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The boundary layer phenomenon in bending of thick annular sector plates

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Summary

In this article, the bending equations of thick annular sector plates are derived based on the third-order shear deformation theory (TSDT). Using a function, called boundary layer function, the coupled system of equations is converted into two decoupled equations and solved analytically. It is shown that the value of the boundary layer function for TSDT is higher than that of the Mindlin theory. Thus, variations of stress components in the edge zone of the plate are more significant. It is seen that there exist no boundary layer, a weak boundary layer, and a strong boundary layer effect for simply supported, clamped, and free edges, respectively.

Introduction

The unusual changes of different parameters in the vicinity of the edges of plates are known as boundary layer phenomenon. The cause of these effects is the existence of the boundary layer function. This function has a significant value near the edges and is zero in the interior zone.

There are some studies in literature dealing with the boundary layer function of Mindlin plates. Nosier et al. [1] described the boundary layer function in rectangular plates. Also, they obtained the same results for the boundary layer function of sector plates [2]. Jomehzadeh et al. [3] developed an analytical solution for bending of functionally graded sector plates using the boundary layer function in polar coordinates.

Apparently, the boundary layer effects are more significant for thick plates. Although some studies have been carried out for the boundary layer phenomenon in Mindlin theory, no studies can be found for the boundary layer phenomenon in the third-order shear deformation theory. In this paper, the governing equations of the third-order shear deformation theory are converted to new uncoupled equations and solved analitically. The new equations are in terms of the boundary layer function and transverse deflection of the plate. The effects of the boundary layer phenomenon in the third-order shear deformation plate theory are considered and compared with those of the Mindlin plate theory.

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Governing Equations

According to the third-order shear deformation plate theory in polar coordinates the displacement components are assumed to be [4]

$$\begin{aligned} u_r &= u + z\left(\psi_r - \frac{4}{3}\left(\frac{z}{h}\right)^2(\psi_r + w_{,r})\right) \\ u_\theta &= v + z\left(\psi_\theta - \frac{4}{3}\left(\frac{z}{h}\right)^2(\psi_\theta + \frac{1}{r}w_{,\theta})\right) \\ u_z &= w \end{aligned} \tag{1}$$

Based on the displacement field (1), the out-of-plane shear stresses on the top and bottom surfaces of the plate are equal to zero. The principle of minimum total potential energy is defined as [5]

$$\delta(U+V) = 0 \tag{2}$$

where δ is the variational symbol, U and V are the strain energy and potential energy of applied forces, respectively. Considering plane-stress state and using the strain-displacement relations and the displacement field (1), the governing equilibrium equations of sector plate in polar coordinate are obtained as follows

$$\begin{aligned} \delta\psi_r : D&\left(\psi_{r,rr} + \frac{3}{r}\psi_{r,r} - \frac{1}{r^2}\psi_r + \frac{1}{r}\psi_{\theta,r\theta} - \frac{1}{r^2}\psi_{\theta,\theta}\right) + C\left(\frac{1}{r^2}\psi_{r,\theta\theta} - \frac{1}{r}\psi_{\theta,r\theta} - \frac{1}{r^2}\psi_{\theta,\theta}\right) \\ &-A(\psi_r + w_{,r}) - F\left(w_{,rrr} + \frac{1}{r}w_{,rr} - \frac{1}{r^2}w_{,r} + \frac{1}{r^2}w_{,r\theta\theta} - \frac{2}{r^3}w_{,\theta\theta}\right) = 0 \\ \delta\psi_\theta : D&\left(\frac{1}{r}\psi_{r,r\theta} + \frac{1}{r^2}\psi_{r,\theta} + \frac{1}{r^2}\psi_{\theta,\theta\theta}\right) + C\left(\frac{1}{r^2}\psi_{r,\theta} - \frac{1}{r}\psi_{r,r\theta} - \frac{1}{r^2}\psi_\theta + \psi_{\theta,rr} + \frac{1}{r}\psi_{\theta,r}\right) \\ &-A\left(\psi_\theta + \frac{1}{r}w_{,\theta}\right) - F\left(\frac{1}{r}w_{,rr\theta} + \frac{1}{r^2}w_{,r\theta} + \frac{1}{r^3}w_{,\theta\theta\theta}\right) = 0 \\ \delta w : F&\left(\psi_{r,rrr} + \frac{2}{r}\psi_{r,rr} - \frac{1}{r^2}\psi_{r,r} + \frac{1}{r^3}\psi_r + \frac{1}{r^2}\psi_{r,r\theta\theta} + \frac{1}{r^3}\psi_{r,\theta\theta}\frac{1}{r}\psi_{\theta,rr\theta} - \frac{1}{r^2}\psi_{\theta,r\theta}\right. \\ &\left. + \frac{1}{r^3}\psi_{\theta,\theta} + \frac{1}{r^3}\psi_{\theta,\theta\theta\theta}\right) - H\left(w_{,rrrr} + \frac{2}{r}w_{,rrr} - \frac{1}{r^2}w_{,rr} + \frac{1}{r^3}w_{,r} + \frac{2}{r^2}w_{,rr\theta\theta} - \frac{2}{r^3}w_{,r\theta\theta}\right. \\ &\left. + \frac{4}{r^4}w_{,\theta\theta} + \frac{1}{r^4}w_{,\theta\theta\theta\theta}\right) + A\left(\psi_{r,r} + \frac{1}{r}\psi_r + \frac{1}{r}\psi_{\theta,\theta} + w_{,rr} + \frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta}\right) + p = 0 \end{aligned} \tag{3}$$

