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Determination of response spectrum of a linear time invariant gyroscopic system to random excitations using finite element method

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Summary

In this work, dynamic analysis of a compliant shaft while it rotates about its axis has been carried out using Finite Element Method. This system is linear time invariant gyroscopic system. The governing equations are formulated for the shaft supported at both ends. Both ends of the shaft are supported to allow rotations by constraining all degrees of freedom, except axial rotations. The shaft is modeled using beam elements, with six degrees of freedom at the nodes. The effect of rotation on displacements is analysed by assessing the total kinetic energy and potential energy. In this work dynamic response of the shaft while rotating, excited by random forces at various points are analysed along with free vibration analysis.

Introduction

In this paper Finite element method is used to analyse a rotating shaft subjected to random excitations, which are ergodic in nature. The shaft is discretised by beam elements. The beam elements are one-dimensional Euler- Bernoulli beam element [4]. The beam element has six degrees of freedom at the nodes- three translations and three rotations. The procedure includes formulation and solving of the fourth order differential equation for shaft bending vibrations with second order equation for axial vibrations and torsional vibrations [2]. The governing equations are

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right]+f(x,t)=m(x)\frac{\partial^2 w(x,t)}{\partial t^2} \quad 0<x<L \tag{1}$$

$$-\frac{\partial}{\partial x}\left[EA(x)\frac{\partial w(x,t)}{\partial x}\right]+f(x,t)=m(x)\frac{\partial^2 w(x,t)}{\partial t^2} \quad 0<x<L \tag{2}$$

In the Euler-Bernoulli beam theory, it is assumed that plane cross sections perpendicular to the axis of the beam, remain plane and perpendicular to the axis after deformation. In this theory the transverse deflections w of the beam is governed by the fourth order differential equation. Beam element is described in figure 1.

The formulation of stiffness and mass coefficient matrices includes the degrees of freedom, at the nodes. Bending deflections are contributed by displacement u_2 and rotation θ_3 at nodes superposed by that contributed by u_3, and θ_2 at nodes.

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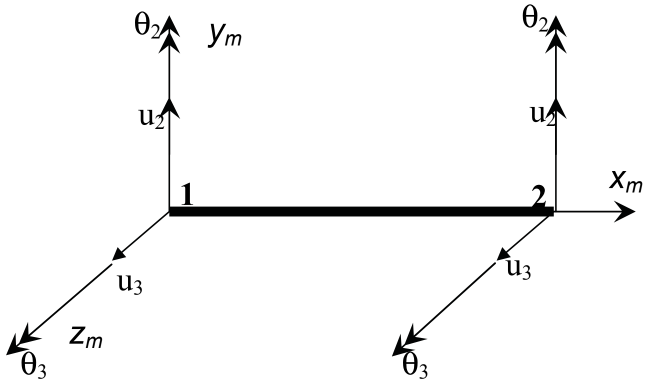

(a) One dimensional beam element showing
bending degrees of freedom (four) at each nodes

(b) One dimensional beam element showing
axial degrees of freedom (two) at each nodes

Figure 1: Beam element

Axial displacement denoted as u_1 and torsion denoted by θ_1. The x_m axis coincides with the centroid axis of the member and is positive from i to j. The y_m and z_m axes are chosen such that $x_m - y_m$ and $x_m - z_m$ planes are principal planes of bending. The member axes x_m, y_m, and z_m are parallel to the global axes and the degrees of freedom in global and local axes system are the same.

FEM procedure is as follows. The weak form of the governing equation is formulated, the domain is discretised into elements, interpolation functions are selected, finite element equations are formulated, elements are assembled for obtaining the matrix equations considering the boundary conditions and finally the matrix equations are solved to get the solutions. The weak form of the bending equations is as follows

$$\int_{x_e}^{x_{e+1}} v\left[\frac{d^2}{dx^2}\left(b\frac{d^2w}{dx^2}\right)-f\right]d_x=0 \tag{3}$$

Interpolation functions are obtained by considering the displacement function, which is to be solved as continuous with its second derivative also continuous. The second derivative describes the curvature. The following equations show the interpolation functions describing the bending (equation 4) followed by axial displacement and torsion (equation 5).

