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ARTICLE

A Hyperfinite Grünwald–Letnikov Representation of the Riemann–Liouville Fractional Derivative: Nonstandard Analysis and Numerical Illustration

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ABSTRACT: This paper develops a hyperfinite Grünwald–Letnikov representation of the Riemann–Liouville fractional derivative within Nelson's Internal Set Theory. For absolutely continuous functions on the unit interval and fractional orders between zero and one, we define a hyperfinite fractional difference operator on an infinitesimal timeline and prove that, for almost every standard point in the interval, its standard part coincides with the corresponding Riemann–Liouville fractional derivative. The proof is based on an exact partial-sum identity for the fractional binomial coefficients, expressed in terms of Gamma functions, which yields the fractional kernel and a uniform singular bound directly from Gamma-function asymptotics, without assuming the classical Grünwald–Letnikov convergence theorem. The exceptional set on which the associated singular convolution may fail to be finite is characterized explicitly and shown to have Lebesgue measure zero. Under additional smoothness assumptions, a first-order consistency estimate is stated, and the construction is extended naturally to the Caputo derivative. Numerical experiments involving smooth monomials, a nonsmooth piecewise-linear function, and a comparison with the L1 scheme support the theoretical results. The contribution lies in the explicit internal realization of the Riemann–Liouville and Caputo derivatives as standard parts of hyperfinite fractional-difference operators, with the singular kernel emerging from an exact fractional-binomial partial-sum identity followed by standardization.

KEYWORDS: Fractional calculus; Riemann-Liouville derivative; nonstandard analysis; Internal Set Theory; hyperfinite calculus; mathematical foundations

MSC: 26A33 (Primary); 03H05; 26E35 (Secondary)

1 Introduction

Fractional calculus extends classical integration and differentiation to non-integer orders, providing a natural framework for memory, hereditary effects, anomalous transport, and nonlocal dynamics. Among the various fractional operators, the Riemann-Liouville derivative remains fundamental. For $0 < \alpha < 1$, it is defined by

$$D^{\alpha}_{\mathrm{RL}} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau)\, d\tau. \tag{1}$$

Its nonlocal character is encoded by the weakly singular kernel $(t-\tau)^{-\alpha}$. Although definition (1) is analytically well established, its connection with an explicitly discrete fractional-difference construction is

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less immediate than in the integer-order case. This has motivated the development of Grünwald-Letnikov-type formulas based on the fractional binomial weights $w_k(\alpha) = (-1)^k\binom{\alpha}{k}$, $k \geq 0$, taking the form

$$\Delta_h^\alpha f(t) = \frac{1}{h^\alpha}\sum_{k=0}^{\lfloor t/h\rfloor} w_k(\alpha) f(t-kh). \tag{2}$$

The relation between Grünwald-Letnikov differences and Riemann-Liouville derivatives has been studied extensively in the classical literature [1–5]. In numerical analysis, these formulas provide the basis for several discretization techniques for fractional differential equations.

The present work studies Formula (2) from the perspective of nonstandard analysis [6–8], which permits rigorous reasoning with infinitesimals, finite hyperreal numbers, and unlimited hyperintegers. Within this framework, classical limiting procedures can be represented by a single internal expression evaluated at an infinitesimal scale and followed by the standard-part operation. Let $H \in {}^*\mathbb{N}\backslash\mathbb{N}$ be an unlimited hyperinteger and set $\Delta t = 1/H$. For an internal function F, we define the hyperfinite fractional difference operator

$$\Delta_H^\alpha F(t) = \frac{1}{(\Delta t)^\alpha}\sum_{k=0}^{\lfloor t/\Delta t\rfloor} (-1)^k\binom{\alpha}{k} F(t-k\Delta t). \tag{3}$$

This is the direct hyperfinite counterpart of the Grünwald-Letnikov formula. The main purpose of the paper is to determine precisely when the standard part of (3) exists and to prove that it coincides with the classical Riemann-Liouville derivative.

Nonstandard methods have previously appeared in the study of fractional models [9–11]. Recent developments have also broadened the landscape of fractional calculus [12–15]. Fractional operators play a central role in anomalous diffusion and stochastic dynamics [16–18]. Such applications provide additional motivation for rigorous formulations of fractional operators and their discrete representations.

The principal mathematical ingredient of our analysis is the exact partial-sum identity

$$\sum_{k=0}^{m}(-1)^k\binom{\alpha}{k} = \frac{\Gamma(m+1-\alpha)}{\Gamma(1-\alpha)\Gamma(m+1)}. \tag{4}$$

This identity makes it possible to derive the hyperfinite kernel directly from Gamma-function estimates, without using the classical Grünwald-Letnikov convergence theorem as an assumption in the proof of the main equivalence result [19].

