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Can the Conventional High-Cycle Multiaxial Fatigue Criteria Be Re-Interpreted in Terms of the Theory of Critical Distances?

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Abstract: This paper reports on an attempt to systematically re-interpret the conventional multiaxial fatigue criteria in terms of the Theory of Critical Distances: in the present study the criteria proposed by Crossland, Dang Van, Papadopoulos, Matake, McDiarmid, respectively, and the so-called Modified Wöhler Curve Method were considered. The procedure devised to re-interpret the above methods in terms of the Theory of Critical Distances was based on the following two assumptions: (i) the critical distance is a material constant to be determined under fully-reversed uniaxial fatigue loading; (ii) the presence of non-zero mean stresses as well as of non-zero out-of-phase loading has to be directly taken into account by the fatigue damage parameters themselves. The constants depending on the material fatigue properties of every considered criterion were re-calculated by considering a cracked plate subjected to Mode I as well as to Mode III loading. The systematic application of the proposed procedure proved the fact that only the critical plane approaches can coherently be re-formulated in accordance with the Theory of Critical Distances. Finally, to check the accuracy of such criteria in predicting fatigue limits of notched components several experimental results, generated under both uniaxial and multiaxial fatigue loading, were selected from the technical literature. This validation demonstrated that the most accurate critical plane approach is the Modified Wöhler Curve Method, giving predictions mainly lying within an error interval of ±20%, independently of geometrical feature and degree of multiaxiality of the stress field in the fatigue process zone.

keyword: Linear Elastic Fracture Mechanics, Multiaxial Fatigue, Theory of Critical Distances.

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Nomenclature

E [%]	Fatigue strength error index
F_I, F_{III}	Geometrical factors for the stress intensity factor due to Mode I and III loading
K_I, K_{III}	Stress intensity factors due to Mode I and III loading
K_i	Notch-stress intensity factor (i=1, 2, 3)
K_t	Stress concentration factor
L	Critical distance for use in the TCD
m	Unit vector defining a direction on a generic plane Δ
m*	Direction of maximum resolved shear stress
M	Generic direction on the Δ plane
M_σ, T_σ	Papadopoulos' integrals
n	unit vector normal to the generic Δ plane
Oxyz	Reference frame
Oabn	Reference frame on the generic plane Δ
r, δ	Polar coordinates
r_n	Notch root radius
t	Generic instant (t∈T)
S_i	Components of the vector expressing $\sqrt{J_2}$ (i=1,2, ..., 5)
$\sqrt{J_2}$	Square root of the second invariant of the stress deviator
R	Load ratio ($\sigma_{min}/\sigma_{max}$)
T	Period of the cyclic load history
X, Y	Parameters depending on the applied loading
$\omega, \omega^*, \eta, \eta^*$	Parameters depending on the material fatigue strength
γ	Out-of-phase angle
ϕ, θ	Angles which define the position of a generic plane Δ
ϕ^*, θ^*	Angles which define the orientation of the critical plane
Δ	Generic plane
$[\sigma_D]$	Deviatoric stress tensor
σ_0	Fully-reversed uniaxial plain fatigue limit
σ_A	Uniaxial plain fatigue limit under R=0
σ_b	Nominal bending stress referred to the net area
σ_H	Hydrostatic stress

σ_n	Stress normal to the plane of maximum shear stress amplitude
$\sigma_1, \sigma_2, \sigma_3$,	Principal stresses
σ_r, σ_δ, $\tau_{r\delta}$	Stress components calculated with reference to the polar coordinates
σ_x, σ_y, σ_z	Normal stresses
σ_T	Tensile strength
τ	Shear stress relative to the plane of maximum shear stress amplitude
τ_0	Fully reversed torsional fatigue limit
τ_r	Resolved shear stress
τ_t	Nominal torsional stress referred to the net area
τ_{xy}; τ_{xz}; τ_{yz}	Shear stresses
ν	Poisson's constant
$\psi_{x,i}$	Phase angle between σ_x(t) and σ_i(t) (i=y, z)
$\psi_{x,i}$	Phase angle between σ_x(t) and τ_i(t) (i=xy, xz, yz)
$\Delta K_{I,th}$	Range of the threshold value of the Mode I stress intensity factor
$\Delta\sigma_0$	Range of the uniaxial plain fatigue limit

Subscripts

a	= amplitude
m	= mean value
max	= maximum value

1 Introduction

The state of the art shows that many researchers have attempted to propose sound criteria suitable for predicting fatigue strength of smooth components subjected to multiaxial fatigue loading (Papadopoulos, Davoli, et al., 1997; Socie and Marquis, 2000; You and Lee, 2006). On the contrary, only a few attempts have been made to provide engineers engaged in practical problems with reliable methods suitable for assessing stress concentration phenomena under complex cyclic loading (Gough, 1949; Lazzarin and Susmel, 2003; Tipton and Nelson, 1997). The main limitation in using the existing methods in situations of practical interest is that their application requires the definition of nominal parameters such as reference section, nominal stress, notch depth, equivalent stress intensity, etc. This aspect makes them not suitable for being systematically used to post-process linear elastic FE results, limiting the possibility of using them in an industrial reality.

In the uniaxial fatigue ambit, this problem has recently been addressed by a re-interpretation of the Theory of Critical Distances (TCD) (Taylor, 1999). The reliability and accuracy of such a method were confirmed by extensive validations carried out by considering both notched specimens (Susmel and Taylor, 2003a; Taylor and Wang, 2000) and real components (Taylor, Bologna, Bel Knani, 2000): the TCD was seen to be capable of fatigue strength predictions lying within an error interval of $\pm20\%$ when used in the presence of stress raisers having notch root radius ranging from 0.01mm up to 8mm and depth from about 0.05mm up to 5mm. Taking advantage of the features of the TCD, the present authors have proposed an extension of such an approach to multiaxial fatigue situations (Susmel and Taylor, 2003b). In this initial work we investigated various strategies for applying the TCD in these circumstances, showing that reasonable predictions could be obtained for various types of multiaxiality. The present paper contains a more systematic examination of the issues related to the use of the TCD in combination with multiaxial fatigue laws. In particular, we considered whether there were any theoretical barriers to the use of particular multiaxial criteria, before carrying out an extensive validation exercise of those criteria for which no such barriers emerged.

2 Some Preliminary Definitions

Consider a cylindrical plain specimen subjected to a multiaxial fatigue loading resulting in a triaxial stress state at the surface point O (Fig. 1a). Such a point is assumed to be the critical one for the component integrity, and, for this reason, it is taken as the centre of the frame of reference Oxyz. In a generic instant t of the cyclic load history (where t∈T) the stress state [σ(t)], determined in O with respect to the introduced frame of reference, is:

$$[\sigma(t)] = \begin{bmatrix} \sigma_x(t) & \tau_{xy}(t) & \tau_{xz}(t) \\ \tau_{xy}(t) & \sigma_y(t) & \tau_{yz}(t) \\ \tau_{xz}(t) & \tau_{yz}(t) & \sigma_z(t) \end{bmatrix} \quad (1)$$

The above stress tensor can be decomposed into two different parts: the deviatoric part and hydrostatic part:

$$[\sigma(t)] = [\sigma_H(t)] + [\sigma_D(t)] \quad (2)$$

where

$$\sigma_H(t) = \frac{1}{3} \cdot tr([\sigma(t)]) = \frac{1}{3} \cdot [\sigma_x(t) + \sigma_y(t) + \sigma_z(t)], \quad (3)$$