$$\begin{aligned} N_1 &= 1-3\frac{(x-x_e)^2}{(l_e)^2}+2\left(\frac{x-x_e}{l_e}\right) & N_3 &= 3\frac{(x-x_e)^2}{(l_e)^2}-2\left(\frac{x-x_e}{l_e}\right)^3 \\ N_2 &= -(x-x_e)\left(1-\frac{x-x_e}{l_e}\right)^2 & N_4 &= -(x-x_e)\left[\left(\frac{x-x_e}{l_e}\right)-\frac{x-x_e}{l_e}\right] \end{aligned} \tag{4}$$

$$\bar{N}_1 = 1-x/l_e \qquad \bar{N}_2 = x/l_e \tag{5}$$

Following equations show the formulation of stiffness and mass matrix [7]

$$K_{ij}^e = \int_{x_e}^{x_{e+1}} b\frac{d^2N_i}{dx^2}\cdot\frac{d^2N_j}{dx^2}dx \quad M_{ij}^e = \int_{x_e}^{x_{e+1}} \rho A N_i^e N_j^e dx$$

The general displacements at any point on the element can be calculated from the nodal displacement values. Here it can be seen that rotations about y_m and z_m adds to the deflections through z_m and y_m respectively. Refer figure 1.

Analysis

As the system is gyroscopic, the governing equations are formulated from basic principles for adopting in FEM. The displacement vector at each node of the element is denoted as

$$\underset{\sim}{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{6}$$

and the velocity vector as

$$\underset{\sim}{\dot{u}} = \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} \tag{7}$$

Global velocity vector of the node is calculated as

$$\underset{\sim}{\dot{U}} = \underset{\sim}{\dot{u}} + \underset{\sim}{\omega} \times \underset{\sim}{u} \tag{8}$$

Kinetic energy of the element with respect to global coordinates can be written as

$$T = \frac{1}{2}\int e dv \left(\underset{\sim}{\dot{U}} \bullet \underset{\sim}{\dot{U}}\right) = \frac{1}{2}\int \rho dv (\alpha+\beta+\gamma) \tag{9}$$

Where $\alpha = \overset{T}{\underset{\sim}{\dot{u}}}\underset{\sim}{u}$, $\beta = 2\overset{T}{\underset{\sim}{\dot{u}}}[R]\underset{\sim}{u}$ and $\gamma = \overset{T}{\underset{\sim}{u}}[R]^T[R]\underset{\sim}{u} = \overset{T}{\underset{\sim}{u}}[S]\underset{\sim}{u}$.

$$[R] = \begin{bmatrix} 0 & -\omega_3 & +\omega_2 \\ +\omega_3 & 0 & -\omega_1 \\ -\omega_2 & +\omega_1 & 0 \end{bmatrix} \quad \text{and} \quad [S] = \begin{bmatrix} \omega_2^2+\omega_3^2 & -\omega_1\omega_2 & -\omega_1\omega_3 \\ -\omega_1\omega_2 & \omega_1^2+\omega_3^2 & -\omega_2\omega_3 \\ -\omega_1\omega_3 & -\omega_2\omega_3 & \omega_1^2+\omega_2^2 \end{bmatrix}$$

Elemental Kinetic energy can be written as

$$T = \frac{1}{2}\overset{eT}{\underset{\sim}{\dot{u}}}[M_\circ^e]\dot{u}^e + \overset{eT}{\underset{\sim}{\dot{u}}}[M_1^e]\overset{e}{\underset{\sim}{u}} + \overset{eT}{\underset{\sim}{u}}[M_2^e]\overset{e}{\underset{\sim}{u}} \tag{10}$$

where $[M_\circ^e] = \int\limits_v N^T N\rho dv$, $[M_1^e] = \int\limits_v N^T[R]N\rho dv$ and $[M_2^e] = \int\limits_v N^T[S]N\rho dv$.

