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General ray method for solution of the Dirichlet boundary value problems for elliptic partial differential equations in domains with complicated geometry

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Summary

New General Ray (*GR*) method for solution of the Dirichlet boundary value problem for the class of elliptic Partial Differential Equations (PDE) is proposed. *GR*-method consists in application of the Radon transform directly to the PDE and in reduction PDE to assemblage of Ordinary Differential Equations (ODE). The class of the PDE includes the Laplace, Poisson and Helmgoltz equations. *GR*-method presents the solution of the Dirichlet boundary value problem for this type of equations by explicit analytical formulas that use the direct and inverse Radon transform. Proposed version of *GR*-method justified theoretically, realized by fast algorithms and MATLAB software, which quality we demonstrate by numerical experiments.

Keywords: partial differential equations, boundary value problems, Radon transform, fast algorithms, MATLAB software.

Introduction

There are two main approaches for solving boundary value problems for partial differential equations in analytical form: the Fourier decomposition and the Green function method [Sobolev (1966)]. The Fourier decomposition is used, as the rule, only in theoretical investigations. The Green function method is the explicit one, but it is difficult to construct the Green function for the complex geometry of the considered domain Ω. The known numerical algorithms are based on the Finite Differences method, Finite Elements (Finite Volume) method and the Boundary Integral Equation method. Numerical approaches lead to solving systems of linear algebraic equations [Samarsky (1977)] that require a lot of computer time and memory.

A new approach for the solution of boundary value problems on the base of the General Ray Principle (*GRP*) was proposed by the author in [Grebennikov (2003)] for the stationary waves field. *GRP* leads to explicit analytical formulas (*GR*-method) and fast algorithms, developed and illustrated by numerical experiments in [Grebennikov (2003); Grebennikov (2004); Grebennikov (2005); Grebennikov (2006)] for solution of the direct and coefficient inverse problems for the equations of mathematical physics.

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Here we extend the proposed approach and construct another version of *GR*-method based on application of the direct Radon transform [Radon (1917)] to the PDE [Sigurdur (1999); Gelfand and Shapiro (1955)]. This version of *GR*-method is justified theoretically, realized as algorithms and program package in MATLAB system, illustrated by numerical experiments.

General Ray Principle

The General Ray Principle (*GRP*) means to construct for considering PDE an analogue as family of ODE describing the distribution of the function $u(x,y)$ along of "General Rays", which are presented by a straight line l with the traditional Radon parameterization due a parameter t: $x = p\cos\varphi - t\sin\varphi$, $y = p\sin\varphi + t\cos\varphi$. Here $|p|$ is a length of the perpendicular from the centre of coordinates to the line l, $\varphi \in [0,\pi]$ is the angle between the axis x and this perpendicular. Using this parameterization, we considered in [Grebennikov (2003); Grebennikov (2004); Grebennikov (2005)] the t-version of *GR*-method that leads to reducing Laplace equation to the assemblage (depending of p, φ) of ordinary differential equations with respect to variable t. This family of ODE was used as the local analogue of the PDE. The numerical justification of t-version of *GR*-method was given for the domain Ω as unit circle [Grebennikov (2003)]. For some convex domains the quality of the method was illustrated by numerical examples. The reduction of the considered PDE to the family of ODE with respect to variable t gives possibilities to satisfy directly boundary conditions, construct the effective and fast numerical algorithms. At the same time, there are problems with its realization for the complicated geometry of the region Ω.

Formulation and Theoretical Foundation of p-Version of *GR*-Method

Let us consider the Dirichlet boundary problem for the elliptic equation:

$$\Delta u(x,y) + k^2 u(x,y) = \psi(x,y),\ (x,y) \in \Omega; \tag{1}$$

$$u(x,y) = f(x,y), \quad (x,y) \in \Gamma. \tag{2}$$

with respect to the function $u(x,y)$ that has two continuous derivatives on bought variables inside the plane domain Ω, bounded with a continuous curve Γ. Here $\psi(x,y)$, $(x,y) \in \Omega$ and $f(x,y)$, $(x,y) \in \Gamma$ are given functions.

