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ARTICLE

On Factorization of N-Qubit Pure States and Complete Entanglement Analysis of 3-Qubit Pure States Containing Exactly Two Terms and Three Terms

Dhananjay P. Mehendale[1,*] and Madhav R. Modak[2]

[1]Department of Physics, Sir Parashurambhau College, Pune, 411030, India

[2]Department of Mathematics, Sir Parashurambhau College, Pune, 411030, India

*Corresponding Author: Dhananjay P. Mehendale. Email: dmehendale032@gmail.com

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ABSTRACT

A multi-qubit pure quantum state is called separable when it can be factored as the tensor product of 1-qubit pure quantum states. Factorizing a general multi-qubit pure quantum state into the tensor product of its factors (pure states containing a smaller number of qubits) can be a challenging task, especially for highly entangled states. A new criterion based on the proportionality of the rows of certain associated matrices for the existence of certain factorization and a factorization algorithm that follows from this criterion for systematically extracting all the factors is developed in this paper. 3-qubit pure states play a crucial role in quantum computing and quantum information processing. For various applications, the well-known 3-qubit GHZ state which contains two nonzero terms, and the 3-qubit W state which contains three nonzero terms, have been studied extensively. Using the new factorization algorithm developed here we perform a complete analysis vis-à-vis entanglement of 3-qubit states that contain exactly two nonzero terms and exactly three nonzero terms.

KEYWORDS

Associated matrix; proportionality of rows; factorization criterion; factorization algorithm

1 Introduction

Entanglement and separability are important concepts in the study of quantum systems. Entanglement is a fundamental concept that describes the correlation among the particles that are in a combined state, such that their individual states cannot be described independently. For a multi-qubit pure state, the entanglement status is determined by the degree to which the state can be factorized into individual qubit states. By definition, a pure state of N-qubits is entangled if it cannot be expressed as the tensor product of individual 1-qubit states. Detecting entanglement in multi-qubit systems is a challenging task and is decided by various measures and criteria. In the field of quantum computing and information pure quantum states are expressed as vectors in some finite dimensional Hilbert space over the field of complex numbers. A pure quantum state exists as some linear combination of basis states. We use standard computational basis made up of basis states $\{|00\ldots 0>, |00\ldots 01>, |00\ldots 10>, \ldots, |11\ldots 1>\}$ to express pure quantum states in terms of the superposition of these basis states. We are aiming in this paper to check separability with respect to single qubits or maximal separability, i.e., we will call a pure state separable if and only if we can express it as the tensor product of single-qubit states. Entanglement and separability are fundamental to understanding the non-classical nature of quantum systems and have important implications for the way particles interact and possess their unusual properties. To determine whether a given multi-qubit pure quantum state is separable or entangled is one of the important problems in quantum information theory [1–8] and we have dealt with it through a new factorization criterion and a new factorization algorithm that follows.

To systematically solve this problem a factorization algorithm was developed in the earlier work on this subject [9].

The first factorization criterion was obtained in [9] in terms of the following theorem which led to the first factorization algorithm developed in [9].

Theorem: The state $|\psi>$ given in Eq. (1) below can be factored as the tensor product, $|\psi_1>\otimes|\psi_2>$ of an m-qubit state $|\psi_1>$ and an n-qubit state $|\psi_2>$ if and only if the rank of the $2^m \times 2^n$ matrix $A = [a_{i_u j_v}]$ associated with $|\psi>$ is equal to unity.

The above theorem was proved by making use of a result from linear algebra (Lemma 2 in [9]). It was further shown that apart from solving the problem of checking the entanglement status of multi-qubit states the factorization algorithms have other applications, for example, it was shown that they are useful to exponentially speed up the process of synthesis of a pure quantum state in the laboratory when that state has large many factors [10].