The energy dissipation is accounted from Raleigh's dissipation function given as

$$\Re = 1/2\overset{eT}{\underset{\sim}{\dot{u}}}[C^e]\overset{eT}{\underset{\sim}{\dot{u}}} \tag{11}$$

where $[C^e] = \alpha_0[M^e] + \beta_0[K^e]$

Potential energy is given as

$$V = 1/2\overset{eT}{\underset{\sim}{u}}[K^e]\overset{e}{\underset{\sim}{u}} \tag{12}$$

Governing equation is obtained using Lagragian formulation.

$$M\underset{\sim}{\ddot{u}} + \left(M_1 - M_1^T\right)\underset{\sim}{\dot{u}} + C\underset{\sim}{\dot{u}} + (K - M_2)\underset{\sim}{u} = Q \tag{13}$$

which is modified as

$$M\underset{\sim}{\ddot{u}}(t) + (C+G)\underset{\sim}{\dot{u}}(t) + (K+H)\underset{\sim}{u}(t) = Q \tag{14}$$

where $[G]$ is the gyroscopic matrix and $[H]$ the circulatory matrix. With this formulation the elemental mass matrix, stiffness matrix, damping matrix, gyroscopic matrix and circulatory matrix are obtained. Here the damping is treated as structural damping, where the coefficient of damping is taken as linear combination of mass and stiffness parameters. The proportional parameters α_0 and β_0 are obtained form the material properties.

To construct the overall system equations the matrices formed for elements are assembled. The mathematical statement for assembly of stiffness is done as follows

$$[K] = \sum_{e=1}^{E}[K]^{(e)}, \quad [M] = \sum_{e=1}^{E}[M]^{(e)}, \quad [C] = \sum_{e=1}^{E}[C]^{(e)},$$

$$[G] = \sum_{e=1}^{E}[G]^{(e)}, \quad [H] = \sum_{e=1}^{E}[H]^{(e)}$$

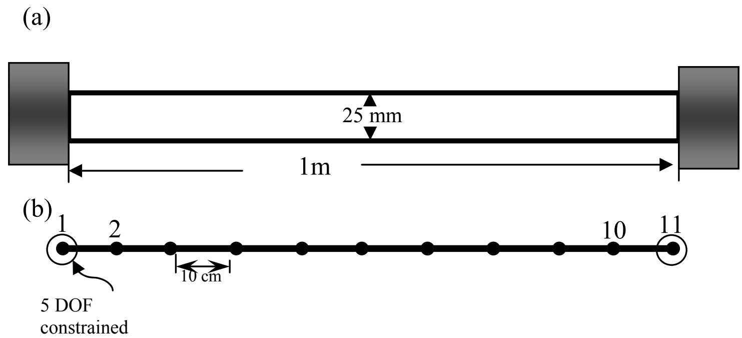

Figure 2: (a) Domain of analysis (b) FEM discretisation by beam elements

Dynamic analysis (Free vibration)

Finite element analysis in the previous section leads to the formulation of equations of motion as shown by equation 15. In free vibration the load vector is zero. For obtaining the natural frequencies and modes of vibration a free vibration analysis is carried out. In this analysis applied forces are assumed to be null and the system does not dissipates energy

$$M \underset{\sim}{\ddot{u}}(t) + (C+G)\, \underset{\sim}{\dot{u}}(t) + (K+H)\, \underset{\sim}{u}(t) = 0 \tag{15}$$

Dynamic Analysis (Random Excitation)

For the analysis a shaft of 1meter length and diameter 25.4mm is considered. The material of the shaft is assumed as C 40 steel. The shaft is assumed to be fixed at the ends with axial rotation degree of freedom is left free. The analysis is carried out using Matlab and the results are plotted as mode shapes. The first four mode shapes are plotted. Figure 2 shows the domain analysed. The mode shapes are plotted by formulating the problem as an Eigen value problem and then finding all Eigen vectors

In order to carry out the forced vibration analysis the equation 14 has to be considered completely. In this section the analysis is pertaining to find the response due to random excitation. The generalised force {Q} is considered to be an ergodic random process. Using the mass, stiffness and damping matrices the complex frequency response matrix is obtained. Theory of random vibration depicts that the spectral density of response can be obtained from spectral density of excitation using the complex frequency response function.[3]. Hence the analysis is carried out in frequency domain.