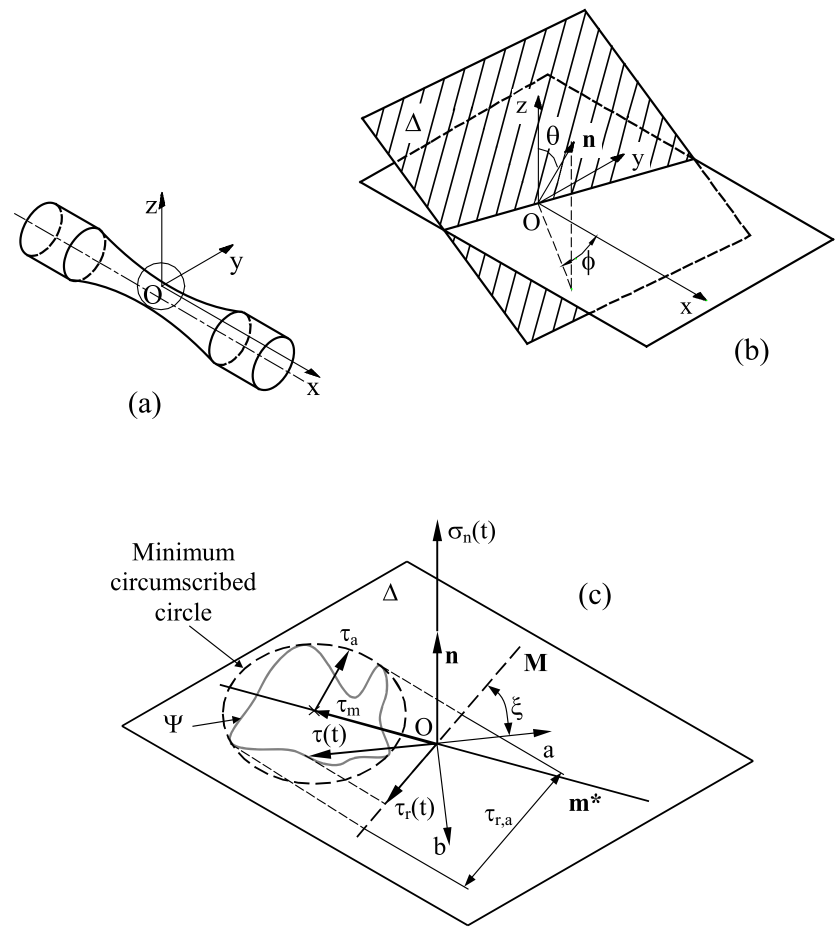

Figure 1 : Frame of reference (a), definition of the polar co-ordinates ϕ and θ (b) for a generic plane Δ and formalisation of the Minimum Circumscribed Circle concept (c).

and

$$[\sigma_D(t)] = \begin{bmatrix} \sigma_x(t)-\sigma_H(t) & \tau_{xy}(t) & \tau_{xz}(t) \\ \tau_{xy}(t) & \sigma_y(t)-\sigma_H(t) & \tau_{yz}(t) \\ \tau_{xz}(t) & \tau_{yz}(t) & \sigma_z(t)-\sigma_H(t) \end{bmatrix} \quad (4)$$

The hydrostatic stress is a quantity widely used in formulating multiaxial fatigue criteria, and, for this reason, it is useful to define its amplitude, its mean value and its maximum value:

$$\sigma_{H,a} = \frac{1}{2}\left\{\max_{t_1\in T}\frac{tr([\sigma(t_1)])}{3} - \min_{t_2\in T}\frac{tr([\sigma(t_2)])}{3}\right\}, \quad (5)$$

$$\sigma_{H,m} = \frac{1}{2}\left\{\max_{t_1\in T}\frac{tr([\sigma(t_1)])}{3} + \min_{t_2\in T}\frac{tr([\sigma({}_2t)])}{3}\right\} \quad (6)$$

$$\sigma_{H,\max} = \sigma_{H,a} + \sigma_{H,m} \quad (7)$$

Another important quantity adopted to formulate multiaxial fatigue criteria is the square root of the second invariant of the stress deviator:

$$\sqrt{J_2(t)} = \sqrt{\frac{1}{2}[\sigma_D(t)]\cdot[\sigma_D(t)]}. \quad (8)$$

According to the suggestions reported in Lemaitre and Chaboche, 1990, the above quantity can even be expressed as follows:

$$\sqrt{J_2(t)} = \sqrt{S_1^2(t)+S_2^2(t)+S_3^2(t)+S_4^2(t)+S_5^2(t)} \quad (9)$$

where

$$S_1(t) = \frac{\sqrt{3}}{2}[\sigma_x(t)-\sigma_H(t)]$$

$$S_2(t) = \frac{1}{2}[\sigma_y(t)-\sigma_z(t)]$$

$$
\begin{aligned}
S_3(t) &= \tau_{xy}(t) \\
S_4(t) &= \tau_{xz}(t) \\
S_5(t) &= \tau_{yz}(t)
\end{aligned}
\tag{10}
$$

The amplitude of such a stress quantity, $\sqrt{J_{2,a}}$, is usually calculated by using the Longest Chord Method (Lemaitre and Chaboche, 1990; Fuchs and Stephens, 1980; Papadopoulos, 1998): to be precise, the amplitude of $\sqrt{J_2}$ is equal to half of the longest chord of the hyper-curve plotted, in a 5-dimensional Euclidian space, by the tip of the vector having instantaneous components equal to S_i(t) (t∈T, i=1, 2, ..., 5).

Consider now a generic material plane Δ (Fig. 1b) having normal unit vector **n**. Its orientation can unambiguously be determined by introducing the following spherical coordinates: φ is the angle between the projection of the unit vector **n** on the x-y plane and the x-axis, whereas θ is the angle between the normal **n** and the z-axis (Papadopoulos, 1998) (Fig. 1b). For any fixed material plane Δ the stress tensor [σ(t)] can be decomposed into two stress components: the normal stress σ_n(t) and the shear stress τ(t).

In general, critical plane approaches are based on the hypothesis that fatigue damage in the high-cycle fatigue regime reaches its maximum value on the plane experiencing the maximum shear stress amplitude (the so-called critical plane). The calculation of the amplitude of the shear stress relative to a generic material plane is a complex problem to be addressed, because the vector τ(t) changes both its magnitude and its direction during the load cycle (Fig. 1c). Even though there exist different proposals to calculate τ_a, it is common opinion that the most rigorous method is the one proposed by Papadopoulos (Papadopoulos, 1998) and based on the use of the minimum circumscribed circle concept. According to the above method, the shear stress amplitude is equal to the radius of the minimum circle that circumscribes the curve plotted by the tip of the shear stress vector τ(t) on the Δ plane during the cyclic load. (Fig. 1c).

The amplitude, the mean value and the maximum value of the stress component perpendicular to the generic plane Δ can be expressed simply as:

$$
\begin{aligned}
\sigma_{n,a}(\phi,\theta) &= \frac{1}{2}\left\{\max_{t\in T}\sigma_n(\phi,\theta,t) - \min_{t\in T}\sigma_n(\phi,\theta,t)\right\} \\
\sigma_{n,m}(\phi,\theta) &= \frac{1}{2}\left\{\max_{t\in T}\sigma_n(\phi,\theta,t) + \min_{t\in T}\sigma_n(\phi,\theta,t)\right\} \\
\sigma_{n,\max}(\phi,\theta) &= \sigma_{n,a}(\phi,\theta) + \sigma_{n,m}(\phi,\theta)
\end{aligned}
\tag{11}
$$

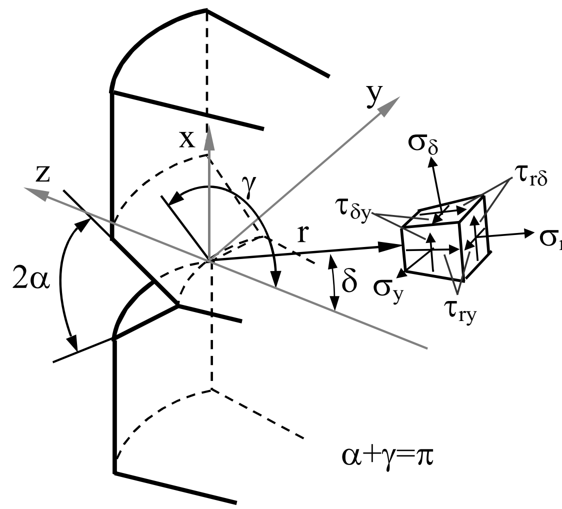

Figure 2 : V-notched specimen, definition of the coordinates r and δ and stress components damaging an elementary material volume in the vicinity of the notch tip.