The Schmidt decomposition, also known as the Schmidt theorem, is a fundamental result in quantum mechanics that applies to bipartite (two-party) pure quantum states [11]. The Schmidt decomposition is used in quantum algorithms like the Quantum Singular Value Decomposition, which is a key component of some quantum machine learning algorithms. The Schmidt decomposition plays a crucial role in various aspects of quantum information theory, including quantum teleportation, dense coding, and quantum error correction.

The second factorization criterion obtained was the following theorem [12] which led to the second factorization algorithm [12].

Theorem: An N-qubit pure quantum state can be factored into the tensor product of an m-qubit quantum state and an n-qubit quantum state, $N = m + n$, if and only this N-qubit state when expressed as a bipartite state with 2^m dimensional first part and 2^n dimensional second part has Schmidt rank equal to unity.

In this paper, we develop one more factorization criterion and the factorization algorithm that follows from this new criterion. This new criterion uses the same $2^m \times 2^n$ matrix, $A = [a_{i_u j_v}]$, used in the first criterion [9]. This new criterion is based on checking the mutual proportionality of nonzero rows of the above matrix A for the existence of a factor. The new factorization algorithm based on this new criterion also finds all the factors that exist and only requires checking the mutual proportionality of the nonzero rows of the associated matrices to check their existence. The proof for this new criterion does not require the support of any outside results in its proof.

3-qubit states have a wide range of applications in quantum computing, quantum cryptography, quantum simulation, and other quantum technologies. Their ability to represent more complex entanglement configurations and quantum interactions makes them indispensable for building sophisticated quantum systems and harnessing the power of quantum mechanics for various computational and cryptographic tasks. We will apply the new factorization algorithm developed in this paper to study the entanglement status of all 3-qubit states containing two nonzero terms and three nonzero terms.

2 New Criterion for the Existence of Factorization

We now proceed to develop the new criterion for factorization of the N-qubit pure quantum state based on the proportionality of the nonzero rows of certain associated matrix in the next section.

Notation: Let $|\psi>$ be an N-qubit pure state:

$$\left|\psi >= \sum_{s=1}^{2^N} a_{r_s}\right| r_s > \tag{1}$$

Expressed in terms of the computational basis. Here the basis vectors $|r_s>$ are ordered lexicographically. That is, the corresponding binary sequences are ordered lexicographically: $r_1 = 00\ldots 00$, $r_2 = 00 \ldots 01, \ldots r_{2^N} = 11 \ldots 11$, so that $|r_1> = |00 \ldots 00>$, $|r_2> = |00 \ldots 01>, \ldots, |r_{2^N}> = |11 \ldots 11>$. Let m, n be any integers such that $1 \leq m, n < N$ and $m + n = N$. Let the corresponding two sets of computational basis vectors ordered lexicographically be $|i_1>, \ldots, |i_{2^m}>$ (each of length m) and $|j_1>, \ldots, |j_{2^n}>$ (each of length n). We rewrite $|\psi>$ thus:

$$\left|\psi >= \sum_{u=1}^{2^m}\sum_{v=1}^{2^n} a_{i_u j_v}\right| i_u > \otimes |j_v > \tag{2}$$

Here in the symbol $a_{i_u j_v}$, the suffix $i_u j_v$ is the juxtaposition of the binary sequences i_u and j_v in that order. Thus, we get a $2^m \times 2^n$ matrix $A = [a_{i_u j_v}]$ which will be called the $2^m \times 2^n$ associated matrix associated to $|\psi>$.

Definition 2.1: The nonzero row of A is that row that contains at least one nonzero element in it.

Definition 2.2: The zero row or the row of zeros of A is that row in which all the elements in that row have zero value.

Remark 2.1: Note that the nonzero rows of A correspond to those basic vectors, $| i_k >$, each of length 2^m, which appear in the first factor in the factorization of $|\psi>$ if and when it exists as per the criterion obtained below (Theorem 2.1) and the rows of zeros, on the other hand, correspond to the basic vectors, $|i_l >$, each of length 2^m, which do not appear in the first factor when factorization exists (as per Theorem 2.1). Thus, only nonzero rows of A contribute and are important in checking the existence of a certain factorization.