Theorem 1 (Main equivalence theorem)*: Let* $0 < \alpha < 1$ *and* $f \in AC([0,1])$*. For every unlimited hyperinteger* $H \in {}^*\mathbb{N}\backslash\mathbb{N}$ *and for almost every standard* $t \in (0,1]$*, the hyperreal number* $\Delta_H^\alpha({}^*f)(t)$ *is finite and*

$$\mathrm{st}(\Delta_H^\alpha({}^*f)(t)) = D_{\mathrm{RL}}^\alpha f(t). \tag{5}$$

The contributions of the paper are summarized as follows. We introduce an internal hyperfinite fractional-difference operator associated with an infinitesimal mesh and derive its exact kernel representation from the partial-sum identity (4). We establish pointwise kernel approximation and a uniform singular bound of the form $0 < K_H(u) \leq C_\alpha u^{-\alpha}$, $u > 0$, which provide the key estimates for the equivalence proof. For $f \in AC([0,1])$, with $g = f'$, we explicitly characterize the exceptional set

$$E_g = \left\{ t \in (0,1] : \int_0^t |g(s)|(t-s)^{-\alpha}\, ds = \infty \right\}, \tag{6}$$

and prove that $|E_g| = 0$. If $f' \in L^\infty(0,1)$, then $E_g = \varnothing$, and the equivalence holds for every $t \in (0,1]$. We prove that the standard part of the hyperfinite operator is independent of the particular unlimited hyperinteger H. We extend the construction to the Caputo derivative by defining $\Delta^\alpha_{H,C}F(t) = \Delta^\alpha_H(F - F(0))(t)$, and show that its standard part equals $D^\alpha_C f(t)$. Under additional smoothness assumptions, we state a first-order consistency estimate and verify the predicted behavior numerically. We provide numerical experiments for several monomial functions, multiple fractional orders, interior evaluation points, and a nonsmooth piecewise-linear test, and compare the Caputo realization with the L1 scheme.

The novelty of the paper does not lie in proposing a new finite-mesh numerical method. On finite grids, the hyperfinite operator reduces to the classical Grünwald-Letnikov formula. The contribution is instead the explicit internal representation of the Riemann-Liouville derivative and the derivation of its singular kernel from the exact fractional-binomial partial sum, followed by standardization.

The remainder of the paper is organized as follows. Section 2 reviews the required background from Riemann-Liouville fractional calculus, nonstandard analysis, and fractional binomial identities. Section 3 introduces the hyperfinite fractional difference operator and establishes its basic internal properties. Section 4 develops the kernel estimates, characterizes the exceptional set, proves the main equivalence theorem, establishes independence from H, and presents the Caputo extension. Section 5 provides numerical illustrations for smooth and nonsmooth test functions and compares the hyperfinite Caputo realization with the L1 method. Section 6 contains the concluding remarks.

2 Preliminaries and Hyperfinite Construction

We begin by recalling the classical fractional operators and then introduce their hyperfinite counterparts. Throughout, $0 < \alpha < 1$ is fixed.

2.1 Classical Fractional Operators

For $f \in AC([0,1])$, the Riemann-Liouville fractional derivative is

$$D^\alpha_{\mathrm{RL}} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau)\, d\tau, \quad t > 0. \tag{7}$$

The Caputo derivative is

$$D^\alpha_C f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\, ds. \tag{8}$$

We shall use the standard decomposition (see e.g., [4])

$$D^\alpha_{\mathrm{RL}} f(t) = \frac{f(0)}{\Gamma(1-\alpha)} t^{-\alpha} + \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\, ds, \tag{9}$$

valid for $f \in AC([0,1])$.

2.2 Hyperfinite Setting

We work in a sufficiently saturated nonstandard extension of the standard superstructure, with the usual Transfer Principle and standard-part map [6,19,20]. Fix an unlimited hyperinteger $H \in {}^*\mathbb{N}\backslash\mathbb{N}$ and set $\Delta t = 1/H$, so Δt is a positive infinitesimal. Let $T_H = \{0, \Delta t, \ldots, H\Delta t = 1\}$ be the hyperfinite partition of $[0,1]$. For finite $x \in {}^*\mathbb{R}$, $\mathrm{st}(x) \in \mathbb{R}$ denotes the standard part, and we write $x \approx y$ when $x - y$ is infinitesimal. We use the standard fact that the nonstandard extension of every L^1-function is S-integrable (Theorem 4.4.1 in [20]).

2.3 Fractional Binomial Identity

Define $w_k(\alpha) = (-1)^k\binom{\alpha}{k}$, where $\binom{\alpha}{0} = 1$ and $\binom{\alpha}{k} = \alpha(\alpha-1)\cdots(\alpha-k+1)/k!$ for $k \geq 1$.