Consider now a specimen subjected to a Mode I and II loading and containing a V-shaped notch (Fig. 2). According to Williams (Williams, 1952), the stress state along the notch bisector (δ=0°) can be calculated by the following equations proposed by Lazzarin and Tovo (Lazzarin and Tovo, 1996):

$$
\left\{\begin{array}{c}\sigma_\delta\\ \sigma_r\\ \tau_{r\delta}\end{array}\right\}_{r_n=0} = \frac{1}{\sqrt{2\pi}}\frac{r^{\lambda_1-1}K_1}{(1+\lambda_1)+\chi_1(1-\lambda_1)} \cdot\left[\left\{\begin{array}{c}(1+\lambda_1)\\ (3-\lambda_1)\\ 0\end{array}\right\} + \chi_1(1-\lambda_1)\left\{\begin{array}{c}1\\ -1\\ 0\end{array}\right\}\right] \tag{12}
$$

$$
\left\{\begin{array}{c}\sigma_\delta\\ \sigma_r\\ \tau_{r\delta}\end{array}\right\}_{r_n=0} = \frac{1}{\sqrt{2\pi}}\frac{r^{\lambda_2-1}K_2}{(1-\lambda_2)+\chi_2(1+\lambda_2)} \cdot\left[\left\{\begin{array}{c}0\\ 0\\ 1\end{array}\right\} + \chi_2(1+\lambda_2)\left\{\begin{array}{c}0\\ 0\\ 1\end{array}\right\}\right] \tag{13}
$$

In the above relationships λ_1, λ_2, χ_1 and χ_2 are constants depending on the opening angle value (Atzori, Lazzrain, Tovo, 1999; Lazzarin and Tovo, 1996; Williams, 1952).

On the other hand, when the specimen of Figure 2 is subjected to anti-plane loading, the stress components along the notch bisector turn out to be (Dun, Suwito, Cunningham, 1997; Quan and Hasebe, 1997):

$$\left\{ \begin{array}{c} \tau_{\delta y} \\ \tau_{ry} \end{array} \right\} = \frac{K_3}{\sqrt{2\pi}} r^{\lambda_3 - 1} \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\} \quad (14)$$

where the value of λ_3 depends on the notch-opening angle. In equations (12)-(14) K_1, K_2 and K_3 are the notch-stress intensity factors due to Mode I, Mode II and Mode III loading, respectively. These factors can easily be calculated by applying the following definitions (Dun, Suwito, Cunningham, 1997; Gross and Mendelson, 1972; Quan and Hasebe, 1997):

$$K_1 = \sqrt{2\pi} \lim_{r\to 0} (\sigma_\delta)_{\delta=0} r^{1-\lambda_1}$$

$$K_2 = \sqrt{2\pi} \lim_{r\to 0} (\tau_{r\delta})_{\delta=0} r^{1-\lambda_2} \quad (15)$$

$$K_3 = \sqrt{2\pi} \lim_{r\to 0} (\tau_{\delta y})_{\delta=0} r^{1-\lambda_3}$$

When the notch-opening angle, 2α, is equal to 0 (i.e. when the notch becomes a crack), and the specimen of Figure 2 is subjected to Mode I loading, equation (12) can be simplified to obtain the following well-known relationships:

$$\sigma_r = \sigma_z = \frac{F_I \cdot K_I}{\sqrt{2\pi r}} \quad (16)$$

$$\sigma_\delta = \sigma_x = \sigma_z = \frac{F_I \cdot K_I}{\sqrt{2\pi r}} \quad (17)$$

$$\tau_{r\delta} = \tau_{zx} = 0 \quad (18)$$

where F_I is the geometrical correction factor for the LEFM stress intensity factor due to Mode I loading, K_I. If the same cracked specimen ($2\alpha = 0$) is loaded in torsion, equation (14) results in the following simplified form:

$$\tau_{\delta z} = \tau_{xy} = \frac{F_{III} \cdot K_{III}}{\sqrt{2\pi r}} \quad (19)$$

In the above relationship F_{III} is the geometrical correction factor for the LEFM stress intensity factor due to Mode III loading, K_{III}.

3 The Point Method: The Simplest Formalisation Of The Theory Of Critical Distances

The TCD uses the elastic stresses in the local region close to the notch. It can be formalised in different ways (see, for example, Taylor, 1999), all of which involve the calculation of a characteristic stress, whose range can be compared to the plain fatigue limit. The multiaxial fatigue criteria considered in the present study were originally formalised by considering smooth components and assuming that the stress state to be used to estimate fatigue damage had to be determined at the crack initiation point. For this reason, in the next sections, the existing multiaxial fatigue criteria will be re-interpreted just in terms of the one particular formulation of the TCD, which is known as the Point Method (PM). Consider a notched specimen subjected to a remote uniaxial fatigue loading (Fig. 2). According to the PM, the specimen is in the fatigue limit condition when the stress range at a point, located a distance L/2 from the notch root, is equal to the plain fatigue limit, i.e.:

$$\Delta\sigma_x \left(\delta = 0, r = \frac{L}{2}\right) = \Delta\sigma_1 \left(\delta = 0, r = \frac{L}{2}\right) = \Delta\sigma_0 \quad (20)$$

In the above equation, L is the material characteristic length, which is calculated, for the correct load ratio, as follows (Taylor, 1999):

$$L = \frac{1}{\pi} \left(\frac{\Delta K_{I,th}}{\Delta\sigma_0} \right)^2 \quad (21)$$

It is worth noticing that the above definition results in a constant similar to the one defined by Topper and co-workers to address the short crack problem (see, for instance, El Haddad, Dowling, Topper and Smith, 1980), though here such a quantity is used in a different way. Equation (21) makes it evident that, L is a function of two material properties, therefore it is also a material property, which is different for different materials and R ratios. Consider now a notched specimen subjected to torsional fatigue loading (Fig. 2). In this case, the PM can be formulated as follows (Susmel and Taylor, 2006):

$$\Delta\tau_{xy} \left(\delta = 0, r = \frac{L}{2}\right) = \Delta\sigma_1 \left(\delta = 0, r = \frac{L}{2}\right) = \Delta\tau_0 \quad (22)$$

This approach assumes that the length constant L is independent of the type of loading. It is not entirely clear from the experimental data whether this is true or not (see, for example, Endo and Murakami, 1987; Susmel 2004; Susmel and Taylor, 2006), however we have shown that accurate predictions for notches in torsion can be obtained using L values determined from tensile testing (Susmel and Taylor, 2006), so this assumption will be used in what follows.

4 The General Procedure To Re-Interpret The Classical Multiaxial Fatigue Criteria In Terms Of The Theory Of Critical Distances

The existing multiaxial fatigue criteria are based on different theoretical assumptions resulting in the use of different stress quantities. In any case, independently of the stress parameters employed to formalise such criteria, they can always be expressed in the following general form:

$$X+\omega Y=\eta \tag{23}$$

In the above relationship, X and Y are stress parameters depending on the applied multiaxial fatigue loading, whereas ω and η are constants depending on the material fatigue properties. In particular, ω and η have to be determined by applying the considered criterion to two experimental cases: the cases normally used are the fully-reversed plain uniaxial fatigue limit, σ_0, and the fully-reversed plain torsional fatigue limit, τ_0. However, if the criterion is generally applicable, and if it can be used in conjunction with the PM, then it should also be possible to obtain these constants from data on a specimen containing a notch or crack, loaded in tension and in torsion, respectively, using Eqs (20) and (22). Suppose that this procedure results in two constants, ω^* and η^* only when $\omega^*=\omega$ and $\eta^*=\eta$ it is possible to affirm that the considered multiaxial fatigue criterion can consistently be used along with the TCD. In fact, if the above identities were not assured, this would result in a criterion sensitive to the sharpness of the notch, and therefore of little practical value.

In what follows, this test for consistency will be applied using the case of a cracked body under Mode I and Mode III loading (Fig. 3), for which we can write, for fully-reversed uniaxial fatigue loading (Fig. 3a):

$$\sigma_{x,a}=\sigma_{z,a}=\frac{F_I\cdot K_{I,a}}{\sqrt{\pi L}}=\sigma_0 \tag{24}$$

and for anti-plane stress (Fig. 3b):

$$\tau_{xy,a}=\frac{F_{III}\cdot K_{III,a}}{\sqrt{\pi L}}=\tau_0 \tag{25}$$

The above considerations make it evident that this paper considers only bi-dimensional situations resulting in biaxial stress field (plane stress) damaging the fatigue process zone.

