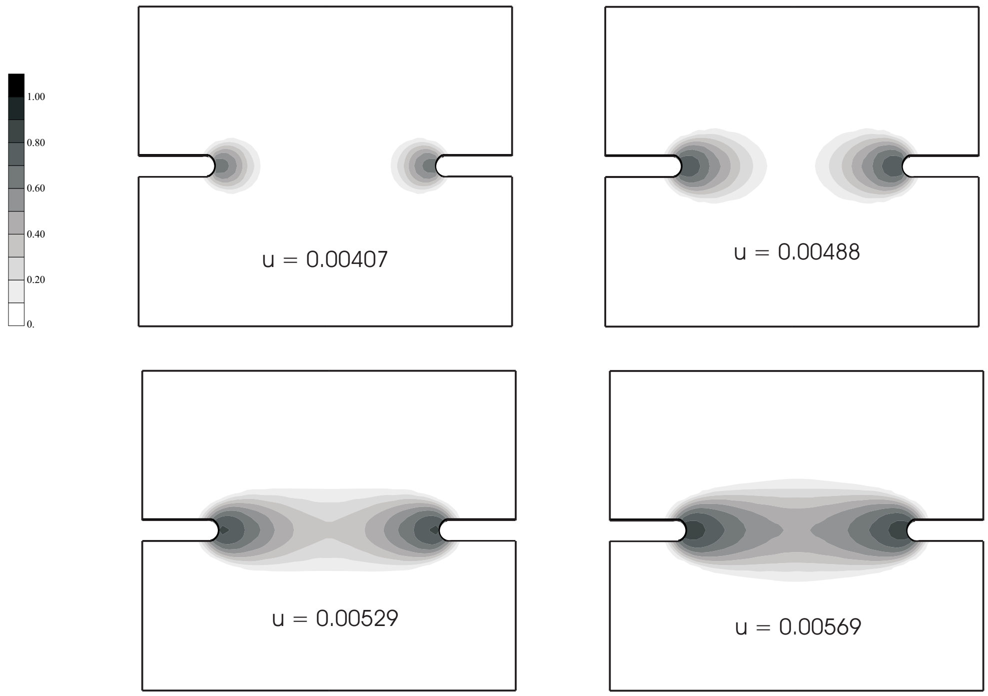

Figure 6 : Damage contour maps at various points of the force-displacement curve.

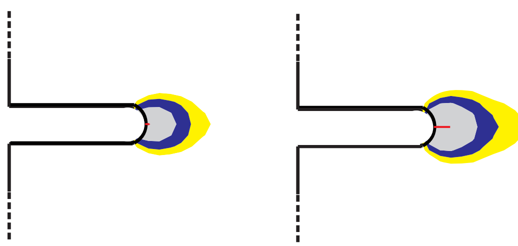

Figure 7 : Damage contour map at u=0.00569 mm and at u=0.00620 mm. Zoom at the left notch.

turns out to be objective with the mesh when coupled with the BEM. Therefore, in this paper the domain discretisation drawn in figure 5 is adopted: 236 quadratic quadrilater cells in the domain and 154 quadratic boundary elements (with 308 boundary nodes) on the external boundary Γ.

Figure 6 shows the regularised damage contour at different load levels up to the occurrence of the first crack line. The last contour plot, corresponding to the reaction force $R = 120.29N$ and to the top displacement $u = 0.00569\,mm$ is the first equilibrium point in which the damage parameter exceeds a pre-defined critical value $\bar{d} = 0.85$.

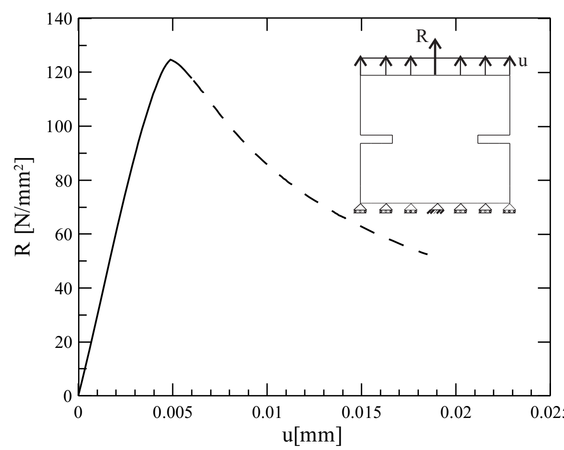

Figure 8 : Diagram showing traction reaction R versus top displacement u.

The points in which such a behaviour occurs coincide with the notches and represent the part of the boundary where the crack initiates. A zoom on the left notch of the slab at the load level $u = 0.00569\,mm$ is given in figure 7. The expanding line corresponds to $d = 0.85$ whereas the remaining three lines correspond to the levels 0.8, 0.75 and 0.7.

In the zoom on the left, the red point gives the point of the crack initiation. The same zoom on the right describes the crack configuration at $u = 0.00620\,mm$, i.e. some load steps further. The crack propagates in the correct direction.

In figure 8 the force-displacement curve, i.e. the resultant reaction R at the top edge versus the imposed displacement is presented. The filled line represents the structural response of the proposed model for the first two steps of the crack propagation. The numerical analyses are still in progress. It is not clear yet how the dissipation can influence the transformation from damaged line to crack line, when the fictitious crack model is not involved. Furthermore, a deeper analysis of the dissipation over the cells close to the propagating crack line would be necessary to prevent crack propagation taking wrong directions. The dashed line of the diagram in figure 8 represents the response of the same slab when the nonlocal damage model is implemented without crack. The line is not drawn in its complete path up to $R = 0$. It is possible to observe that the damage plus crack response follows the same equilibrium path at least in the first load steps.

6. Conclusions

A BE formulation involving both nonlocal damage and discrete crack has been developed. The model intends to analyse the formation of the crack and its propagation in brittle materials, such as concrete. The nonlinear phenomenon is described by a nonlocal damage model which shifts to the macro-crack when the level of damage exceeds a critical value. A numerical example is presented in order to show the efficiency of the formulation when 1) the macro-crack is set as bilateral 2) the geometry and the load conditions generate a mode I behaviour. In order to deal correctly with the first steps of transition between damage and macro-crack a fictitious crack model is proposed.

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