Definition 2.3: Let A be a $p \times q$ matrix over the field of complex numbers C. Two nonzero rows of A, say $[a_1 \ldots a_p]$ and $[b_1 \ldots b_p]$, are said to be proportional if their nonzero elements correspond i.e., $a_i \neq 0$ if and only if $b_i \neq 0$, $1 \leq i \leq p$, and these elements have the same constant ratio, i.e., there is a constant $k \neq 0$ such that $a_i/b_i = k$ whenever $a_i \neq 0$, $1 \leq i \leq p$.

Definition 2.4: The nonzero rows of a matrix A are said to be mutually proportional if any two nonzero rows of A are proportional.

We now give a simple criterion for factorization of a state into two factor states. The proof for this criterion is based on just checking the proportionality of the nonzero rows of certain associated matrix. We show that if the nonzero rows are proportional then the factor exists and can be extracted and the given quantum state can be expressed as the tensor product of two factors and when the proportionality of the nonzero rows is broken even because of some one among the nonzero rows then the factor does not exist and so cannot be extracted and the factorization of the given state into two factors fails.

Theorem 2.1 (New Factorization Criterion): The state $|\psi>$ given by (1) can be factored as the product, $|\psi_1>\otimes|\psi_2>$ of an m-qubit state $|\psi_1>$ and an n-qubit state $|\psi_2>$ if and only if the nonzero rows of the $2^m \times 2^n$ matrix $A = [a_{i_u j_v}]$ associated to $|\psi>$ are mutually proportional.

Proof: With the above notation, let $|r_w>$ be the first basic vector such that $a_{r_w} \neq 0$. Choose integers m, n such that $1 \leq m, n < N$, and $m + n = N$.

Let the corresponding two sets of computational basis vectors ordered lexicographically be $|i_1>$, $|i_2>, \ldots, |i_2m>$ (each of length 2^m) and $|j_1>, |j_2>, \ldots, |j_2n>$ (each of length 2^n). Then we can write:

$$|\psi>= \sum_{u=1}^{2^m} \left[|i_u> \otimes \sum_{v=1}^{2^n} a_{i_u j_v} |j_v> \right] \tag{3}$$

Consider the associated $2^m \times 2^n$ matrix $A = [a_{i_u j_v}]$. Suppose $|r_w>=|i_p>\otimes|j_q>$ so that the first nonzero element of A is the q-th element in the p-th row, namely $a_{i_p j_q}$. Thus, the p-throw of A is the first nonzero row. Since only nonzero rows contribute to the factorization as per the above-stated remark (Remark 2.1) we can assume hereafter without loss of generality that all rows of A are nonzero. Now, suppose that the rows of this associated matrix A are mutually proportional. It meant that if we choose some row then all the other rows will be proportional to this chosen row. Let us take that chosen row to be the p-th row. Then there will exist numbers $k_1, k_2, \ldots, k_2m$ such that $k_p = 1$ and (row u) $= k_u \times$ (row p) where $u = 1, 2, \ldots, 2^m$, i.e., $a_{i_u j_v} = k_u a_{i_p j_v}$ ($u = 1, 2, \ldots, 2^m$ and $v = 1, 2, \ldots, 2^n$). Hence Eq. (3) can be written as:

$$|\psi>= \sum_{u=1}^{2^m} k_u |i_u> \otimes \sum_{v=1}^{2^n} a_{i_p j_v} |j_v>= |\psi_1> \otimes |\psi_2> \tag{4}$$

Thus $|\psi>$ factors as stated.

Conversely, suppose $|\psi>$ factors as:

$$|\psi>= |\psi_1> \otimes |\psi_2> \tag{5}$$

where $|\psi_1>= \sum_{u=1}^{2^m} k_u |i_u>, |\psi_2>= \sum_{v=1}^{2^n} l_v |j_v>$.