5 Criteria Based On The Stress Invariants

The first class of criteria to be considered are those which are formulated as linear combinations of two scalar parameters: the amplitude of the square root of the second invariant of the stress deviator and the hydrostatic stress. These classical methods have widely been employed to assess real components, even though they seem not to be capable of satisfactorily accounting for the presence of non-zero out-of-phase angles (Petrone and Susmel, 1999). Another limitation of these approaches is that they do not supply any information on the orientation of the crack initiation plane. Amongst such criteria (Papadopoulos, 1997) the most popular one is probably that proposed by Sines in 1959. Such a criterion is formalised as a linear combination of $\sqrt{J_{2,a}}$ and the mean hydrostatic stress $\sigma_{H,m}$:

$$\sqrt{J_{2,a}}+\omega\cdot\sigma_{H,m}=\eta \tag{26}$$

The constants of the above criterion can be determined considering the fully-reversed torsional case and a uniaxial case characterised by a load ratio, R, equal to 0. If the plain fatigue limit under R=-1 is expressed as a function of the plain fatigue limit under R=0 by using Goodman's equation, then the above constants turn out to be (Papadopoulos, 1997):

$$\omega=\frac{\sqrt{3}\sigma_A}{\sigma_T};\quad \eta=\tau_0 \tag{27}$$

In our experience (Petrone and Susmel, 1999) this criterion is not capable of satisfactorily accounting for the presence of both non-zero out-of-phase angles and non-zero mean stresses: it has been reported in the present section for completeness, but it will not be considered in the final re-analysis. Another well-known criterion based on the calculation of the stress invariants is the one proposed by Crossland in 1956, who uses the maximum hydrostatic stress, giving:

$$\sqrt{J_{2,a}}+\omega\cdot\sigma_{H,\max}=\eta \tag{28}$$

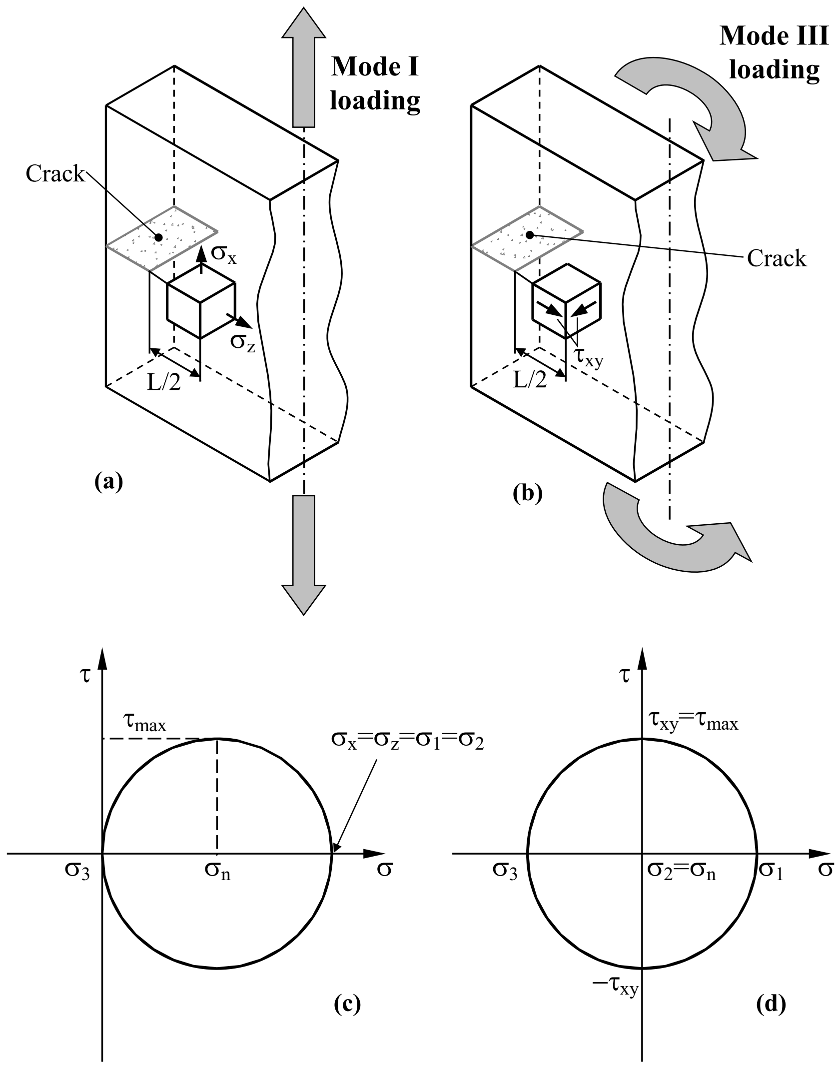

Figure 3 : Specimens subjected to tensile stress (a) and to anti-plane stress (b) and corresponding stress states at the critical point plotted in terms of Mohr's circles (c, d).

Observing that under fully-reversed torsional fatigue loading the relevant stress quantities are equal to: $\sqrt{J_{2,a}} = \tau_0; \sigma_{H,\max} = 0$, and under fully-reversed uniaxial fatigue loading to: $\sqrt{J_{2,a}} = \sigma_0/\sqrt{3}; \sigma_{H,\max} = \sigma_0/3$, the constants of Eq. (28) turn out to be:

$$\omega = \frac{3\tau_0}{\sigma_0} - \sqrt{3}; \eta = \tau_0 \tag{29}$$

To re-formulate the above criteria in terms of the TCD, the condition expressed by Eqs (24) and (25) must be assured. In particular under fully-reversed uniaxial fatigue loading $\sqrt{J_{2,a}}$ and $\sigma_{H,max}$ turn out to be:

$$\sqrt{J_{2,a}} = \frac{\sigma_0}{\sqrt{3}}; \quad \sigma_{H,\max} = \frac{2\sigma_0}{3} \tag{30}$$

whereas under fully-reversed torsional fatigue loading the above quantities are equal to:

$$\sqrt{J_{2,a}} = \tau_0; \quad \sigma_{H,\max} = 0 \tag{31}$$

Considering the conditions expressed by the above identities, the constants in equation (28) result in the following values:

$$\omega* = \frac{1}{2}\left(\frac{3\tau_0}{\sigma_0} - \sqrt{3}\right); \eta* = \tau_0 \tag{32}$$

The fact that $\omega^* \neq \omega$ proves that Crossland's criterion cannot rigorously be re-interpreted in terms of the TCD: constant ω^* is different for plain and cracked specimens and therefore will, in general, depend on the K_t value of the notch.

6 Criteria Based On The Mesoscopic Approach

Taking advantage of the work of Dang Van published in 1973, many researchers have attempted to propose criteria based on an interpretation at a mesoscopic level of the material fatigue behaviour. In the present section the criteria proposed by Dang Van himself and Papadopoulos will be considered. Dang Van's criterion summarises a sophisticated attempt to describe the fatigue damage evolution in a metal at a mesoscopic scale resulting in the use of the macroscopic shear stress and hydrostatic stress. The rigorous in field application of such an approach is not trivial, due to the complex procedure which has to be applied to account for the presence of non-proportional loading as well as non-zero mean stress (Dang Van, Cailletaud, Flavenot, Le Douaron and Lieurade, 1989; Dang Van, Griveau and Messagge, 1989). In the present section, the criterion is considered in its simplest form (Ballard, Dang Van, Deperrois and Papadopoulos, 1995):

$$\frac{[\sigma_1(t)-\sigma_{1,m}]-[\sigma_3(t)-\sigma_{3,m}]}{2}+\omega\cdot\sigma_H(t)=\eta \quad (t\in T) \tag{33}$$

Applying the above equation to the fully-reversed uniaxial and fully-reversed torsional case, constants ω and η turn out to be:

$$\omega=3\left(\frac{\tau_0}{\sigma_0}-\frac{1}{2}\right);\quad \eta=\tau_0 \tag{34}$$

On the contrary, when applied to a cracked body using Eqs (24) and (25), the constants take the following values:

$$\omega*=\frac{3}{2}\left(\frac{\tau_0}{\sigma_0}-\frac{1}{2}\right);\quad \eta*=\tau_0 \tag{35}$$

Again, by showing that $\omega^*\neq\omega$ we demonstrate that Dang Vang's criterion cannot be re-interpreted in terms of the TCD.