Lemma 1: *For every integer $m \geq 0$,*

$$\sum_{k=0}^{m} w_k(\alpha) = \frac{\Gamma(m+1-\alpha)}{\Gamma(1-\alpha)\Gamma(m+1)}. \tag{10}$$

Proof: The binomial identity $\binom{\alpha}{k} = \binom{\alpha-1}{k} + \binom{\alpha-1}{k-1}$ gives $w_k(\alpha) = (-1)^k\binom{\alpha-1}{k} - (-1)^{k-1}\binom{\alpha-1}{k-1}$; summing telescopes to $(-1)^m\binom{\alpha-1}{m}$, whose Gamma form is immediate. □

2.4 Hyperfinite Fractional Difference

For an internal function $F : {}^*\mathbb{R} \to {}^*\mathbb{R}$, define

$$\Delta_H^\alpha F(t) = \frac{1}{(\Delta t)^\alpha} \sum_{k=0}^{\lfloor t/\Delta t \rfloor} w_k(\alpha) F(t - k\Delta t), \quad t \in [0,1]. \tag{11}$$

This is the hyperfinite Grünwald-Letnikov operator. For $F = {}^*f$ with $f \in AC([0,1])$, we have the kernel representation

$$\Delta_H^\alpha({}^*f)(t) = {}^*f(0)K_H(t) + \int_0^t {}^*f'(s)K_H(t-s)\,ds, \tag{12}$$

where

$$K_H(u) = \frac{1}{(\Delta t)^\alpha} \sum_{k=0}^{\lfloor u/\Delta t \rfloor} w_k(\alpha). \tag{13}$$

This identity follows by transferring (9) and interchanging summation and integration.

The Caputo version is obtained by subtracting the initial value:

$$\Delta_{H,C}^\alpha F(t) = \Delta_H^\alpha(F - F(0))(t). \tag{14}$$

The main result of this paper is that for almost every standard $t \in (0,1]$,

$$\mathrm{st}(\Delta_H^\alpha({}^*f)(t)) = D_{\mathrm{RL}}^\alpha f(t), \tag{15}$$

and similarly $\mathrm{st}(\Delta_{H,C}^\alpha({}^*f)(t)) = D_C^\alpha f(t)$.

The proof requires estimates on the kernel K_H. These are developed in Section 4.

3 Hyperfinite Representation and Main Equivalence

Fix an unlimited hyperinteger $H \in {}^*\mathbb{N}\backslash\mathbb{N}$ and set $\Delta t = 1/H$. Let $T_H = \{0, \Delta t, \ldots, H\Delta t = 1\}$.

3.1 Hyperfinite Fractional Difference

For $\alpha \in (0,1)$ and an internal function $F : {}^*\mathbb{R} \to {}^*\mathbb{R}$, define

$$\Delta_H^\alpha F(t) = \frac{1}{(\Delta t)^\alpha} \sum_{k=0}^{M(t)} w_k(\alpha) F(t - k\Delta t), \qquad M(t) = \left\lfloor \frac{t}{\Delta t} \right\rfloor, \tag{16}$$

where $w_k(\alpha) = (-1)^k\binom{\alpha}{k}$. Since the sum is hyperfinite and all quantities are internal, $\Delta_H^\alpha F$ is internal.

3.2 Kernel Estimates

For $h > 0$, $u > 0$, define

$$K_h(u) = \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor u/h \rfloor} w_k(\alpha). \tag{17}$$

The hyperfinite kernel is $K_H(u) := K_{\Delta t}(u)$.

Lemma 2 (Exact kernel representation): *For $h > 0$ and $m = \lfloor u/h \rfloor$,*

$$K_h(u) = \frac{1}{h^\alpha} \frac{\Gamma(m+1-\alpha)}{\Gamma(1-\alpha)\Gamma(m+1)}. \tag{18}$$

Consequently, for every standard $u \in (0,1]$,

$$K_H(u) \approx \frac{u^{-\alpha}}{\Gamma(1-\alpha)}. \tag{19}$$

Moreover, there exists $C_\alpha > 0$ such that for all $h > 0$, $u > 0$,

$$0 < K_h(u) \le C_\alpha u^{-\alpha}. \tag{20}$$

Proof: Eq. (18) follows from (10). For (19), take $m = \lfloor u/\Delta t \rfloor$; as $H \to \infty$, $m \to \infty$ and $m\Delta t \to u$, and Stirling gives $\Gamma(m+1-\alpha)/\Gamma(m+1) \sim m^{-\alpha}$. For (20), if $m \ge 1$, Wendel's inequality yields $K_h(u) \le C_\alpha h^{-\alpha}(m+1)^{-\alpha}$. Since $u < (m+1)h$, we get $h^{-\alpha}(m+1)^{-\alpha} < u^{-\alpha}$. If $m = 0$, then $u < h$ and $K_h(u) = h^{-\alpha} < u^{-\alpha}$. □

Corollary 1: *For every standard $\delta \in (0,1)$,*

$$\sup_{\delta \le u \le 1} \left| K_H(u) - \frac{u^{-\alpha}}{\Gamma(1-\alpha)} \right| \approx 0. \tag{21}$$

3.3 Hyperfinite Convolution Identity

Let $f \in AC([0,1])$ and $g = f' \in L^1(0,1)$. From the decomposition (9) and the definition of Δ_H^α,

$$\Delta_H^\alpha({}^*f)(t) = {}^*f(0)K_H(t) + \int_0^t {}^*g(s)K_H(t-s)\,ds. \tag{22}$$

Indeed, substituting $f(t-k\Delta t) = f(0) + \int_0^{t-k\Delta t} g(s)\,ds$ into (16) and interchanging the hyperfinite sum with the transferred integral gives the kernel representation. The interchange is justified since K_H is internal and bounded on compact intervals away from zero; the singular behavior is controlled by (20).