Now by collecting the coefficients of $|i_u> \otimes |j_v>$ using Eq. (5), we construct the associated matrix A (as given above Eq. (2)) as follows:

$$A = \begin{bmatrix} k_1 l_1 & \cdots & k_1 l_{2^n} \\ \vdots & \ddots & \vdots \\ k_{2^m} l_1 & \cdots & k_{2^m} l_{2^n} \end{bmatrix} \tag{6}$$

It is straightforward to see that the rows of the above-constructed matrix A in Eq. (6) are mutually proportional as desired, hence the Theorem.

An alternative proof for the converse is as follows:

Conversely, suppose that all the nonzero rows of the associated matrix A are not mutually proportional. It meant that if we choose some nonzero row, say p-th row, then all the other nonzero rows are not proportional to this p-th row. Then, as a consequence, there will exist at least some one nonzero row, say s-th row for which there will not exist a constant, say k_s, such that $a_{i_s j_v} = k_s a_{i_p j_v}$ for this s-th row ($v = 1, 2, \ldots, 2^n$). Hence, we cannot have:

$$\left[|i_s> \otimes \sum_{v=1}^{2^n} a_{i_u j_v} |j_v> \right] = \left[k_s |i_s> \otimes \sum_{v=1}^{2^n} a_{i_p j_v} |j_v> \right] = k_s |i_s> \otimes |\psi_2> \tag{7}$$

Hence $|\psi_2>$ will not be a common factor for at least one of the basis states, $|i_s>$, which is one of the basis states, present among the basis states with nonzero coefficients, whose superposition forms the state $|\psi_1>$. Therefore, it is not possible to have the factorization of $|\psi>$ as $|\psi>=|\psi_1>\otimes|\psi_2>$, such that:

$$|\psi_1>=\sum_{u=1}^{2^m} k_u|i_u>, |\psi_2>=\sum_{v=1}^{2^n} a_{i_p j_v}|j_v>$$

Thus $|\psi>$ cannot be factored as stated. This proves the theorem.

3 The New Factorization Algorithm

The new factorization algorithm based on the above theorem for complete factorization of an arbitrary N-qubit pure quantum state will be exactly similar to the above-mentioned first factorization algorithm developed in [9]. The only difference in this new algorithm will be as follows. Instead of checking the rank of the associated matrices we just need to check the proportionality of the rows of the same associated matrices to find the factors and achieve the complete factorization as was achieved through previous algorithms. To avoid repetition, we skip the writing down of the steps of this new algorithm and just demonstrate these steps through a few examples given below. Before that, we mention below some implications of this new factorization algorithm:

1) Using this factorization algorithm one can split the multipartite state into tensor product of non-factorable factor states.
2) These factor states are completely independent from each other and one can perform the partial measurement on the qubits belonging to any of these factor states without affecting the other factor states.
3) These non-factorable factor states could be entangled states individually but these factor states are completely disentangled from one another.
4) The factorization algorithm can work as a useful tool to study the entanglement structure of a multi-qubit pure quantum state.
5) The factorization algorithm can work as a useful tool for quick detection of the entanglement status and its type for multi-qubit pure quantum states.

We now discuss two examples to illustrate the working of the new factorization criterion and the new factorization algorithm that follows from it to see how the factor is extracted when the rows of the associated matrix are proportional and how thus the entanglement status of a pure state is determined.

Examples:

(i) $|\psi>=\frac{1}{\sqrt{2}}[|01>-|10>]$

We now proceed to check whether $|\psi>$ has a linear factor (on the left). For this, we check the proportionality of nonzero rows of the associated matrix:

$$A=\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

The nonzero rows of A are not proportional, so that |ψ> has no linear factor and therefore, the state |ψ> is entangled.