Papdopoulos' mesoscopic approach postulates that multiaxial fatigue strength of ductile materials depends on the following two stress parameters (Papadopoulos, 1987):

$$M_\sigma=\sqrt{\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}T_\sigma^2(\phi,\theta)\,sin\theta\cdot d\theta\cdot d\phi} \tag{36}$$

$$\sigma_{H,\max}=\max_{t\in T}\left[\frac{\sigma_1(t)+\sigma_2(t)+\sigma_3(t)}{3}\right] \tag{37}$$

where

$$T_\sigma(\phi,\theta)=\sqrt{\int_{\xi=0}^{2\pi}\tau_{r,a}^2(\phi,\theta,\xi)\,d\xi} \tag{38}$$

If the critical point of the component is subjected to a triaxial stress state, integral M_σ takes the following form (Papadopoulos, 1995):

$$M_\sigma=\sqrt{\frac{1}{3}}\left[\sigma_{x,a}^2+\sigma_{y,a}^2+\sigma_{z,a}^2+3\tau_{xy,a}^2+3\tau_{xz,a}^2+3\tau_{yz,a}^2 -\sigma_{x,a}\cdot\sigma_{y,a}\cos\psi_{x,y}-\sigma_{y,a}\cdot\sigma_{z,a}\cos\psi_{y,z} -\sigma_{x,a}\cdot\sigma_{z,a}\cos\psi_{x,z}\right]^{\frac{1}{2}} \tag{39}$$

According to Papadopoulos' criterion, a smooth component ($0.5\leq\frac{\tau_0}{\sigma_0}\leq 0.6$) is in the fatigue limit condition when the following condition is assured (Papadopoulos, 1987; Papadopoulos, 1995):

$$M_\sigma+\omega\cdot\sigma_{H,\max}=\eta \tag{40}$$

If the previous equation is calibrated by using the fully-reversed uniaxial and torsional plain fatigue limit, the constants of Eq. (40) turn out to be (Papadopoulos, 1995):

$$\omega=\frac{\tau_0-\sigma_0/\sqrt{3}}{\sigma_0/3};\eta=\tau_0 \tag{41}$$

To reinterpret Papadopoulos' criterion in terms of the TCD, M_σ and $\sigma_{H,max}$ have to be determined by considering the stresses at the critical distance, L/2, when the cracked specimens of Figure 3 are subjected to Mode I as well as to Mode III loading. In particular, under fully-reversed uniaxial fatigue loading Papadopoulos' stress quantities turn out to be:

$$M_\sigma=\sqrt{\frac{\sigma_{x,a}^2+\sigma_{z,a}^2-\sigma_{x,a}^2\cdot\sigma_{z,a}^2}{3}};\qquad \sigma_{H,\max}=\frac{\sigma_{x,a}+\sigma_{z,a}}{3} \tag{42}$$

whereas under fully-reversed Mode III loading they turn out to be:

$$M_\sigma=\tau_{xy,a};\quad \sigma_{H,\max}=0 \tag{43}$$

Applying these parameters to calculate the constants of Eq. (40) for cracked specimens gives the following result:

$$\omega* = \frac{1}{2}\left[\frac{\tau_0 - \sigma_0/\sqrt{3}}{2\sigma_0/3}\right]; \eta* = \tau_0 \tag{44}$$

These relationships make it evident that:

$$\omega* = \frac{1}{2}\omega \tag{45}$$

This means that Papadopoulos' criterion cannot always be successfully re-interpreted in terms of the TCD. In particular, if constants ω^* and η^* were used to predict the fatigue strength of a plain specimen under tension-compression the value of the plain fatigue limit would result in a quantity depending on the τ_0 to σ_0 ratio. To be precise, under fully-reversed uniaxial fatigue loading the condition postulated by Eq. (38) would result in the following one:

$$\sigma_{x,a} \le \frac{2\sqrt{3}\tau_0}{1+\sqrt{3}\frac{\tau_0}{\sigma_0}} \tag{46}$$

It is interesting to observe that, if the τ_0 to σ_0 ratio were expressed according to Von Mises' hypothesis (Atzori, Meneghetti and Susmel, 2005), the above condition would turn out to be:

$$\sigma_{x,a} \le \sigma_0 \tag{47}$$

This fact confirms that Papadopoulos' criterion can coherently be re-interpret in terms of the TCD only when the τ_0 to σ_0 ratio is equal to $1/\sqrt{3}$. Unfortunately, experimental evidence shows that this ratio varies in the range 0.5-1.0 (Atzori, Meneghetti and Susmel, 2005); we conclude that, in the general case, the criterion of Papadopoulos cannot be used in conjunction with the TCD.

7 Criteria Based On The Critical Plane Approach

Many different researchers have proposed multiaxial high-cycle fatigue criteria based on the critical plane concept. Even though these methods have been formalised in many different ways, they take as their common starting point the assumption that crack initiation occurs on the plane experiencing the maximum shear stress amplitude (critical plane). They postulate that fatigue damage depends on the combined effect of the shear stress amplitude and the normal stress relative to the critical plane (Papadopoulos, Davoli, et al., 1997; Socie and Marquis, 2000; You and Lee, 2006). In the present study, three different critical plane approaches were considered: Matake's criterion (Matake, 1977), McDiarmid's criterion (McDiarmid, 1991 and 1994) and, finally, the Modified Wöhler Curve Method (MWCM) (Susmel and Lazzarin, 2002; Lazzarin and Susmel, 2003). We deliberately did not review the well-know criterion proposed by Findley in 1959, because, as highlighted by Papadopoulos (Papadopoulos, 1997) it predicts that non-zero mean torsional stress affects the torsional fatigue limit value: unfortunately, the validity of this dependence is refuted by the experimental evidence, both for plain (Sines, 1959) and notched specimens (Gough, 1949). The criteria due to Matake and McDiarmid can be formalised as follows:

$$\tau_a(\phi^*,\theta^*) + \omega\frac{\sigma_{n,\max}}{\tau_a}(\phi^*,\theta^*) = \eta \tag{48}$$

According to the hypotheses these criteria are based on, when applied to smooth components, constants ω and η take the following form:

Matake's criterion:

$$\omega = \frac{2\tau_0}{\sigma_0} - 1; \eta = \tau_0 \tag{49}$$

McDiarmid's criterion:

$$\omega = \frac{\tau_0}{2\sigma_T}; \eta = \tau_0 \tag{50}$$

It is important to highlight that McDiarmid's criterion is an empirical formula based on the assumption that a plain component subjected to fully-reversed uniaxial fatigue loading is in its fatigue limit condition when:

$$\sigma_{x,a} = \frac{2\tau_0}{\left(1+\frac{\tau_0}{2\sigma_T}\right)} \tag{51}$$

This condition is rigorously assured only when the right-hand side of Eq. (51) equals the material's fully-reversed plain fatigue limit: unfortunately, this is not always true. The MWCM, on the other hand, is written as (Susmel and Lazzarin, 2002):

$$\tau_a(\phi*,\theta*) + \left[\tau_{A\infty} - \frac{\sigma_{A\infty}}{2}\right]\frac{\sigma_{n,\max}}{\tau_a}(\phi*,\theta*) \le \tau_{A\infty}. \tag{52}$$

where

$$\omega = \tau_0 - \frac{\sigma_0}{2}; \eta = \tau_0 \quad (53)$$

To re-interpret the above criteria in terms of the TCD, it is possible to make use of Mohr's circles as shown in Figures 3c and 3d. In particular, under fully-reversed uniaxial fatigue loading, for a cracked plate in the fatigue limit condition, it is trivial to write the following identities:

$$\tau_a = \sigma_{n,\max} = \frac{\sigma_{x,a}}{2} = \frac{\sigma_{z,a}}{2} = \frac{\sigma_0}{2} \quad (54)$$

Whereas, according to Eq. (25), the plate of Figure 3b is in the fatigue limit condition, when the relevant stress parameters, that is, τ_a and $\sigma_{n,max}$, are equal to:

$$\tau_a = \tau_0; \sigma_{n,\max} = 0 \quad (55)$$

The above identities can directly be used to calculate constants ω^* and η^* of Matake's criterion when re-interpreted in terms of the TCD, obtaining:

$$\omega* = \frac{2\tau_0}{\sigma_0} - 1; \eta* = \tau_0 \quad (56)$$

In the same way, constants ω^* and η^* can be calculated for the MWCM when applied in terms of the PM:

$$\omega* = \tau_0 - \frac{\sigma_0}{2}; \eta* = \tau_0 \quad (57)$$

Finally, according to the consideration reported above, to coherently calculate constants ω^* and η^* for McDiarmid's criterion it is necessary to form the following hypothesis on the plain fatigue limit value:

$$\sigma_0 = \frac{2\tau_0}{1 + \frac{\tau_0}{2\sigma_T}} \quad (58)$$

By using the above expression for σ_0, when the conditions expressed by Eqs (24) and (25) are assured, constants ω^* and η^* in McDiarmid's criterion expressed in terms of the TCD turn out to be:

$$\omega* = \frac{\tau_0}{2\sigma_T}; \eta* = \tau_0 \quad (59)$$

These considerations show that these critical plane approaches can successfully be re-interpreted in terms of the TCD: for these criteria the condition $\omega^* = \omega$ and $\eta^* = \eta$ is always assured. We conclude then, that the only multiaxial fatigue laws which can be consistently used with the TCD are the critical plane criteria.