Decoupling the Governing Equilibrium Equations

Introducing a function which will be referred to the boundary layer as

$$\phi(r,\theta) = \frac{1}{r}[\psi_{r,\theta} - (r\psi_\theta)_{,r}] \tag{4}$$

and using some algebraic operations, three highly coupled governing equations (3) can be converted into two independent equations as

$$\begin{aligned} &C\nabla^2\phi - A\phi = 0 \\ &I\nabla^6 w - \bar{D}\nabla^4 w - \frac{D}{A}\nabla^2 p + p = 0 \end{aligned} \tag{5}$$

It can be seen that the rotation functions ($\psi_r$and ψ_θ) can be written in terms of transverse deflection and the boundary layer function as

$$\begin{aligned} \psi_r &= \frac{DI}{A(D+F)}(\nabla^4 w)_{,r} - \frac{D+F}{A}(\nabla^2 w)_{,r} - w_{,r} + \frac{1}{r}\frac{C}{A}\phi_{,\theta} - \frac{D^2}{A^2(D+F)}p_{,r} \\ \psi_\theta &= \frac{DI}{A(D+F)}\frac{1}{r}(\nabla^4 w)_{,\theta} - \frac{D+F}{A}\frac{1}{r}(\nabla^2 w)_{,\theta} - \frac{1}{r}w_{,\theta} - \frac{C}{A}\phi_{,r} - \frac{D^2}{A^2(D+F)}\frac{1}{r}p_{,\theta} \end{aligned} \tag{6}$$

Solution of the Bending Equations of Sector Plate

Consider an annular sector plate of inner radius a, outer radius b,sector angle θ_0 and uniform thickness h.It is assumed that two opposite edges of the plate at $\theta = 0$ and $\theta = \theta_0$ are simply supported. For the bending analysis of the sector plate, the boundary layer function, transverse deflection and distributed load have been considered as

$$\begin{aligned} \phi(r,\theta) &= \sum_{m=1}^{\infty} \Phi_m(r)\cos(\beta_m\theta) \\ w(r,\theta) &= \sum_{m=1}^{\infty} W_m(r)\sin(\beta_m\theta) \\ p(r,\theta) &= \sum_{m=1}^{\infty} P_m\sin(\beta_m\theta), \quad \beta_m = m\pi/\theta_0 \end{aligned} \tag{7}$$

Substituting series solutions (7) into reformulated Eqs. (5), two ordinary differential equations are obtained. The solutions of these equations are as

$$\Phi_m(r) = C_{m1} I_{\beta_m}(\mu_1 r) + C_{m2} K_{\beta_m}(\mu_1 r)$$
$$W_m^H(r) = C_{m3} r^{\beta_m} + C_{m4} r^{-\beta_m} + C_{m5} r^{\beta_m+2} + C_{m6} r^{-\beta_m+2} + C_{m7} I_{\beta_m}(\mu_2 r) + C_{m8} K_{\beta_m}(\mu_2 r)$$
$$\mu_1 = \sqrt{A/C}, \quad \mu_2 = \sqrt{\bar{D}/I} \tag{8}$$

where W_m^H is homogeneous solution of w. The particular solution of w depends on the distributed load.

Numerical Results and Discussion

In this section, the numerical results are presented for an annular sector plate with $b/a = 2$ and $\theta_0 = \pi/6$. It is assumed that the plate is an isotropic plate with $h = a/10$ and $\nu = 0.3$.