3.4 Exceptional Set and Finiteness

Define

$$E_g = \left\{ t \in (0,1] : \int_0^t |g(s)|(t-s)^{-\alpha}\,ds = \infty \right\}. \tag{23}$$

Lemma 3: *$|E_g| = 0$, and for every standard $t \in (0,1]\backslash E_g$, $\Delta_H^\alpha({}^*f)(t)$ is finite.*

Proof: By Tonelli, $\int_0^1 \int_0^t |g(s)|(t-s)^{-\alpha}\,ds\,dt = \frac{1}{1-\alpha}\int_0^1 |g(s)|(1-s)^{1-\alpha}\,ds < \infty$; hence $|E_g| = 0$. For $t \notin E_g$, (19) makes the first term in (22) finite, and (20) bounds the integral by $C_\alpha \int_0^t |g(s)|(t-s)^{-\alpha}\,ds < \infty$. □

3.5 Main Equivalence Theorem

Theorem 2: *For $f \in AC([0,1])$ and $g = f'$, for every standard $t \in (0,1]\backslash E_g$ and every unlimited H,*

$$\mathrm{st}(\Delta_H^{\alpha}({}^*f)(t)) = D_{\mathrm{RL}}^{\alpha} f(t). \tag{24}$$

Since $|E_g| = 0$, (24) holds for almost every $t \in (0,1]$.

Proof: Write $\Delta_H^{\alpha}({}^*f)(t) = T_1 + T_2$ as in (22), where $T_1 = {}^*f(0)K_H(t)$ and $T_2 = \int_0^t {}^*g(s)K_H(t-s)\,ds$. By (19), $\mathrm{st}(T_1) = f(0)t^{-\alpha}/\Gamma(1-\alpha)$.

For T_2, fix standard $\delta \in (0,t)$. On $[0, t-\delta]$, (21) gives uniform kernel approximation. Since ${}^*g$ is S-integrable and K_H is bounded there by a standard constant, the S-integrability of ${}^*g$ permits passage of the standard part inside the integral:

$$\mathrm{st}\left(\int_0^{t-\delta} {}^*g(s)K_H(t-s)\,ds\right) = \frac{1}{\Gamma(1-\alpha)}\int_0^{t-\delta} g(s)(t-s)^{-\alpha}\,ds. \tag{25}$$

The tail integral is bounded by $C_\alpha \int_{t-\delta}^{t} |g(s)|(t-s)^{-\alpha}\,ds$, which tends to zero as $\delta \downarrow 0$ since $t \notin E_g$. Combining with (25) and using (9) gives (24). □

Remark 1: *The standard part in (24) is independent of the particular unlimited hyperinteger H: if H_1, H_2 are unlimited, then both have standard part $D_{\mathrm{RL}}^{\alpha} f(t)$, so their difference is infinitesimal.*

Corollary 2: *If $f' \in L^{\infty}(0,1)$, then $E_g = \varnothing$, and (24) holds for every $t \in (0,1]$.*

Corollary 3 (Caputo extension): *With $\Delta_{H,C}^{\alpha} F(t) = \Delta_H^{\alpha}(F - F(0))(t)$, for $f \in AC([0,1])$ and $g = f'$,*

$$\mathrm{st}(\Delta_{H,C}^{\alpha}({}^*f)(t)) = D_C^{\alpha} f(t) \tag{26}$$

for every standard $t \in (0,1]\backslash E_g$.

Proof: Set $q = f - f(0)$. Then $q(0) = 0$, $q' = g$, and $D_C^{\alpha} f = D_{\mathrm{RL}}^{\alpha} q$. Applying Theorem 2 to q gives (26). □

4 Main Equivalence Theorem and Caputo Extension

This section establishes the main equivalence between the hyperfinite fractional difference operator and the classical Riemann-Liouville derivative. Throughout, $0 < \alpha < 1$, $H \in {}^*\mathbb{N}\backslash\mathbb{N}$, and $\Delta t = 1/H$.

4.1 Kernel Representation

For $h > 0$ and $u \geq 0$, define

$$K_h(u) = \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor u/h \rfloor} w_k(\alpha), \qquad w_k(\alpha) = (-1)^k \binom{\alpha}{k}. \tag{27}$$

In particular, $K_h(0) = h^{-\alpha}$. The hyperfinite counterpart is $K_H(u) := K_{\Delta t}(u)$.