8 Extending The Approach To Open V-Notches And Triaxial Stress States

The aim of the present section is to briefly investigate accuracy and reliability of the critical plane approaches in the presence of stress states having a higher level of complexity than the ones considered in the previous section.

Initially, attention can be focused on open V-notches subjected to nominal uniaxial fatigue loading. To correctly address this problem, it is important to observe that, when dealing with blunt V-notches, in the vicinity of the stress concentrator apex the influence of the notch opening angle on the stress filed distribution is negligible. In the light of this fact, it is logical to believe that the TCD correctly predicts the high-cycle fatigue strength of blunt notches also in the presence of large values of the notch opening angle (Taylor and Susmel, 2003; Taylor and Wang, 2000).

On the contrary, when V-notches become sharp, the stress distribution ahead of the stress concentrator apex is strongly influenced by the value of the notch opening angle itself: the exponents of William's equations change as the opening angle value increases. We never investigated this problem systematically, but whenever we tried to use the TCD to predict the high-cycle fatigue strength in the presence of sharp V-notches having a large value of the opening angle, predictions were always highly accurate (see, for instance, the sound agreement between estimations and experimental results obtained when applying the TCD to welded joints (Taylor, Barrett and Lucano, 2002), that is, to sharp V-notches having opening angle equal to 135 °).

It is also worth noticing here that, in 2005, Atzori, Lazzarin and Meneghetti deeply investigated the problem of predicting the high-cycle fatigue strength of sharp V-notches characterised by different values of the opening angle. In such a study, they critically reviewed several methods, coming to the conclusion that the PM (but also other approaches based on different theoretical frameworks) works very well in the presence of large values of the notch opening angle (even though the critical distance is defined considering a crack, that is, a stress raiser having notch opening angle equal to zero).

Consider now a sharp V-notch subjected to a remote uniaxial fatigue loading. When the notch opening angle, 2α, is larger than zero, Eqs (12) and (13) show that σ_x becomes larger than σ_z. The corresponding stress state is

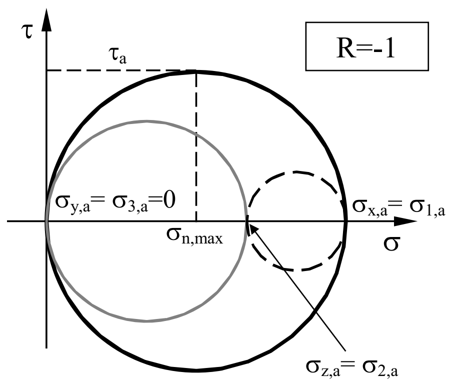

Figure 4 : Mohr's circles for a sharp notch having opening angle larger than zero.

described by Mohr's circles sketched in Figure 4. In such a situation, the PM postulates that a component is in the fatigue limit condition when:

$$\sigma_{x,a} = \sigma_{1,a} = \sigma_0 > \sigma_{z,a} \tag{60}$$

and the validity of the above relationship has been confirmed by systematic re-analyses done considering specimens of different material and characterised by different geometrical features (Taylor and Susmel, 2003; Taylor and Wang, 2000). According to the symbolism of Figure 4, it is possible now to easily write the following identities (Susmel and Taylor, 2003):

$$\tau_a = \sigma_{n,\max} = \frac{\sigma_{1,a}}{2} = \frac{\sigma_0}{2} \tag{61}$$

which are still valid even in the presence of notch root radii different from zero, because the stress field at any point along the notch bisector is always described by Mohr's circles similar to those sketched in Figure 4 (Lazzarin and Tovo, 1996). In other words, even when the root radius of a notch subjected to a remote uniaxial fatigue loading is larger than zero, the stress field along the notch bisector is always biaxial. According to the above remarks, it is trivial to demonstrate that the critical plane approaches considered in the previous section can successfully be re-interpreted in terms of the TCD even in the presence of non-zero notch-opening angles and non-zero root radii: the condition $\omega^* = \omega$ and $\eta^* = \eta$ is always assured.

Now it can be observed again that all the considerations on the reviewed criteria reported in the previous sections were formalised considering bi-dimensional problems resulting in a plane stress condition at the tip of the stress concentrator. Of course, dealing with more general situations, a triaxial stress state should have been considered. Unfortunately, at this stage this problem is difficult to be address, due to the lack of experimental results and to some added complications in the theoretical analysis. The TCD was originally formalised in terms of the maximum principal stress (Taylor, 1999), whose distribution ahead of cracks is the same both under plane stress and under plain strain: for this reason, under uniaxial fatigue loading the TCD can successfully be applied to predict the fatigue limit of stress concentrators independently of the multiaxiality of the stress field in the fatigue process zone.

On the contrary, if one wanted to correctly account for the triaxiality of the stress field ahead of a crack under plane strain (for instance, by defining a convenient equivalent stress), the situation would become much more complex: as far as the authors are aware there are still no sound approaches which can be used to address this complex and tricky problem.

In any case, to conclude this section, it is possible to say that, unfortunately, the problem is far from being completely solved: more effort is needed in this area, both from an experimental and a theoretical point of view, to propose sound methods suitable for assessing three-dimensional stress concentrators subjected to triaxial stress states. All the considerations reported in the present paper were instead based on the assumption that the reference configuration was that of a crack subjected to plane stress, and all the calculations summarised above were done in accordance with this simplifying hypothesis. It is worth noting, however, (as will be demonstrated below) that the present bi-dimensional approach turned out to be very successful for predicting the available experimental data.

9 Quantitative Comparison Of The Selected Criteria

The theoretical arguments reported above demonstrated that the critical plane approaches are the only multiaxial fatigue criteria which can rigorously be re-interpreted in terms of the TCD. In the present section, to check the practical accuracy of such criteria, they are employed to predict fatigue limits of notched specimens subjected to multiaxial fatigue loading. Table 1 summarises the experimental data used for this systematic validation (Gough, 1949; Susmel and Taylor, 2003; Kurath, Dow-

Table 1 : Summary of the experimental data generated under multiaxial fatigue loading.

Material	Ref.	σ_T [MPa]	σ_0 [MPa]	τ_0 [MPa]	L [mm]	Spec. type	Load type
SAE 1045	Kurath et al., 1989	621(2)	304(2)	176(1)	0.159(2)	Fig. 5a	Be/To
Ck 45 (SAE 1045)	Sonsino, 1994	621(2)	304(2)	176(1)	0.159(2)	Fig. 5b	Be/To
S65A	Gough, 1949	1003	584	371	0.056(3)	Fig. 5c	Be/To
0.4% C Steel (Norm.)	Gough, 1949	648	332	207	0.178(4)	Fig. 5d	Be/To
3% Ni Steel	Gough, 1949	526	343	205	0.144(4)	Fig. 5d	Be/To
3/3.5% Ni Steel	Gough, 1949	722	352	267	0.516(4)	Fig. 5d	Be/To
Cr-Va Steel	Gough, 1949	752	429	258	0.101(4)	Fig. 5d	Be/To
3.5% NiCr steel (NI)	Gough, 1949	895	540	352	0.150(4)	Fig. 5d	Be/To
3.5% NiCr steel (LI)	Gough, 1949	897	509	324	0.109(4)	Fig. 5d	Be/To
NiCrMo steel (75-80 tons)	Gough, 1949	1243	594	343	0.106(4)	Fig. 5d	Be/To
BS040A12	Susmel et al., 2003	410	273	171	0.200	Fig. 5e	MI-II

(1) Calculated using Von Mises' criterion
(2) Material properties taken from DuQuesnay, Yu, Topper, 1988;
(3) Determined by using the fully-reversed uniaxial fatigue test (Susmel, 2004);
(4) Determined by using both the uniaxial and the torsional fatigue results (Susmel, 2004).
Be= bending; To= torsion; MI-II= in-phase Mode I and II loading

ing, Galliart, 1989) - the assumptions made to estimate some of the material properties reported in Table 1 are widely discussed in (Susmel, 2004), whereas the geometries of the considered specimens are sketched in Figure 5. The accuracy in estimating fatigue limits of the different multiaxial fatigue criteria have been quantified according to the fatigue strength error index proposed by Papadopoulos (Papadopulos, 1995):

$$E[\%] = \left(\frac{X+\omega\cdot Y}{\eta}-1\right)\cdot 100 \tag{62}$$

The above definition implies that when E=0 fatigue life estimations are exact; when E<0 predictions are non-conservative and when E>0 predictions are conservative. According to the usual accuracy shown by the TCD (Taylor and Susmel, 2003; Taylor and Wang, 2000), estimations were considered to be acceptable when falling within an error interval of ±20%. The critical plane approaches should be capable of accounting for both the mean stress effect and the presence of non-proportional loading. Regarding the mean stress effect, we did not address this problem while re-interpreting multiaxial fatigue criteria from a TCD point of view; all the calculations were based on the assumption that the cracked specimens were subjected to fully-reversed loading (R=-1). Thus the material characteristic length, L, was determined by using fatigue properties (i.e., σ_0 and K_{th}) determined under R ratios equal to –1.

