 ICCES, vol.14, no.1, pp.37-42

Finite element-based flow simulations using exponential weighting functions

K. Kakuda[1], **Y. Maeda**[1]

Summary

The applications of a finite element scheme to one-dimensional linear advection-diffusion equation, the incompressible Navier-Stokes equations, and compressible Euler system of equations are presented. The mesh-based scheme is the Petrov-Galerkin weak formulation with exponential weighting functions. Some numerical results demonstrate the workability and the validity of the present approach.

Keywords: Finite element, Petrov-Galerkin method, exponential weighting functions, CFD.

Introduction

From a simulation-based design's point of view, computational fluid dynamics (CFD) is now indispensable in the fields of engineering and science. Numerical instabilities have been experienced in the solution of the advection-dominated system of equations in fluid flow [Brooks and Hughes (1982)]. Various robust schemes have been successfully presented in the mesh-based and meshless-based frameworks for CFD.

In our previous work, we have proposed a finite element-based scheme for solving effectively the problems of incompressible viscous fluid flow [Kakuda and Tosaka (1992)]. The scheme is based on the Petrov-Galerkin weak formulation using exponential weighting functions.

The purpose of this paper is to apply the Petrov-Galerkin finite element-based scheme to one-dimensional linear advection-diffusion equation, the incompressible Navier-Stokes equations, and compressible Euler system of equations. The workability and the validity of the present approach are demonstrated through some numerical results.

Linear advection-diffusion equation

Statement of the problem

Let us first consider the one-dimensional advection-diffusion equation in spatial coordinate, x, given by

$$u\varphi_{,x} = k\varphi_{,xx} \tag{1}$$

[1]College of Industrial Technology, Nihon University, Narashino, Chiba 275-8575, Japan

with the adequate boundary conditions, where u and k are the given velocity and diffusivity, respectively. Now, we define the flux $f = u\varphi$ in Eq. (1). With this definition, Eq. (1) is given as follows :

$$f_{,x} = k\varphi_{,xx} \tag{2}$$

Petrov-Galerkin finite element formulation

In order to solve the flux in a stable manner, we shall adopt the Petrov-Galerkin finite element formulation using exponential weighting functions [2]. On the other hand, the conventional Galerkin finite element formulation can be applied to solve numerically Eq. (2).

First of all, we start with the weighted integral expression of the flux in a subdomain $\Omega_i = [x_{i-1}, x_i]$ with respect to weighting functions M_α. By applying the divergence theorem to the weighted integral expression, we obtain the following integral form :

$$\int_{\Omega_i} M_\alpha N_\beta dx f_\beta - u \int_{\Omega_i} M_\alpha N_\beta dx \varphi_\beta = 0 \tag{3}$$

The weighting function M_α can be chosen as a general solution which satisfies

$$uM_\alpha + \Delta x_i \sigma(u) M_{\alpha,x} = 0 \tag{4}$$

where $\Delta x_i = x_i - x_{i-1}$, and $\sigma(u)$ denotes some functions described by Yee et al.[3], which is sometimes referred to as the coefficient of numerical viscosity. The solution of Eq. (4) is

$$M_\alpha = e^{-a(x-x_\alpha)} \qquad (\alpha = x_{i-1}, x_i) \tag{5}$$

where $a = \frac{u}{\Delta x_i \sigma(u)}$.

Here, applying an element-wise mass lumping to the first term of the left-hand side of Eq. (3), and carrying out exactly those integrals in Eq. (3), we have the following finite element equation

$$\tilde{c}\delta_{\alpha\beta} f_\beta = -uH_{\alpha\beta}\varphi_\beta \tag{6}$$

where $\delta_{\alpha\beta}$ is the Kroneker's delta, and

$$\tilde{c} = e^{-\gamma} - e^{\gamma} \quad , \quad H_{\alpha\beta} = \begin{bmatrix} (e^{\gamma} + \frac{\tilde{c}}{2\gamma}) & -(e^{-\gamma} + \frac{\tilde{c}}{2\gamma}) \\ (e^{\gamma} - \frac{\tilde{c}}{2\gamma}) & -(e^{-\gamma} - \frac{\tilde{c}}{2\gamma}) \end{bmatrix} \quad , \quad \gamma = \frac{u}{2\sigma(u)} \tag{7}$$