$$S_u(\omega) = H(-\omega)\, H(\omega)\, S_f(\omega) \tag{16}$$

where $S_u(\omega)$ is the spectral density of response and $S_f(\omega)$is the spectral density of excitation and $H(\omega)$ is the complex frequency response function. For a multi degree freedom system the equation takes the form,

$$[S]_u = [H]^T [S]_f [H] \tag{17}$$

where $[H] = \left[[K+H] + [C+G] - \omega^2 [M]\right]^{-1}$ is the frequency response matrix for a non-natural system. The system state matrices are formed. For time history the following procedure is done. The transfer function of the multi input multi output system is obtained. The time history of excitation is transformed to Laplace domain and is input to the system. The system response obtained is transformed back to time domain and results are plotted. This analysis shows that any non-natural system can be analysed by this formulation using finite element method. The formation of the gyroscopic matrix circulatory matrix for non-natural system is made simple using the finite element technique.

Results and Discussion

The analysis is carried using Matlab programming. The complex frequency response matrix for the multi degree freedom system is calculated. The mode shapes for first five frequencies are shown in figures 3 to 6. The results obtained are in *good agreement with actual modes of vibration obtained by closed form solutions*. The transverse modes of vibration are harmonics of the form $(1 - \cos(x))$. The results plotted are the same harmonics. The results shown are normalised with respect to the maximum value.

White noise of unit amplitude [1] is considered as the input excitation (Figure 7). the shaft is excited at the midpoint. Results are tabulated and responses at various locations of the shaft are plotted. The results show the response function in frequency domain and also resonance state. The program developed can extract response spectrum at any node, through any degree of freedom. the spectral density of response are shown in figures 8 9 and 10. The time history of response is also analysed and the results are plotted (figure 11). In this analysis the time history is obtained using state variable approach. A comparison is also made (figure 7) in the result, to show the difference between the response at two different points, for the same excitation at the nodes.

Conclusion

Finite element method based technique is developed for dynamic analysis of Gyroscopic system .The numerical method proposed for dynamic analysis of flexible rotating members can be applied to any kind of members with any geometry. This method uses basic dynamic principles. The response spectrum for the ergodic excitations are obtained for the system considered. free vibration analysis has also been carried out. Results are obtained and plotted. As this is a fundamental work, it can be used for applied dynamic analysis.