Lemma 4 (Exact kernel representation): *For $h > 0$, $u > 0$, and $m = \lfloor u/h \rfloor$,*

$$K_h(u) = \frac{1}{h^{\alpha}} \frac{\Gamma(m+1-\alpha)}{\Gamma(1-\alpha)\Gamma(m+1)}. \tag{28}$$

Proof: Immediate from Lemma 1. □

Lemma 5 (Pointwise kernel approximation): *For every standard* $u \in (0,1]$,

$$K_H(u) \approx \frac{u^{-\alpha}}{\Gamma(1-\alpha)}. \tag{29}$$

Proof: For standard $h \downarrow 0$, let $m = \lfloor u/h \rfloor$. Then $m \to \infty$, $mh \to u$, and Stirling gives $\Gamma(m+1-\alpha)/\Gamma(m+1) \sim m^{-\alpha}$. Hence $K_h(u) \to u^{-\alpha}/\Gamma(1-\alpha)$. Transfer to $h = \Delta t$ gives (29). □

Lemma 6 (Uniform kernel bound): *There exists* $C_\alpha > 0$ *such that for all* $h > 0$ *and* $u > 0$,

$$0 < K_h(u) \le C_\alpha u^{-\alpha}. \tag{30}$$

Consequently, $0 < K_H(u) \le C_\alpha u^{-\alpha}$ *for every positive hyperreal* u.

Proof: If $m = \lfloor u/h \rfloor \ge 1$, Wendel's inequality gives $\Gamma(m+1-\alpha)/\Gamma(m+1) \le C_\alpha (m+1)^{-\alpha}$. Thus $K_h(u) \le C_\alpha h^{-\alpha}(m+1)^{-\alpha}$. Since $u < (m+1)h$, we have $h^{-\alpha}(m+1)^{-\alpha} < u^{-\alpha}$. If $m = 0$, then $u < h$ and $K_h(u) = h^{-\alpha} < u^{-\alpha}$. □

Corollary 4 (Uniform convergence away from zero): *For every standard* $\delta \in (0,1)$,

$$\sup_{\delta \le u \le 1} \left| K_H(u) - \frac{u^{-\alpha}}{\Gamma(1-\alpha)} \right| \approx 0. \tag{31}$$

Proof: Using $\Gamma(m+1-\alpha)/\Gamma(m+1) = m^{-\alpha}(1 + O(m^{-1}))$ and $m \ge \delta/h - 1$, we have $m^{-1} = O(h)$ uniformly for $u \in [\delta, 1]$. Hence the convergence is uniform. □

4.2 Convolution Identity and Exceptional Set

Let $f \in AC([0,1])$ and $g = f' \in L^1(0,1)$. By absolute continuity,

$$f(t) = f(0) + \int_0^t g(s)\, ds. \tag{32}$$

Substituting into (16) and interchanging the hyperfinite sum with the transferred integral yields the convolution identity

$$\Delta_H^\alpha({}^*f)(t) = {}^*f(0) K_H(t) + \int_0^t {}^*g(s) K_H(t-s)\, ds, \tag{33}$$

where the interchange uses

$$\sum_{k=0}^{\lfloor (t-s)/\Delta t \rfloor} w_k(\alpha) = (\Delta t)^\alpha K_H(t-s). \tag{34}$$

Define the exceptional set

$$E_g = \left\{ t \in (0,1] : \int_0^t |g(s)|(t-s)^{-\alpha}\, ds = \infty \right\}. \tag{35}$$

Lemma 7: $|E_g| = 0$, *and for every standard* $t \in (0,1] \backslash E_g$, $\Delta_H^\alpha({}^*f)(t)$ *is finite.*

Proof: By Tonelli,

$$\int_0^1 \int_0^t |g(s)|(t-s)^{-\alpha}\, ds\, dt = \frac{1}{1-\alpha} \int_0^1 |g(s)|(1-s)^{1-\alpha}\, ds < \infty, \tag{36}$$

so $|E_g| = 0$. If $t \notin E_g$, then $q_t(s) := |g(s)|(t-s)^{-\alpha} \in L^1(0,t)$. Thus ${}^*q_t$ is S-integrable; using (30), the integral in (33) is bounded by $C_\alpha \int_0^t {}^*q_t(s)\,ds$, which is finite. □

4.3 Main Equivalence Theorem

Theorem 3 (Main equivalence): *For $f \in AC([0,1])$ and $g = f'$, for every standard $t \in (0,1]\backslash E_g$ and every unlimited H,*

$$\mathrm{st}(\Delta_H^\alpha({}^*f)(t)) = D^\alpha_{\mathrm{RL}} f(t). \tag{37}$$

Since $|E_g| = 0$, (37) holds for almost every $t \in (0,1]$.

