

Numerical solution of fractional derivative equations in mechanics: advances and problems

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Summary

This report is to make a survey on the numerical techniques for fractional derivative equations in mechanical and physical fields, including numerical integration of fractional time derivative and emerging approximation strategies for fractional space derivative equations. The perplexing issues are highlighted, while the encouraging progresses are summarized. We also point out some emerging techniques which will shape the future of the numerical solution of fractional derivative equations.

The frequency scaling power law of fractional order appears universal in physical behaviors of complex solids, fluids and soft matter (e.g., polymers, colloids, emulsions, foams, biomaterials, rock layers, sediments, plastics, glass, rubber, oil, soil, DNA) and is considered “anomalous” compared with those of the ideal solids and fluids, in that the various gradient laws of physics, mechanics and chemistry (e.g., Fickian diffusion, Fourier heat conduction, Darcy’s law) are broken. The particular examples are viscous dampers in seismic isolation of buildings, anomalous diffusions in porous media and turbulence, power law dissipation of medical ultrasonic imaging, inelastic dissipative vibration of polymers and soil, just to mention a few. The standard mathematical modeling approach using integer-order time-space derivatives can not accurately reflect fractional power law, while the fractional derivatives are instead found an irreplaceable modeling approach [1]. For example, anomalous diffusion equation of fractional derivatives has been recognized as a master equation in nature for multidisciplinary applications (e.g., transport, relaxation, heat conduction) and is stated as [2]

$$\frac{\partial^\eta s}{\partial t^\eta} + \gamma(-\Delta)^{\mu/2} s = 0, \quad 0 < \eta \leq 1, \quad 0 < \mu \leq 2 \quad (1)$$

where s is the physical quantity of interest (e.g., temperature, density, pressure), γ the corresponding physical coefficient, $(-\Delta)^{\mu/2}$ represents the symmetric non-local positive definite fractional Laplacian. Note that η and μ are in general real numbers, and “fractional” in this letter is traditional misnomer in academic nomenclature. For the ideal solids and fluids (e.g., water, crystals), $\eta=1$ and $\mu=2$; while for complex fluids and solids, μ ranges from 0 to 2 and η is from 0 to 1.

Despite the fast growing research in fractional derivative modeling in recent decades, little has been achieved in the developing efficient numerical algorithms

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for fractional order equation systems because of difficulties in numerical solutions and less attention from scientific computing community. The respective numerical modeling, however, is very important and in high demand by both scientific and industrial practitioners, in particular, relating to biomechanics and geophysics. The fractional derivative equations are known to be computationally expensive, since they are non-local in nature and the numerical discretization will always produce a full matrix equation. The traditional numerical techniques such as finite element, finite difference, finite volume methods and step-by-step time integration schemes, which are originally designed for integer-order partial differential equations, can not well handle such challenging problems. Therefore, the innovative numerical techniques are imperative. On the other hand, numerical analysis theories of stability and error estimate in this regard are still largely an open question and require in-depth research. In addition, a careful design and implementation of numerical software for the fractional order equations can not be found in existing commercial codes.

For fractional time derivative equations, various standard finite-difference algorithms have been developed in the recent decade based respectively on the Grunwald-Letnikov definition, Riemann-Liouville definition, and Caputo definition of fractional time derivative. Our study finds that the algorithm based on the Caputo definition exhibits higher numerical accuracy and better stability. Recent developments such as explicit, implicit and Crank-Nicholson schemes are also noteworthy and have been discussed in this report, Through transforming fractional differential equations into Volterra integral equations, Volterra integral equation method can avoid singularity of calculation and significantly simplifies computing complexity.

On the other hand, little research has been done on the numerical approximation of fractional derivative in space. We observe that the finite difference methods based on the Grunwald-Letnikov definition appear more stable than those based on Riemann-Liouville definition. The shift Grunwald-Letnikov approach performs better than the former two, while the algorithm designed from the Riesz-Feller definition can better reflect the physical significance of the equation model.

This survey also discusses some recent algorithms such as Homotopy perturbation method, variational iteration method, Adomian decomposition method, random walk model based on statistical process. These methods, however, have their respective merits of high accuracy and stability and demerits of complex mathematics and tricky skills to define parameters of iterative function.

In the conclusions of this report, we will point out the following emerging techniques to tackle the bottleneck problems in the numerical solution of fractional derivative equations:

- Preconditioning techniques to reduce memory requirements and CPU time

for long history and large spatial domain problems such as fast multipole method, panel clustering method, etc;

- Global meshfree methods for fractional derivative equations with complex-shaped boundary such as radial basis function based on the new definition of fractional Laplacian [3];
- Development of the standard software code for fractional-order systems.

It is expected that the in-depth and systematic study in numerical fractional derivative equation systems will open new frontiers in scientific computing research.

References

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