(ii) $|\psi>= \frac{1}{\sqrt{3}}[|00> - |01] + \frac{1}{\sqrt{6}}[|10> - |11>]$

We now proceed to check whether |ψ> has a linear factor (on the left). For this, we check the proportionality of nonzero rows of the associated matrix:

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{bmatrix}$$

$Row2 = \frac{1}{\sqrt{2}} \times row1$, so the rows of A are indeed proportional with $k_1 = 1$ and $k_2 = \frac{1}{\sqrt{2}}$. Therefore, |ψ> gets factorized as a tensor product of two linear factors as follows:

$$|\psi>= \left(|0> + \frac{1}{\sqrt{2}}|1>\right) \otimes \left(\frac{1}{\sqrt{3}}\{|0> - |1>\}\right)$$

Hence the state |ψ> is separable.

A Remark: We normalize any given pure state $|\psi>$ by replacing it with state $\frac{|\psi>}{|||\psi>||}$ where $|||\psi>||$ stands for the positive square root of the inner product of $|\psi>$ with itself. Therefore, let $|\psi>$ be a normalized state. Suppose, after applying the above factorization algorithm we get:

$$|\psi>= |\phi_1> \otimes |\phi_2> \otimes \ldots \otimes |\phi_k>$$

Here, each of the above factors $|\phi_1>, |\phi_2>, \ldots, |\phi_k>$ may not be normalized states individually, but we can always normalize them individually by replacing them with states $\frac{|\phi_1>}{|||\phi_1>||}, \frac{|\phi_2>}{|||\phi_2>||}, \ldots,$ $\frac{|\phi_k>}{|||\phi_k>||}$, respectively.

4 Complete Entanglement Analysis of All 3-Qubit Pure Quantum States Containing (I) Exactly Two Nonzero Terms, and (II) Exactly Three Nonzero Terms

We now proceed with carrying out a complete analysis of entanglement of all the general 3-qubit states, (A) Containing exactly two nonzero terms, and (B) Containing exactly three nonzero terms using the new factorization criterion and the new algorithm that follows from it in the next section.

We have seen above that by applying the factorization algorithm we can determine the entanglement status of any given pure quantum state. Studying 3-qubit pure quantum states can be a fascinating exploration into the complexities of quantum entanglement and quantum information theory. 3-qubit entanglement is essential for implementing advanced quantum gates, which are the building blocks of quantum algorithms and circuits. In quantum computing, qubits are susceptible to errors due to noise and other environmental factors. 3-qubit states can be used as part of quantum error correction codes to protect the integrity of quantum information against errors. 3-qubit states are integral to certain quantum cryptographic protocols, such as quantum key distribution (QKD) schemes like the 3-qubit GHZ state. Some quantum algorithms, like the quantum error correction algorithms and certain quantum optimization algorithms, rely on the manipulation and analysis of 3-qubit states to achieve their computational advantages. Some key references related to the work done about 3-qubit states are references [13–16]. In this section, we illustrate the utility of the factorization algorithm developed in section III by considering all (A) 3-qubit states containing exactly two nonzero terms and (B) 3-qubit states containing exactly three nonzero terms and determine complete information about the entanglement status of all states belonging to these categories.

(A) The analysis of 3-qubit states containing exactly two nonzero terms is easy and is as follows:

There are in all $\binom{8}{2} = 28$ states containing exactly two nonzero terms. We write them in a sequence and also factorize them and write their entanglement status. Let $\{a, b\} \in C$, the field of complex numbers, are nonzero constants such that $\|a\|^2 + \|b\|^2 = 1$.

(1) $|\psi_1> = a|000> + b|001> = (|00>) \otimes (a|0> + b|1>)$, a separable state.

(2) $|\psi_2> = a|000> + b|010> = (|0>) \otimes (a|0> + b|1>) \otimes (|0>)$, a separable state.

(3) $|\psi_3> = a|000> + b|011> = (|0>) \otimes (a|00> + b|11>)$, an entangled state.

(4) $|\psi_4> = a|000> + b|100> = (a|0> + b|1>) \otimes |00>$, a separable state.

(5) $|\psi_5> = a|000> + b|101>$, a genuinely entangled state.

(6) $|\psi_6> = a|000> + b|110> = (a|00> + b|11>) \otimes (|0>)$, an entangled state.