Proof: Write $\Delta_H^\alpha({}^*f)(t) = T_1 + T_2$ as in (33), with $T_1 = {}^*f(0)K_H(t)$ and $T_2 = \int_0^t {}^*g(s)K_H(t-s)\,ds$. By (29),

$$\mathrm{st}(T_1) = \frac{f(0)t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{38}$$

For T_2, fix standard $\delta \in (0,t)$. On $[0, t-\delta]$, (31) gives uniform kernel approximation. Since ${}^*g$ is S-integrable,

$$\mathrm{st}\left(\int_0^{t-\delta} {}^*g(s)K_H(t-s)\,ds\right) = \frac{1}{\Gamma(1-\alpha)}\int_0^{t-\delta} g(s)(t-s)^{-\alpha}\,ds. \tag{39}$$

The tail satisfies

$$\left|\int_{t-\delta}^t {}^*g(s)K_H(t-s)\,ds\right| \le C_\alpha \int_{t-\delta}^t |{}^*g(s)|(t-s)^{-\alpha}\,ds. \tag{40}$$

Since $q_t(s) = |g(s)|(t-s)^{-\alpha} \in L^1(0,t)$, its nonstandard extension is S-integrable; hence the standard part of the right-hand side of (40) is bounded by

$$C_\alpha \int_{t-\delta}^t q_t(s)\,ds \to 0 \qquad (\delta \downarrow 0). \tag{41}$$

Thus

$$\mathrm{st}(T_2) = \frac{1}{\Gamma(1-\alpha)}\int_0^t g(s)(t-s)^{-\alpha}\,ds. \tag{42}$$

Combining (38) and (42) with (8) yields (37). □

Remark 2: *The standard part in (37) is independent of H: if H_1, H_2 are unlimited, then both have standard part $D^\alpha_{\mathrm{RL}} f(t)$, so $\Delta^\alpha_{H_1}({}^*f)(t) \approx \Delta^\alpha_{H_2}({}^*f)(t)$.*

Corollary 5 (Bounded derivative): *If $f' \in L^\infty(0,1)$, then $E_g = \varnothing$, and (37) holds for every $t \in (0,1]$.*

Corollary 6 (Caputo extension): *With $\Delta^\alpha_{H,C} F(t) = \Delta^\alpha_H(F - F(0))(t)$, for $f \in AC([0,1])$,*

$$\mathrm{st}(\Delta^\alpha_{H,C}({}^*f)(t)) = D^\alpha_C f(t) \tag{43}$$

for every standard $t \in (0,1]\backslash E_g$.

Proof: Set $q = f - f(0)$. Then $q(0) = 0$, $q' = g$, and $D^\alpha_C f = D^\alpha_{\mathrm{RL}} q$. Applying Theorem 3 to q gives (43). □

5 Numerical Illustration

This section illustrates the finite-mesh behavior associated with Theorem 3. Since the hyperfinite operator reduces to the classical Grünwald–Letnikov formula when the infinitesimal step is replaced by a finite mesh size h, the experiments are intended as numerical illustrations of the theoretical representation rather than as a new numerical method. All computations were performed using standard double-precision arithmetic.

5.1 Test Problems and Discretization

For $\beta > 0$, consider $f_\beta(t) = t^\beta$, $t \in (0,1]$. The exact Riemann-Liouville fractional derivative is [5]

$$D^\alpha_{\mathrm{RL}} f_\beta(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, \qquad \beta > \alpha - 1. \tag{44}$$

We choose $\beta \in \{1,2,3\}$ and $\alpha \in \{0.25, 0.50, 0.75\}$.

To examine behavior when f' is discontinuous, we consider $f(t) = |t - \frac{1}{2}|$, $t \in (0,1]$. This function belongs to $AC([0,1])$, with $f'(t) = -1$ for $t < \frac{1}{2}$ and $f'(t) = 1$ for $t > \frac{1}{2}$. Using (8), we obtain

$$D^\alpha_{\mathrm{RL}} \left|t - \tfrac{1}{2}\right| = \frac{1}{2\Gamma(1-\alpha)} t^{-\alpha} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}, \qquad 0 < t < \tfrac{1}{2}, \tag{45}$$

$$D^\alpha_{\mathrm{RL}} \left|t - \tfrac{1}{2}\right| = \frac{1}{2\Gamma(1-\alpha)} t^{-\alpha} + \frac{-t^{1-\alpha} + 2(t-\frac{1}{2})^{1-\alpha}}{\Gamma(2-\alpha)}, \qquad t > \tfrac{1}{2}. \tag{46}$$

The approximation is evaluated at dyadic points $t \in \{0.375, 0.625, 0.875\}$, which avoid $t = \frac{1}{2}$.

For the Caputo extension, we use $f(t) = 1 + t^2$. Since $f(0) = 1$, this example displays the difference between Riemann-Liouville and Caputo derivatives. The exact Caputo derivative is

$$D^\alpha_C(1+t^2) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}, \tag{47}$$

whereas $D^\alpha_{\mathrm{RL}}(1+t^2) = t^{-\alpha}/\Gamma(1-\alpha) + 2t^{2-\alpha}/\Gamma(3-\alpha)$.