In works [Gelfand and Shapiro (1955); Sigurdur (1999)] there are presented investigations of the possibility to reduce solution of PDE to the family of ODE using the direct Radon transform. This reduction leads to ODE with respect to variable p and can be interpreted in the frame of the introduced General Ray Principle. However, using the variable p for the first point of view makes it impossible to satisfy directly to the boundary conditions expressed in (x,y) variables. Possibly, by this reason the mentioned and other related investigations were concentrated only at theoretical aspect in constructing some basis of general solutions of PDE. Unfortunately, this approach was not used for construction of numerical methods and algorithms for solution of boundary value problems, except some simple examples [Sigurdur (1999)]. The important new element, introduced here into this scheme, consists in satisfaction of boundary condition by its reduction to homogeneous one.

The$p-$version of the *GR*-method we explain as the consequence of the next steps: 1) reduce boundary condition to homogeneous one; 2) describe the distribution of the potential function along the general ray (a straight line l) by its direct Radon transform $u_{\varphi}(p)$; 3) construct the family of ODE on the variable $p$with respect the function $u_{\varphi}(p)$; 4) solution of the constructed ODE with the zero boundary conditions; 5) calculate the inverse Radon transform of the obtained solution; 6) regress to the initial boundary condition. We present bellow the realization of this scheme.

We suppose that the boundary Γ can be described in the polar coordinates (r,α) by some one-valued positive function that we denote $r_0(\alpha)$, $\alpha \in [0,2\pi]$. It is always possible for the simple connected star region Ω with the centre at the coordinate origin. Let us write the boundary function as $\bar{f}(\alpha) = f(r_0(\alpha)\cos\alpha, r_0(\alpha)\sin\alpha)$. Supposing that functions $r_0(\alpha)$ and $\bar{f}(\alpha)$ have the second derivative, we introduce for all $(x,y) \in \Omega$, $\alpha = arctg(y/x)$, functions: $f_0(\alpha) = \bar{f}(\alpha)/r_0^2(\alpha)$, $\psi_0(x,y) = \psi(x,y) - 4f_0(\alpha) - f_0''(\alpha) - k^2r^2f_0(\alpha)$, $u_0(x,y) = u(x,y) - r^2f_0(\alpha)$.

To realize the first step of the scheme we can write the boundary problem with respect the function $u_0(x,y)$ as the next two equations:

$$\Delta u_0(x,y) + k^2u_0(x,y) = \psi_0(x,y), (x,y) \in \Omega; \tag{3}$$

$$u_0(x,y) = 0, \quad (x,y) \in \Gamma. \tag{4}$$

To make the second and the third steps we apply the direct Radon transform to the equation (3) and obtain, using formula (2) at the pp. 3 of the book [Sigurdur (1999)], the family of the ODE with respect the variable p:

$$\frac{d^2u_{\varphi}(p)}{dp^2} + k^2u_{\varphi}(p) = R[\psi_0](p,\varphi), \quad (p,\varphi) \in \hat{\Omega}; \tag{5}$$

where $\hat{\Omega}$ is the domain of the change of parameters p,φ. As the rule, $\varphi \in [0,\pi]$, module of the parameter p is equal to the radius in the polar coordinates and changes in the limits, determined by the boundary curve Γ. In the considered case for some fixed φ parameter p is in the limits: $-r_0(\varphi-\pi) < p < r_0(\varphi)$.

Unfortunately, boundary condition (4) cannot be modified directly by Radon transform to the corresponding boundary conditions for the every equation of the family (5). For the fourth step, we propose to use the next boundary conditions for every fixed $\varphi \in [0,\pi]$:

$$u_\varphi\left(-r_0(\varphi-\pi)\right)=0; \quad u_\varphi\left(r_0(\varphi)\right)=0. \tag{6}$$

Let us designate $\hat{u}_\varphi\left(p\right)$solution of the problem (5)-(6), that can be univocally determined as function of variable p for every $\varphi \in [0,\pi]$, $p \in (-r_0(\varphi-\pi), r_0(\varphi))$, and out of this interval we extend $\hat{u}_\varphi\left(p\right) \equiv 0$ for all φ with continuity on p.

Let us denote the inverse Radon transform [Sigurdur (1999)] as operator R^{-1}. The foundation of the fifth and sixth steps of the scheme is concentrated in the next theorem.