(7) $|\psi_7> = a|000> + b|111>$, a genuinely entangled state and becomes the well-known GHZ (Greenberger-Horne-Zelinger) state when $a = b = \frac{1}{\sqrt{2}}$ [16].

(8) $|\psi_8> = a|001> + b|010> = (|0>) \otimes (a|01> + b|10>)$, an entangled state.

(9) $|\psi_9> = a|001> + b|011> = (|0>) \otimes (a|0> + b|1>) \otimes (|1>)$, a separable state.

(10) $|\psi_{10}> = a|001> + b|100>$, a genuinely entangled state.

(11) $|\psi_{11}> = a|001> + b|101> = (a|0> + b|1>) \otimes |01>$, a separable state.

(12) $|\psi_{12}> = a|001> + b|110>$, a genuinely entangled state.

(13) $|\psi_{13}> = a|001> + b|111> = (a|00> + b|11>) \otimes (|1>)$, an entangled state.

(14) $|\psi_{14}> = a|010> + b|011> = (|01>) \otimes (a|0> + b|1>)$, a separable state.

(15) $|\psi_{15}> = a|010> + b|100> = (a|01> + b|10>) \otimes (|0>)$, an entangled state.

(16) $|\psi_{16}> = a|010> + b|101>$, a genuinely entangled state.

(17) $|\psi_{17}> = a|010> + b|110> = (a|0> + b|1>) \otimes (|10>)$, a separable state.

(18) $|\psi_{18}> = a|010> + b|111>$, a genuinely entangled state.

(19) $|\psi_{19}> = a|011> + b|100>$, a genuinely entangled state.

(20) $|\psi_{20}> = a|011> + b|101> = (a|01> + b|10>) \otimes (|1>)$, an entangled state.

(21) $|\psi_{21}> = a|011> + b|110>$, a genuinely entangled state.

(22) $|\psi_{22}> = a|011> + b|111> = (a|0> + b|1>) \otimes (|11>)$, a separable state.

(23) $|\psi_{23}> = a|100> + b|101> = (|10>) \otimes (a|0> + b|1>)$, a separable state.

(24) $|\psi_{24}> = a|100> + b|110> = (|1>) \otimes (a|0> + b|1>) \otimes (|0>)$, a separable state.

(25) $|\psi_{25}>= a|100> + b|111> = (|1>) \otimes (a|00> + b|11>)$, an entangled state.

(26) $|\psi_{26}>= a|101> + b|110> = (|1>) \otimes (a|01> + b|10>)$, an entangled state.

(27) $|\psi_{27}>= a|101> + b|111> = (|1>) \otimes (a|0> + b|1>) \otimes (|1>)$, a separable state.

(28) $|\psi_{28}>= a|110> + b|111> = (|11>) \otimes (a|0> + b|1>)$, a separable state.

(B) We now proceed with the analysis of all normalized 3-qubit states containing exactly three nonzero terms. We choose the following compact notation for kets: $|1> = |000>$, $|2> = |110>$ $|3> = |101>$, $|4> = |011>$, $|5>= |111>$, $|6> = |001>$, $|7> = |010>$, and $|8> = |100>$.

We now consider all normalized 3-qubit states, $|ijk>$ such that $|ijk> = a|i> + b|j> +c|k>$ where the nonzero constants $\{a, b, c\} \in C$, the field of complex numbers, and $\{i, j, k\} \in \{1, 2, ..., 8\}$. There are in all $\binom{8}{3} = 56$ states of this type. For these states three scenarios are possible some of these states could be separable, some others could be a tensor product of a 1-qubit state and a 2-qubit entangled state, and some others could be genuinely entangled states not having a 1-qubit factor or a 2-qubit factor. To check the type of entanglement possessed by these states using conventional methods one requires to carry out several calculations. To check the entanglement status and the type of that entanglement possessed by the state, say $|P> = |ijk>$, as given above such that $|ijk> = a|i>+ b|j>+ c|k>$ where the nonzero constants $\{a, b, c\} \in C$, the field of complex numbers, and $\{i, j, k\} \in \{1, 2, ..., 8\}$ one will require to perform the following steps:

(i) To form the density matrix, $|P><P|$, and its corresponding reduced density matrix.