Let $t_n = nh$, $n = 0,1,\ldots,N$, $N = \lfloor 1/h \rfloor$. The Grünwald-Letnikov approximation is

$$\Delta^\alpha_h f(t_n) = \frac{1}{h^\alpha} \sum_{k=0}^{n} w_k(\alpha) f(t_{n-k}), \qquad w_k(\alpha) = (-1)^k \binom{\alpha}{k}. \tag{48}$$

Coefficients are generated recursively: $w_0(\alpha) = 1$, $w_k(\alpha) = (1 - (\alpha+1)/k) w_{k-1}(\alpha)$, $k \geq 1$. This is the finite-step realization of (16) with Δt replaced by h.

For the Caputo case, we use the finite-mesh GL-Caputo approximation

$$\Delta^\alpha_{h,C} f(t_n) := \Delta^\alpha_h (f - f(0))(t_n). \tag{49}$$

We compare this with the classical L1 approximation [21]:

$$L^\alpha_{1,h} f(t_n) = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k (f(t_{n-k}) - f(t_{n-k-1})), \tag{50}$$

where $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}$. For sufficiently smooth functions, the L1 approximation has consistency order $O(h^{2-\alpha})$ [22].

For Riemann-Liouville tests, the absolute error is $E_h(t) = |\Delta_h^\alpha f(t) - D_{\mathrm{RL}}^\alpha f(t)|$. For Caputo tests,

$$E_h^{\mathrm{GLC}}(t) = |\Delta_{h,C}^\alpha f(t) - D_C^\alpha f(t)|, \qquad E_h^{L1}(t) = |L_{1,h}^\alpha f(t) - D_C^\alpha f(t)|. \tag{51}$$

The experimental order of convergence is $\mathrm{EOC} = \log(E_h/E_{h/2})/\log 2$. We use mesh sizes $h = 2^{-5}, 2^{-6}, \ldots, 2^{-10}$.

5.2 Smooth Tests

Table 1 reports errors at $t = 1$ for $\beta \in \{1, 2, 3\}$ and $\alpha \in \{0.25, 0.50, 0.75\}$. The EOC is computed from errors at $h = 2^{-7}$ and $h = 2^{-8}$. The errors approximately halve under mesh refinement, and the EOC values are close to one for all tested (α, β), confirming the expected first-order behavior of the unshifted Grünwald approximation.

Table 1: Errors and EOC for $f_\beta(t) = t^\beta$ at $t = 1$.

β	α	$h = 2^{-5}$	$h = 2^{-6}$	$h = 2^{-7}$	$h = 2^{-8}$	EOC
1	0.25	3.1747e−3	1.5909e−3	7.9587e−4	3.9814e−4	0.9993
1	0.50	4.3985e−3	2.2019e−3	1.1012e−3	5.5074e−4	0.9996
1	0.75	3.2356e−3	1.6174e−3	8.0843e−4	4.0416e−4	1.0002
2	0.25	8.4709e−3	4.2431e−3	2.1235e−3	1.0622e−3	0.9994
2	0.50	1.7573e−2	8.8011e−3	4.4041e−3	2.2030e−3	0.9994
2	0.75	2.5803e−2	1.2915e−2	6.4614e−3	3.2312e−3	0.9996
3	0.25	1.4456e−2	7.2568e−3	3.6360e−3	1.8200e−3	0.9986
3	0.50	3.4918e−2	1.7545e−2	8.7942e−3	4.4025e−3	0.9982
3	0.75	6.1403e−2	3.0865e−2	1.5474e−2	7.7474e−3	0.9981

Table 2 presents average EOC values over the three finest refinement steps for $\alpha = 0.5$ at several evaluation points. The same first-order behavior is observed at the interior points.

Table 2: Average EOC for $f_\beta(t) = t^\beta$, $\alpha = 0.5$.

β	$t = 0.25$	$t = 0.50$	$t = 0.75$	$t = 1.00$
1	0.9992	0.9996	0.9997	0.9998
2	0.9986	0.9993	0.9995	0.9997
3	0.9959	0.9979	0.9986	0.9990

Fig. 1 shows the log-log error plot for $f(t) = t^2$, $\alpha = 0.5$, $t = 1$. The data lie approximately on a straight line with slope one, confirming first-order convergence.