From Eq. (6) we can obtain the following numerical flux $f_{i-1/2}$ in the subdomain Ω_i

$$f_{i-1/2} = f_i + \frac{u}{2}[1 + \{sgn(\gamma) coth|\gamma| - \frac{1}{\gamma}\}](\varphi_{i-1} - \varphi_i) \tag{8}$$

and similarly in another subdomain Ω_{i+1}, we have

$$f_{i+1/2} = f_i + \frac{u}{2}[-1 + \{sgn(\gamma)coth|\gamma| - \frac{1}{\gamma}\}](\varphi_i - \varphi_{i+1}) \tag{9}$$

where $sgn(\gamma)$ denotes the signum function.

Let us now consider the Galerkin finite element model for the weighted integral form of Eq. (2). After that, we assume a uniform mesh $\Delta x_i = \Delta x$ for simplicity of the formulation. Taking into consideration the continuity of $\varphi_{,x}$ at nodal point i, we can obtain the following discrete form :

$$f_{i-1/2} - f_{i+1/2} + \frac{k}{\Delta x}(\varphi_{i-1} - 2\varphi_i + \varphi_{i+1}) = 0 \tag{10}$$

Substituting Eq. (8) and Eq. (9) into Eq. (10) and after some manipulations, we obtain the following finite difference form :

$$\frac{u}{2\Delta x}(\varphi_{i+1} - \varphi_{i-1}) = (k + \tilde{k})\frac{\varphi_{i-1} - 2\varphi_i + \varphi_{i+1}}{\Delta x^2} \tag{11}$$

where for any velocity u

$$\tilde{k} = \frac{|u|\Delta x}{2}\{coth|\gamma| - \frac{1}{|\gamma|}\} \tag{12}$$

There exist some cases for possible choice of $\sigma(u)$ in Eq. (4). Now if we assume $\sigma(u) = |u|/\alpha$ in which an adhoc parameter $\alpha = \Delta x|u|/k$, then γ in Eq. (12) is given as

$$\gamma = \frac{u}{2\sigma(u)} = Pe \quad (\equiv \frac{\Delta x u}{2k} : element\, Peclet\, number) \tag{13}$$

Using the element Peclet number Pe as γ, we reduce Eq. (11) to the following form :

$$\{sgn(Pe) - coth|Pe|\}\varphi_{i+1} + 2coth|Pe|\varphi_i - \{sgn(Pe) + coth|Pe|\}\varphi_{i-1} = 0 \tag{14}$$

This equation has the same structure as the SUPG scheme developed by Hughes et al.[1], and it leads to nodally exact solutions for all values of Pe.

Incompressible Navier-Stokes equations

Petrov-Galerkin finite element formulation

The motion of an incompressible viscous fluid flow is governed by the Navier-Stokes equations in dimensionless form. By applying the time splitting technique to the set of equations, we can split formally the problem into the nonlinear system of advection-diffusion equations and the linear Euler's system of equations.

It is well known that the conventional Galerkin finite element scheme using coarse meshes leads to spurious oscillatory solutions for flow simulations at high Reynolds number. Therefore, let us now consider the Petrov-Galerkin finite element formulation using exponential weighting functions [2] to the nonlinear advection-diffusion equations with a Reynolds number Re. By applying the divergence theorem to the weighted residual form in a subdomain Ω_e of the whole domain Ω, and after some manipulations, we have the following weak form:

$$\int_{\Omega_e} \left\{ \dot{u}_i(\tilde{u}_i, u_i^n) + u_j u_{i,j} \right\} M_\alpha d\Omega + \int_{\Omega_e} \frac{1}{Re} u_{i,j} N_{\alpha,j} d\Omega = \int_{\Gamma_e} \tau_i N_\alpha d\Gamma \qquad (15)$$