In Fig. 1, the boundary layer function is plotted for an annular sector plate with different boundary conditions on its edges. It can be seen that only at the near of the plate edges, the boundary layer function has significant value and at the interior zone of the plate, the value of this function is equal to zero. Because of this effect, it is called the boundary layer function.

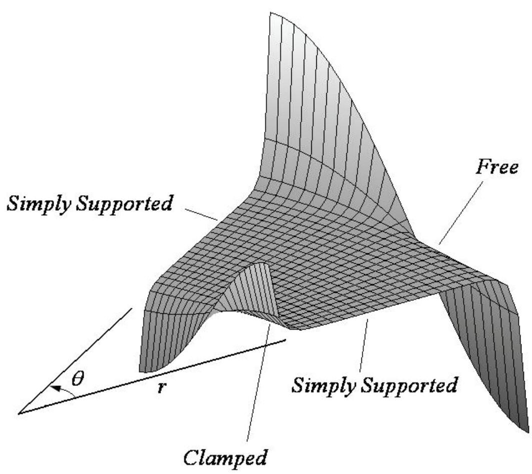

Figure 1: The boundary layer function of TSDT for an annular sector plate

The variation of the out-of-plane shear stress $\sigma_{\theta z}$ is depicted across the radial edges in Fig. 2 for both FSDT and TSDT. The annular sector plate is simply supported in the inner edge and clamped along the outer edge. In this figure, the variation of $\sigma_{\theta z}$ is also plotted by omitting the boundary layer function to show the

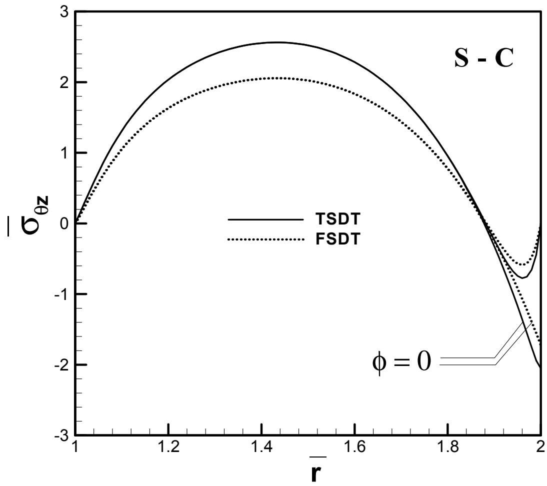

Figure 2: Variation of the out-of-plane shear stress along the radial direction for an annular sector plate with simply supported and clamped edges

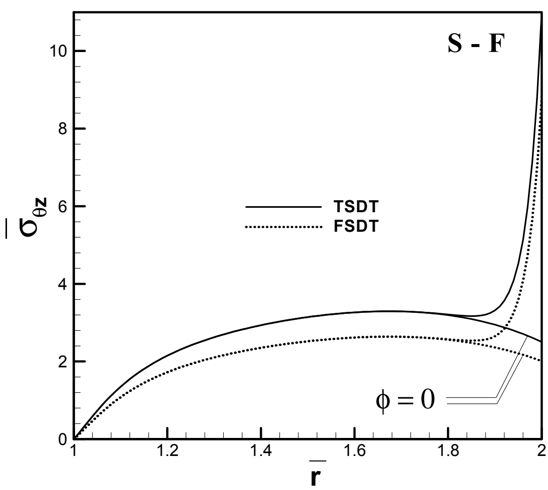

Figure 3: Variation of the out of plane shear stress along the radial direction for an annular sector plate with simply supported and free edges

effects of this function on the stress distribution. It can be concluded that with vanishing boundary layer function, the sudden change of the stress in the edge zones of the plate vanishes in both theories.

The variation of the out of plane shear stress $\sigma_{\theta z}$ is depicted in Fig. 3 for an annular sector plate with simply supported inner edge and free outer edge. By comparing Fig. 3 with Fig. 1, it can be seen that the intensity and limitation of the sudden changes of the stress are similar to that of the boundary layer function, and a vanishing boundary layer function will result in vanishing of the rapid stress changes. As the effect of the boundary layer function, the value of the stress at the vicinity of the edges for TSDT is higher than that of FSDT.

As it has been seen previously, the boundary layer function is zero at simply supported edges. Therefore, in this case the stress has no rapid change at the edges of the plate.

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