However, the rigorous application of the TCD under uniaxial fatigue loading requires the determination of L under the correct load ratio: the L value of a given material changes as the load ratio, R, changes (Taylor, 1999; Atzori, Meneghetti and Susmel, 2005). Therefore, the use in the presence of non-zero mean stresses of multiaxial fatigue criteria re-interpreted in terms of the PM should require the definition of a correct value for L. Unfortunately, this would result in an increase of the problem complexity, making these methods not suitable for systematically post-processing FE linear elastic results. To overcome this problem, we have previously argued (Susmel, 2004) that L must be always determined under fully-reversed uniaxial fatigue loading, because the presence of non-zero mean stresses as well as non-zero out-of-phase angles must be directly accounted for by the criterion used to estimate fatigue damage. In other words, L determined under R=-1 was considered to give representative information of the “pure” material cracking behaviour. Therefore, according to the above considerations, the accuracy of the considered critical plane approaches was checked in the presence of non-zero mean stresses by always using the L value determined under R=-1.

Initially, to check the validity of the above assumption, the reviewed multiaxial fatigue criteria were applied to predict fatigue limits of notched specimens subjected to

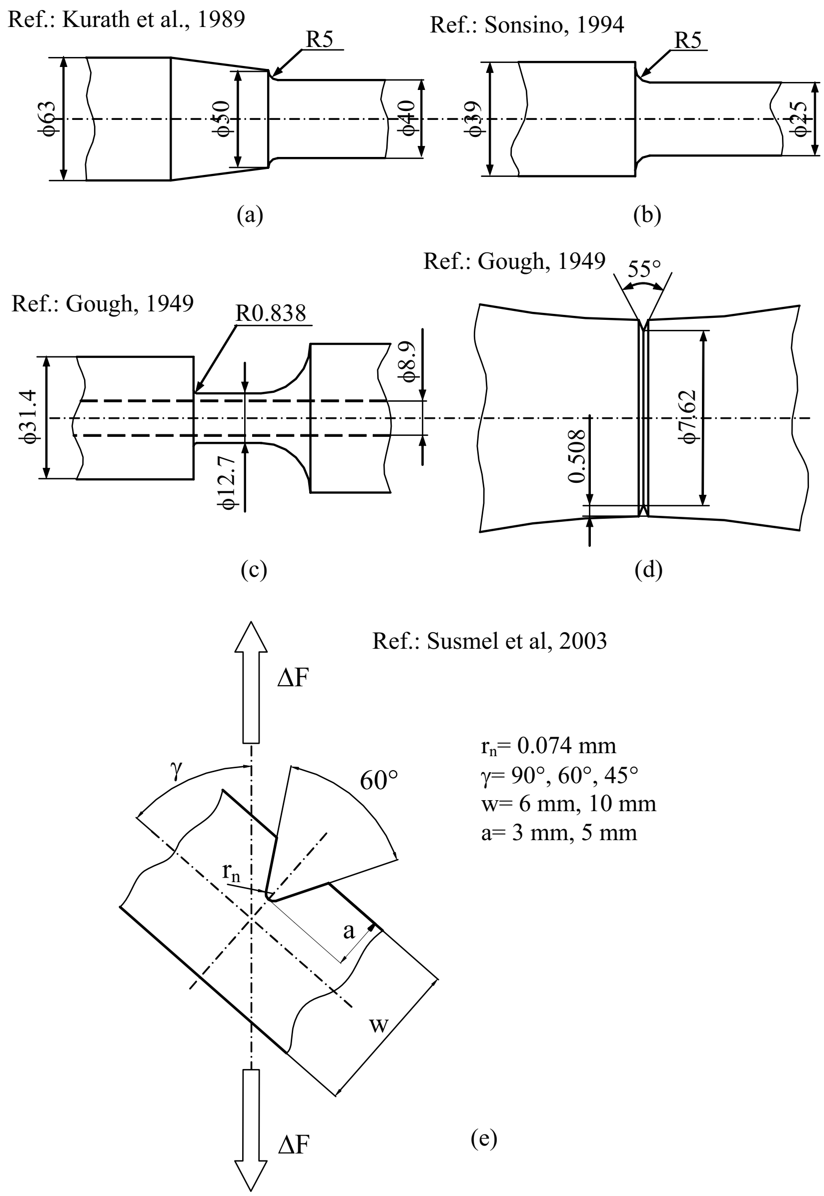

Figure 5 : Geometries of the notched specimens tested under multiaxial fatigue loading (dimensions in millimetres).

uniaxial fatigue loading. Table 2 summarises the performed estimations. In particular, the results listed in Table 2 were generated testing flat specimens, made of both steel - SAE 1045 (DuQuesnay, Yu and Topper, 1988) and SM41B (Tanaka and Nakai, 1983) - and aluminium - 2024-T351 (DuQuesnay, Yu and Topper, 1988) - , with a central hole of radius ranging from 0.12mm up to 2.5mm.

It is important to highlight that, according to the main hypothesis the speculations reported above were based on, stress fields in the vicinity of the notch tips of the considered specimens were determined by linear-elastic FE analyses carried out assuming a plane stress condition.

The accuracy obtained in predicting notch fatigue limits under fully-reversed uniaxial fatigue loading was not surprising: as demonstrated in the previous section, under fully-reversed uniaxial fatigue loading there exists a perfect correspondence between the PM and the critical plane approaches re-interpreted in terms of the TCD. More interesting is the observation that, apart from one prediction, all the criteria were capable of estimations falling within an error interval of ±20% when applied to predict fatigue limits under load ratios, R, larger than –1.

Table 2 : Accuracy of the critical plane approaches in predicting notch fatigue limits under uniaxial fatigue loading (M=Matake, McD=McDiarmid and MWCM= Modified Wöhler Curve Method).

Material	Ref.	σ_0 [MPa]	τ_0(1) [MPa]	σ_T [MPa]	r_n [mm]	Notch Type	R	Error M [%]	Error McD [%]	Error MWCM [%]
AL-2024-T351	DuQuesnay et al., 1988	124	72	460	0.12	Short	-1	-5.2	-11.9	-4.4
					0.25	Short	-1	-5.5	-12.2	-4.7
					0.5	Sharp	-1	13.1	5.0	11.3
					1.5	Blunt	-1	-7.1	-13.7	-6.1
					0.12	Short	0	16.3	1.7	15.7
					0.25	Short	0	-0.8	-13.3	2.8
					0.5	Sharp	0	13.7	-0.6	13.8
					1.5	Blunt	0	7.8	-5.7	9.3
SAE 1045	DuQuesnay et al., 1988	304	176	720	0.12	Short	-1	-19.1	-21.6	-16.5
					0.25	Short	-1	-10.0	-12.8	-8.6
					0.5	Sharp	-1	-2.3	-5.3	-2.0
					1.5	Blunt	-1	2.2	-1.0	1.9
					2.5	Blunt	-1	8.2	4.8	7.0
					0.12	Short	0	-16.6	-21.2	-9.4
					0.25	Short	0	2.6	-3.0	5.2
					0.5	Sharp	0	11.0	5.0	11.6
					1.5	Blunt	0	6.3	0.5	8.1
					2.5	Blunt	0	10.7	4.7	11.4
SMB41	Tanaka, Nakai, 1983 Tanaka, Akiniwa, 1987	163	94	423	0.16	Sharp	-1	-13.7	-20.3	-11.9
					0.39	Sharp	-1	-1.0	-8.5	-0.9
					0.83	Sharp	-1	-9.9	-16.8	-8.6
					3	Sharp	-1	2.1	-5.7	1.8
					0.16	Sharp	0	-39.6	-44.2	-34.4
					0.16	Sharp	0.4	-13.6	-30.4	1.5

(1) Calculated using Von Mises' criterion

The obtained accuracy is very promising, strongly supporting the idea that L can always be determined under fully-reversed fatigue loading, the detrimental effect due to non-zero mean stress being correctly accounted for by the multiaxial fatigue criteria themselves.