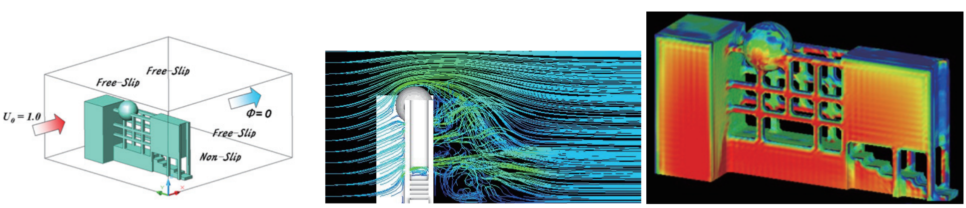
in which $\tau_i \equiv u_{i,j} n_j / Re$, Γ_e is the boundary on the subdomain, and M_α denotes the weighting functions given by

$$M_\alpha(\boldsymbol{x}) = \sum_{\gamma,i} N_\alpha(\boldsymbol{x}) e^{-a_i(N_\gamma x_i^\gamma - x_i^\alpha)} \quad , \quad a_i = \frac{\alpha_i}{\mid L_i \mid} sgn(v_i) \qquad (16)$$

where N_α is the shape function in three dimensions, v_i is the velocity vector averaged in Ω_e, L_i is the reference length for x_i-directions, and α_i is the upwinding parameters which control an effect of the upwinding.

Numerical example

As a numerical example, we shall consider the flow around a building in Japan. Fig. (1) shows the geometrical configuration and the numerical results at $Re = 7,900$. The number of nodes is $473,964$, $\Delta t = 0.01$, and $\alpha_i = 0.2$.



(a) Geometry (b) Streamline (c) Pressure

Figure 1: Geometrical configuration and numerical results

Compressible Euler system of equations

Statement of the problem

In this section, in order to develop a high-resolution scheme based on the TVD we have an adhoc function $\boldsymbol{G}$. As a result, we obtain the following modified hyperbolic systems of conservation laws :

$$\mathbf{W} = \mathbf{S}^{-1}\mathbf{U} \quad , \quad \boldsymbol{G} = h\boldsymbol{W}_{,x} \tag{17}$$

$$\tilde{\boldsymbol{F}} = \boldsymbol{\Lambda}\boldsymbol{W} + \hat{\boldsymbol{\Sigma}}\boldsymbol{G} \tag{18}$$

$$\tilde{\boldsymbol{F}}^{*} = \boldsymbol{S}\tilde{\boldsymbol{F}} \quad , \quad \boldsymbol{U}_{,t} + \tilde{\boldsymbol{F}}^{*}_{,x} = \mathbf{0} \tag{19}$$

where $\hat{\boldsymbol{\Sigma}}$ is a $m \times m$ diagonal matrix associated with the limiter functions.

Petrov-Galerkin finite element formulation

As before, the weighted integral form of Eq. (18) in the subdomain $\Omega_i = [x_{i-1}, x_i]$ is given by $\int_{\Omega_i} \{\tilde{\boldsymbol{F}} - \boldsymbol{\Lambda}_{i-1/2}\boldsymbol{W} - \hat{\boldsymbol{\Sigma}}_{i-1/2}\boldsymbol{G}\}\boldsymbol{M}_{\alpha} dx = \mathbf{0}$ in which $\boldsymbol{M}_{\alpha}$ is the exponential weighting functions as follows :

$$\boldsymbol{M}_{\alpha} = e^{-\boldsymbol{a}_{i-1/2}(x-x_{\alpha})} \quad , \quad \boldsymbol{a}_{i-1/2} = \frac{\boldsymbol{\Sigma}^{-1}_{i-1/2}\boldsymbol{\Lambda}_{i-1/2}}{h_{i-1/2}} \tag{20}$$