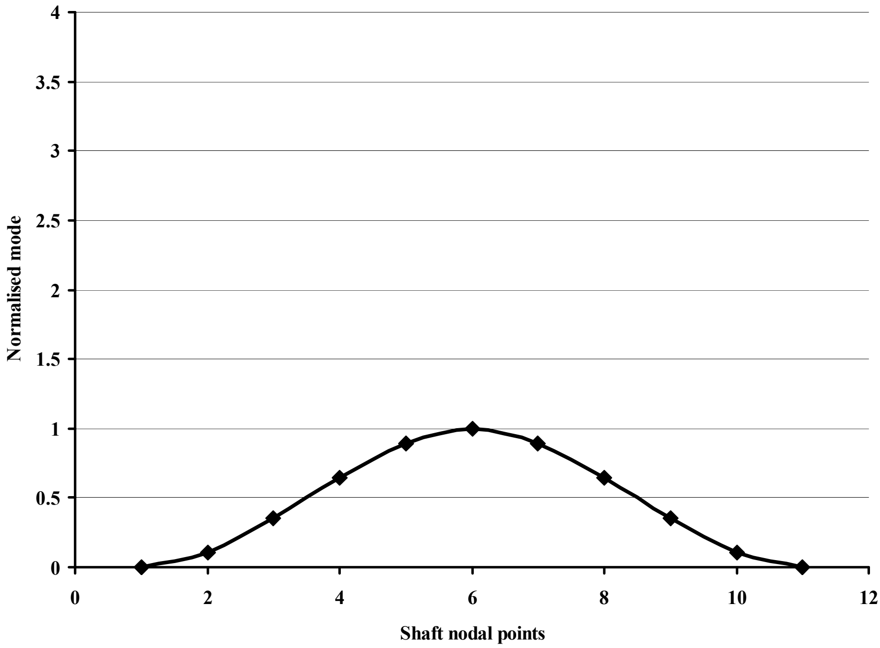

Figure 3: First mode of transverse vibration (normalised); ω_n=32 Hz

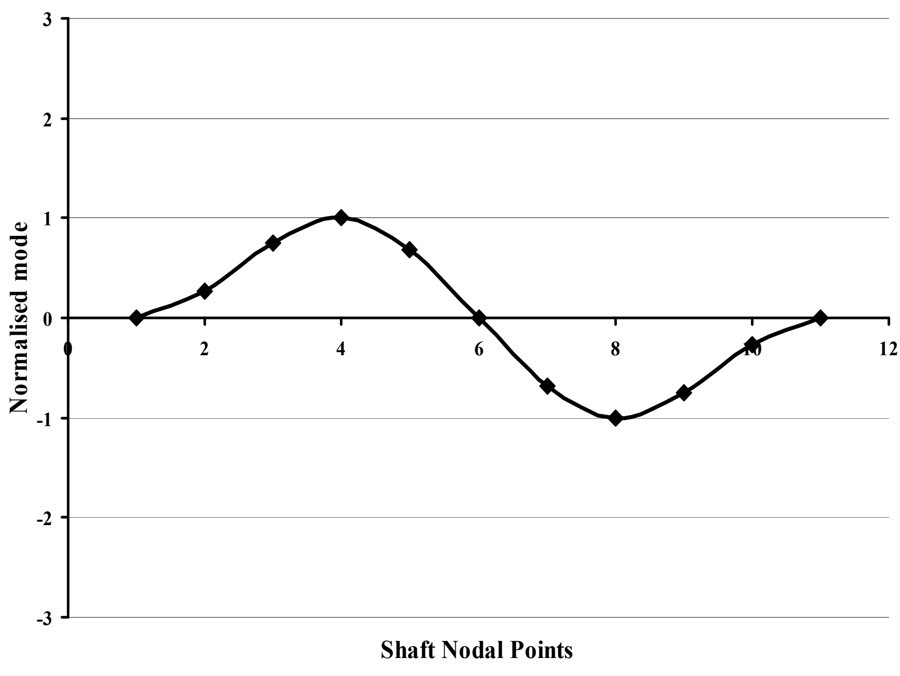

Figure 4: Second mode of transverse vibration (normalised); ω_n= 67 Hz

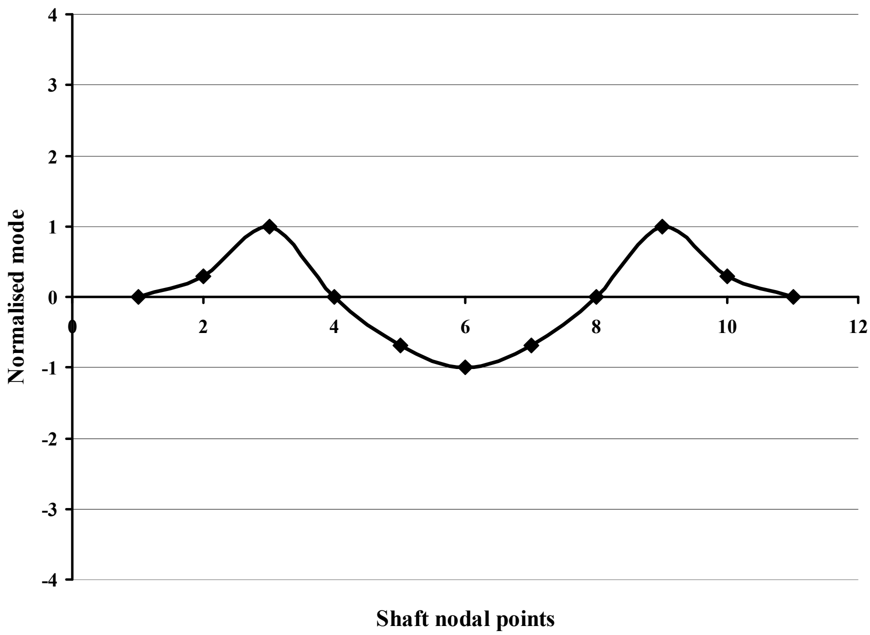