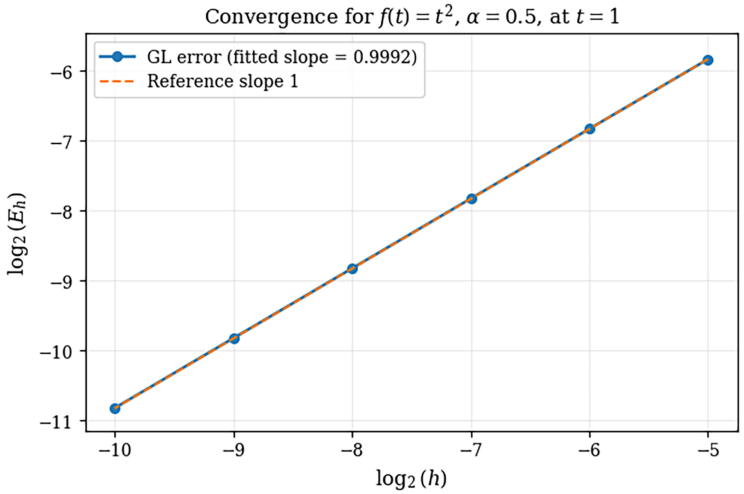


Figure 1: Log-log error plot for $f(t) = t^2$, $\alpha = 0.5$, at $t = 1$. Dashed reference line has slope one.

5.3 Nonsmooth Test

Table 3 reports errors for $f(t) = |t - \frac{1}{2}|$, $\alpha = 0.5$, at the three selected dyadic points. Exact values are computed using (45) and (46). EOC is calculated from errors at $h = 2^{-7}$ and $h = 2^{-8}$.

Table 3: Errors and EOC for $f(t) = |t - \frac{1}{2}|$, $\alpha = 0.5$.

t	$h = 2^{-5}$	$h = 2^{-6}$	$h = 2^{-7}$	$h = 2^{-8}$	EOC
0.375	2.3861e−3	1.1964e−3	5.9903e−4	2.9971e−4	0.9990
0.625	2.1152e−2	1.0692e−2	5.3725e−3	2.6925e−3	0.9966
0.875	1.0959e−2	5.4976e−3	2.7532e−3	1.3777e−3	0.9988

The largest errors occur at $t = 0.625$, where the history interval crosses the point of nondifferentiability, while the EOC remains close to one at all tested points.

5.4 Caputo Comparison with the L1 Scheme

For $f(t) = 1 + t^2$, the finite-mesh GL-Caputo realization (49) is compared with the L1 scheme (50). Table 4 gives errors and EOC values at $t = 1$, $\alpha = 0.5$.

Table 4: Comparison of the GL-Caputo and L1 approximations for $f(t) = 1 + t^2$, $\alpha = 0.5$, at $t = 1$.

h	GL-Caputo Error	GL EOC	L1 Error	L1 EOC
2^{-5}	1.7573e−2	–	2.4999e−3	–
2^{-6}	8.8011e−3	0.9976	8.9335e−4	1.4846
2^{-7}	4.4041e−3	0.9988	3.1822e−4	1.4892
2^{-8}	2.2030e−3	0.9994	1.1310e−4	1.4924
2^{-9}	1.1018e−3	0.9997	4.0108e−5	1.4946
2^{-10}	5.5092e−4	0.9999	1.4203e−5	1.4962

Thus the GL-Caputo and L1 approximations exhibit the expected first- and $(2-\alpha)$-order behavior, respectively.

Overall, the experiments are consistent with Theorem 3. The GL approximation displays approximately first-order behavior for the smooth monomials, both at the endpoint and at interior points, and convergence persists for the piecewise-linear test at points distinct from the kink. For the Caputo test, the GL-Caputo and L1 approximations exhibit the expected first- and $(2-\alpha)$-order behavior, respectively. These computations illustrate the hyperfinite representation rather than introduce a new high-order numerical method.

6 Conclusion

We developed a rigorous hyperfinite Grünwald–Letnikov representation of the Riemann–Liouville fractional derivative within a saturated nonstandard extension of the standard superstructure. For $f \in AC([0,1])$ and $0 < \alpha < 1$, we proved that the standard part of the hyperfinite fractional difference coincides with $D^{\alpha}_{\mathrm{RL}} f(t)$ for almost every standard $t \in (0,1]$. The exceptional set was characterized explicitly and shown to have measure zero; when $f' \in L^{\infty}(0,1)$, the equivalence holds at every $t \in (0,1]$. The resulting standard part is also independent of the particular unlimited hyperinteger used in the construction.

The proof is based on an exact partial-sum identity for the fractional binomial coefficients, from which the singular kernel and its required estimates are derived directly. Thus, the main equivalence is obtained without assuming the classical Grünwald–Letnikov convergence theorem. The construction also yields a natural Caputo counterpart through subtraction of the initial value, whose standard part agrees with the classical Caputo derivative. Numerical experiments with smooth monomials, a piecewise-linear test function, and the Caputo case support the theoretical results; the finite-mesh Grünwald approximations exhibit approximately first-order behavior, while the L1 comparison displays its expected higher order for the smooth Caputo test.

The results provide a direct connection between hyperfinite fractional differences and classical fractional derivatives through the standard-part map. Possible extensions include other fractional operators, higher and variable orders, and fractional evolution equations, where the present kernel-based approach may provide a useful nonstandard framework.

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