By calculating the integrals and using the flux lumping technique such as the mass lumping one, we find the solutions of $\tilde{\boldsymbol{F}}_{i-1/2}$ in Ω_i as follows :

$$\tilde{\boldsymbol{F}}_{i-1/2} = \hat{\boldsymbol{F}}_{i-1/2} + \hat{\boldsymbol{\Sigma}}_{i-1/2}[\boldsymbol{G}_i + \frac{1}{2}\{\boldsymbol{I} + \tilde{\boldsymbol{\zeta}}_{i-1/2}\}(\boldsymbol{G}_{i-1} - \boldsymbol{G}_i)] \tag{21}$$

and similarly in an adjacent subdomain Ω_{i+1}

$$\tilde{\boldsymbol{F}}_{i+1/2} = \hat{\boldsymbol{F}}_{i+1/2} + \hat{\boldsymbol{\Sigma}}_{i+1/2}[\boldsymbol{G}_i - \frac{1}{2}\{\boldsymbol{I} - \tilde{\boldsymbol{\zeta}}_{i+1/2}\}(\boldsymbol{G}_i - \boldsymbol{G}_{i+1})] \tag{22}$$

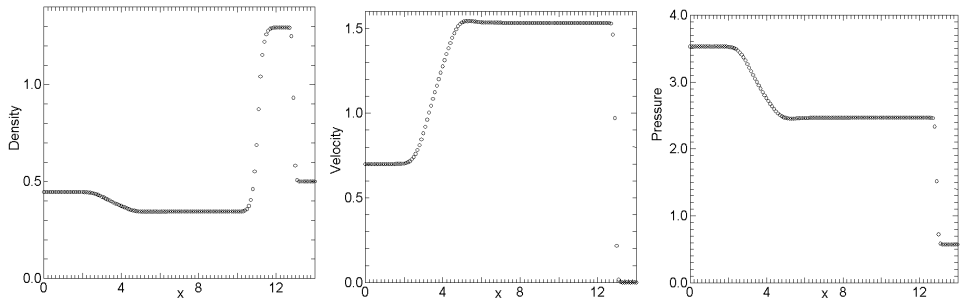
Numerical example

Let us now consider a Riemann problem, namely the shock-tube problem, for the above Euler system of equations in order to demonstrate the workability and the validity of the present approach. The initial data in a field $\{x|0 < x < 14\}$ is given as follows : $\boldsymbol{U}(x,0) = \boldsymbol{U}_L \quad if\ x < 8 \quad = \boldsymbol{U}_R \quad if\ x > 8$, where $\boldsymbol{U}_L = (0.445, 0.311, 8.928)^T$, $\boldsymbol{U}_R = (0.5, 0.0, 1.4275)^T$.

In Fig. (2) we show the numerical results obtained by the finite element scheme with the Harten's limiter [Yee, Warming and Harten (1985)] and the Roe's linearization. The calculations were performed with 100 time steps under the CFL restriction of 0.95. The number of elements is 140.

Conclusions

We have presented a finite element-based scheme using exponential weighting functions for solving numerically the system of equations in fluid flow, such as incompressible Navier-Stokes equations and compressible Euler's equations. The numerical results demonstrated that the present approach was capable of solving in a stable manner the system of equations.



(a) Density (b) Velocity (c) Pressure

Figure 2: Numerical results

References

1. **Brooks, A.; Hughes, T. J. R.** (1982): Streamline upwind/Petrov-Galerkin formulations for convection dominated flow with particular emphasis on the incompressible Navier-Stokes equations. *Comp. Meths. Appl. Mech. Engng.*, vol.32, pp.199–259.

2. **Kakuda, K.; Tosaka, N.** (1992): Finite element approach for high Reynolds number flows. *Theoretical and Applied Mechanics*, vol.41, pp.223–232.

3. **Yee, H. C.; Warming, R. F.; Harten, A.** (1985): Implicit total variation diminishing (TVD) schemes for steady-state calculations. *J. Comput. Phys.*, vol.57, pp.327–360.