Table 3 shows that a similar level of accuracy was also obtained when applying the considered fatigue damage parameters to the results generated by Gough testing notched specimens under bending and torsion with superimposed static stresses (Gough, H. J.). Table 3 confirms that, independently of the employed criterion, predictions fell always within the usual error interval of $\pm 20\%$. As to the accuracy in predicting out-of-phase loading situations, the last two rows of Table 3 show that all the critical plane based methods examined here, were capable of correctly predicting the multiaxial high-cycle fatigue strength of the only two results found in the technical literature and generated under out-of-phase loading. These results are promising, even though it is evident that more experimental work has to be done to better investigate the fatigue behaviour of notched components under complex stress states.

The error in the systematic application of the three considered critical plane approaches to the 66 sets of experimental data found in the literature, is summarised in Figure 6: this diagram was constructed assuming a normal distribution of the errors (mean values and standard deviations are reported in Table 4). The error distributions

Table 3 : Accuracy of the considered critical plane approaches in predicting multiaxial fatigue strength due to out-of-phase loading as well as due to non-zero mean stresses (M=Matake, McD=McDiarmid and MWCM= Modified Wöhler Curve Method).

Material	Ref.	$\sigma_{b,a}$	$\sigma_{b,m}$	$\tau_{t,a}$	$\tau_{t,m}$	$\psi_{x,xy}$	Error M	Error McD	Error MWCM
		[MPa]	*[MPa]*	*[MPa]*	*[MPa]*	*[°]*	*[%]*	*[%]*	*[%]*
S65A	Gough, 1949	347.3	266.3	0	0	0	16.4	4.2	16.8
		361.2	0	0	169.8	0	3.7	-3.5	3.1
		318	266.3	0	169.8	0	8.0	-3.6	11.8
		0	0	276.3	169.8	0	-7.6	-7.6	-7.6
		0	266.3	243.9	0	0	14.5	4.1	13.3
		0	266.3	230	169.8	0	9.8	-0.6	10.6
		294.15	0	196.1	169.8	0	12.1	2.8	10.6
		270.9	266.3	180.6	169.8	0	18.0	4.8	19.4
SAE 1045	Kurath et al., 1989	156.9	0	100.6	0	90	-12.2	-14.2	-7.1
CK45	Sonsino, 1994	188.9	0	131.6	0	90	9.5	7.2	9.4

Table 4 : Mean values and Standard Deviations of the error distribution for the three considered critical plane approaches.

Criterion	Mean Value	Standard Deviation
Matake	-7.47	16.67
McDiarmid	-12.94	18.27
MWCM	-1.82	11.33

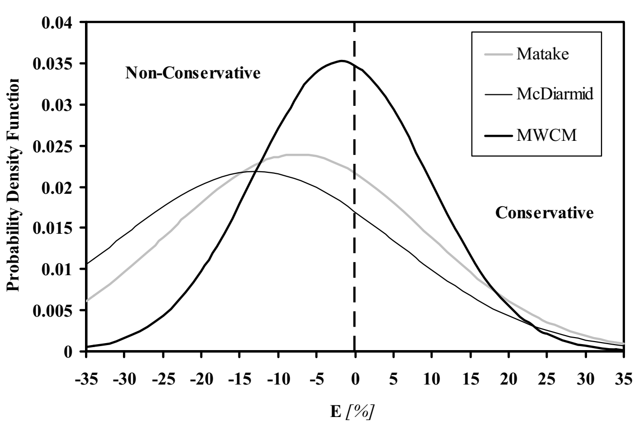

Figure 6 : Probability Density Function vs. Error diagram for the three considered critical plane approaches.

shown by Figure 6 prove that the best accuracy was obtained by applying the MWCM: about 92% of all the predictions fell within our limits of 20% error. Moreover, it can be seen that the three peaks of the error distributions fell within the non-conservative zone (Tab. 3), but the mean value (equal to about –2%) obtained by applying the MWCM was very close to zero, making this the most valid criterion.

To conclude this section, it is possible to say that the only criteria which can rigorously be re-interpreted in terms of the TCD are the critical plane approaches and, among them, the systematic validation discussed above proved that the most accurate one is the MWCM.

10 Discussion

This paper summarises an attempt to use the conventional multiaxial fatigue criteria in conjunction with the TCD. By examining the stress fields at the fatigue limits and thresholds for plain and cracked specimens, it was shown that self-consistent predictions could be made, i.e. that the material constants in the criteria remained truly constant, only for the critical plane criteria. Other criteria, using stress invariants or mesoscopic approaches, were shown to be inconsistent with the TCD by this same test. This argument was then generalised to include also V-shaped notches and notches with non-zero root radius.

It is important to highlight that the above calculations were done assuming as reference configuration the one of a cracked plate where the crack tip was subjected to plane stress. This assumption was a consequence of the fact that, in this paper, we considered only bi-dimensional problems. As far as the authors are aware there are no methods strictly devised to assess three-dimensional stress concentrators subjected to triaxial stress states: it is evident that a great effort has to be made in order to formalise fatigue life estimation techniques which are suitable for addressing such a complex problem.

A second important assumption in this analysis was that the critical distance L is a material constant, independent of the type of loading. In fact we know that this is not strictly true: L has been shown to vary with R ratio (Taylor, 1999; Atzori, Meneghetti and Susmel, 2005) and so we can expect that, in general, it will also vary with the degree of multiaxiality. However, our previous work suggested that this was not a major factor in the predictive accuracy of the model, and this has been borne out by the accuracy of the predictions made in the much larger assessment of experimental data presented here. The likely reason for this is that, due to the nature of the stress gradient at the notch, a change in the value of L tends to have only a small effect on the value of the predicted fatigue strength. Thus this simplifying assumption seems to be valid, at least for the whole range of notch geometries which are likely to be encountered in practice.

Having examined a number of different critical plane approaches, the one which was found to be more accurate, and therefore the one which is recommended from this study, is the MWCM. When combined with the TCD in the form of the Point Method, this criterion was able to account for the effect of multiaxiality in notches with a very wide range of geometries and K_t factors, loaded at different R ratios.

Despite this success, it is clear that the effect of R ratio cannot be completely described by the MWCM, or indeed by any one criterion, as it is well known that different criteria are needed to describe mean-stress effects in different materials (Frost, Marsh and Pook 1974; Susmel, Tovo and Lazzarin, 2005) - e.g. Gerber's parabola, Dietman's parabola, Goodman's straight line, Soderberg's relationship, etc.. Consider a plain specimen subjected to a remote uniaxial fatigue loading characterised by a load ratio larger than –1. If the MWCM were a "perfect" method, it should predict the plain fatigue limit of this specimen, even if calibrated by using fatigue properties generated under fully-reversed loading. Unfortunately, this is not always true, because, as the classical criteria, its accuracy depends on the material sensitivity to superimposed static stresses (Susmel, Tovo and Lazzarin, 2005).

Consider now a notched specimen subjected to Mode I loading with a load ratio, R, different from -1. Even assuming that the MWCM might correctly predict the uniaxial plain fatigue limit under the applied load ratio, the notch fatigue limit prediction would not be correct in any case, in fact, as mentioned above, the L value changes as the load ratio changes (Taylor, 1999; Atzori, Meneghetti and Susmel, 2005).

These considerations highlight the fact that, even though the MWCM performs predictions which are accurate from an engineering point of view, from a theoretical point of view this method is just a pragmatic approximation of a complex phenomenon.

Finally, to conclude this section, it is possible to highlight the high accuracy shown by the critical plane approaches re-interpreted in terms of the TCD when used to predict notch fatigue limits under out-of-phase loading: the obtained results are promising, but the only two experimental results found in the literature are not enough to express a final verdict about this aspect.

11 Conclusions

The critical plane approaches are the only multiaxial fatigue criteria which can coherently be re-interpreted in terms of the TCD; the use of other criteria leads to inconsistencies in the values of the material constants for plain, cracked and notched specimens;

Among the reviewed critical plane approaches, the MWCM was found to be the most accurate one, giving predictions falling within an error interval of about $\pm 20\%$;

A number of assumptions were made in obtaining these predictions, notably the constancy of the critical distance L and the response of the material to different R ratios. These assumptions, though clearly questionable in the general case, can be justified by the high accuracy of the predictions obtained.

More work has to be done in this area to better understand the behaviour of three-dimensional stress concentrators subjected to in-phase and out-of-phase triaxial stress states.

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