(ii) To calculate the value of corresponding partial entropy.

(iii) To check whether the partial entropy is zero or nonzero.

(iv) To use this value to decide about the entanglement status and the type of the entanglement.

On the other hand, by making use of the new factorization algorithm, based on just checking whether the rows of the associated matrices are proportional or not, one can easily check the desired entanglement status and its type.

For example, let

$|U> = |127> = a|000> + b|110> + c|010>$

$|V> = |678> = a|001> + b|010> + c|100>$

where $\{a, b, c\} \in C$, the field of complex numbers.

By applying the factorization algorithm, we can easily check that out of these two states in the state $|U>$, the first two qubits are entangled while the third qubit is disentangled from the first two qubits, i.e.,

$|U>= |127> = (a|00> + b|11> + c|01>) \otimes (|0>)$

On the other hand, in the state $|V >$ all the qubits are mutually entangled. Thus, by applying the factorization algorithm we can easily check that out of these two states the first two qubits are entangled while the third qubit is disentangled from the first two qubits in the state $|U>$, while in state $|V>$ all qubits are mutually entangled. Also, note that this state $|V >$ becomes the so-called well-known $|W>$ (Wolfgang Dur) state when $a = b = c = \frac{1}{\sqrt{3}}$ [16]. This $|W>$ state is the representative of one of the two genuinely entangled (non-bi-separable) classes of 3-qubit states, the other being the famous $|GHZ>$ (Greenberger-Horne-Zelinger) state [15], which cannot be transformed (not even probabilistically) into each other by local quantum operations [16]. Thus in |W⟩W> and |GHZ> states |GHZ⟩GHZGHZ there exist two very different kinds of tripartite entanglements.

Now, by proceeding along similar lines and applying the above-mentioned new factorization algorithm we can easily determine that out of these 56 states, 24 states are 2-qubit entangled and 1-qubit disentangled states while the remaining other 32 states are genuine 3-qubit entangled states.

In brief, our findings are as follows:

(1) All the 3-qubit states, |ijk>, such that |ijk> = a|i> + b|j> + c|k>, where nonzero constants {a, b, c} ∈ C, the field of complex numbers, and {i, j, k} ∈ {1, 2, ..., 8} are entangled.
(2) Out of total $\binom{8}{3} = 56$ such states, 24 states are 2-qubit entangled and 1-qubit disentangled states.
(3) Out of total $\binom{8}{3} = 56$ states, 32 states are genuine 3-qubit entangled states.
(4) 2-qubit entangled and 1-qubit disentangled states are:
|127>, |128>, |136>, |138>, |146>, |147>, |167>, |168>, |178>, |235>, |238>, |245>, |247>, |257>, |258>, |278>, |345>, |346>, |356>, |358>, |368>, |456>, |457>, |467>
(5) Genuine 3-qubit entangled states are:
|123>, |124>,|125>, |126>, |134>, |135>, |137>, |145>, |148>, |156>, |157>, |158>, |234>, |236>, |237>, |246>, |248>, |256>, |267>, |268>, |347>, |348>, |357>, |367>, |378>, |458>, |468>, |478>, |567>, |568>, |578>, |678>

5 Conclusion

The factorization algorithm developed in this paper provides an easy tool to understand the entanglement status of multi-qubit pure quantum states. It systematically extracts all possible factors of a multi-qubit pure state by checking the proportionality of rows of certain corresponding matrices and thus find a representation for any given N-qubit pure state in terms of the tensor product of its factor states.

6 Future Scope of the Research

By proceeding on similar lines as done above one can go further and understand the rich entanglement structure of 3-qubit pure quantum states containing more than three nonzero terms. More importantly, this algorithm can be used on similar lines for the analysis of 4-qubit states containing two, three, or more nonzero terms, etc.

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