Theorem. *The following formula for the solution of boundary value problems (1)-(2) is true*

$$\bar{u}(x,y) = R^{-1}[\hat{u}_\phi\left(p\right)] + r^2 f_0(\alpha), \ (x,y) \in \Omega \tag{7}$$

Proof. Substituting function defined by (7) into left hand side of equation (5) we obtain from Lemma 2.1 at the pp. 3 of the book [Sigurdur (1999)] the next relations:

$$\Delta\bar{u}_0\left(x,y\right) \ = R^{-1}[\frac{d^2\hat{u}_\phi\left(p\right)}{dp^2} \ + k^2 u_\phi\left(p\right)] = R^{-1}[R[\psi_0]\left(p,\phi\right)] = \psi_0\left(x,y\right) \tag{8}$$

that convinces us of the satisfaction of the equation (5). From the condition $\hat{u}_\varphi\left(p\right) \equiv 0, p \notin (-r_0(\varphi-\pi), r_0(\varphi))$, $\varphi \in [0,\pi]$ and Theorem 2.6 (the support theorem) at page 10 of the book [Sigurdur (1999)] it follows that $\bar{u}_0\left(x,y\right) \equiv 0$ for $(x,y) \notin \Omega$ and, due its continuity, it satisfies boundary conditions (6). Then, using relation between $u_0\left(x,y\right)$ and $u\left(x,y\right)$we obtain (7). This finishes the proof.

The inverse Radon transforms in explicit formula (7) can be realized numerically by fast Fourier discrete transformation (FFDT) that guarantees the rapidity of proposed method and developed algorithmic realization.

Results of Numerical Experiments

We have constructed the fast algorithmic and program realization of *GR*-method for considered problem in MATLAB system. We used the uniform discretization of variables $p \in [-1,\ 1]$, $\quad \varphi \in [0,\pi]$, so as for variables x,y , with $\boldsymbol{N}$ nodes. We made testes on mathematically simulated model examples with known exact functions $u\left(x,y\right)$, $f(x,y)$, $\psi\left(x,y\right)$. Some numerical experiments for the Laplace equation with exact solution $u = x + y$ are presented at at Fig. 1, 2,3, particularly in comparison with results obtained by program *pdemodel* from PDE toolbox of MATLAB system.

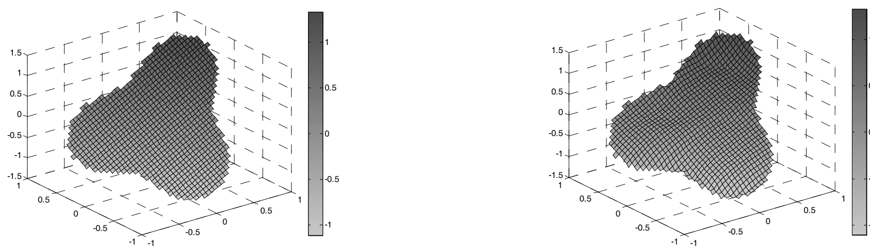

Figure 1: exact $u = x + y$ and reconstructed by *GR*-method

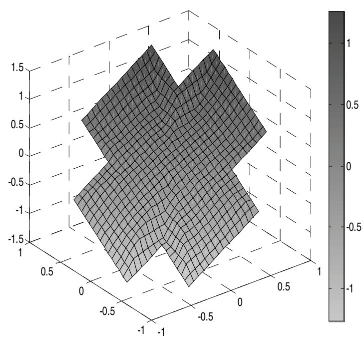

Figure 2: reconstruction by pdemodel, calculation time 12.8750 sec

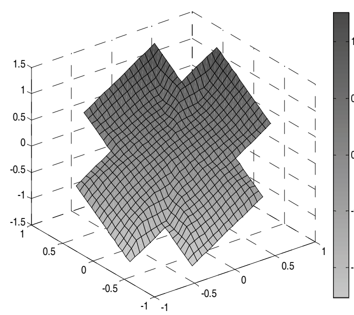

Figure 3: reconstruction by *GR*-method, calculation time 1.2970

Conclusion

New version of *GR*-method is constructed. It is based on application of the Radon transform directly to the partial differential equation. This version of *GR*-method for arbitrary simple connected star domains is justified theoretically, realized as algorithms and program package in MATLAB system, illustrated by numerical experiments.

Proposed version can be applied for solution of boundary value problems for another PDE with constant coefficients.

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