Figure 5: Third mode of transverse vibration (normalised); ω_n=197 Hz

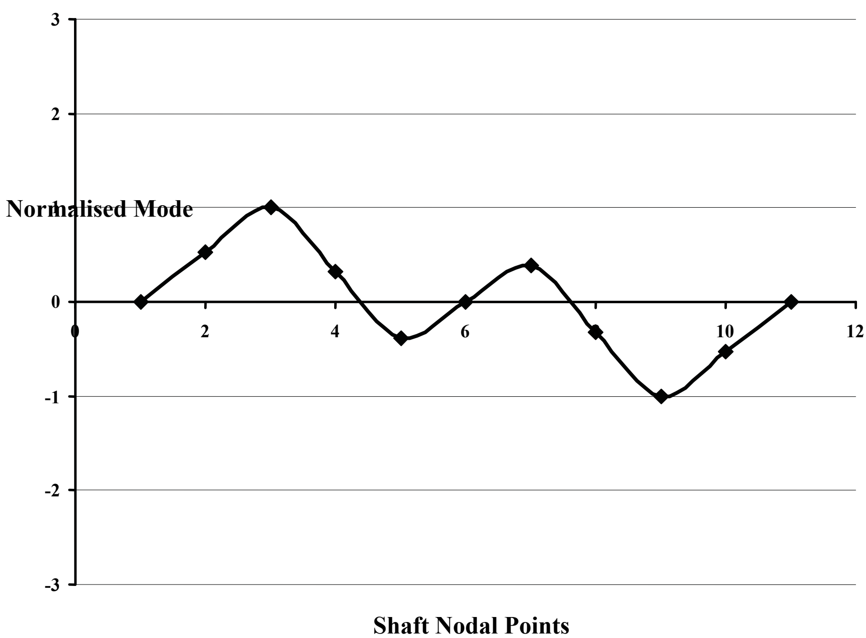

Figure 6: Forth mode of transverse vibration (normalised); ω_n=288 Hz

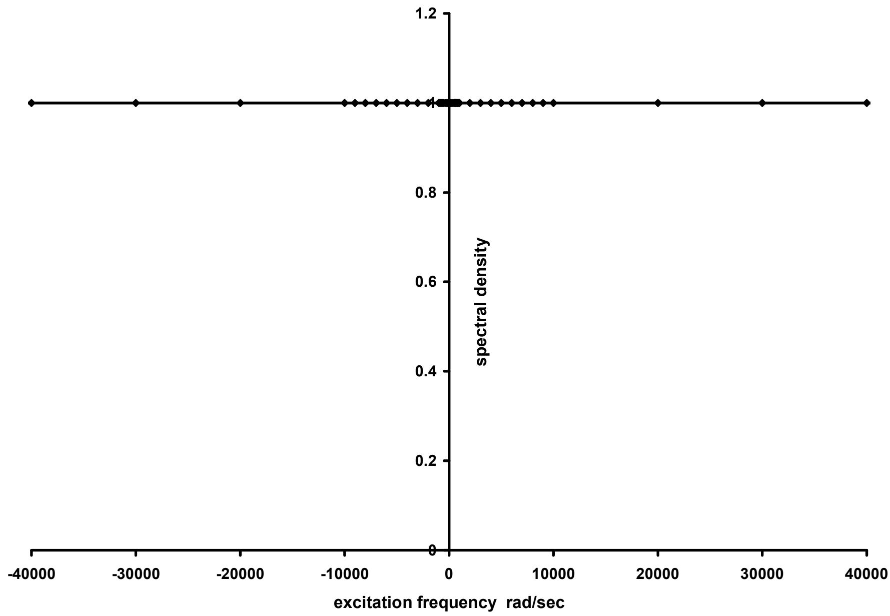

Figure 7: Input unit white noise excitation given at all transverse nodes

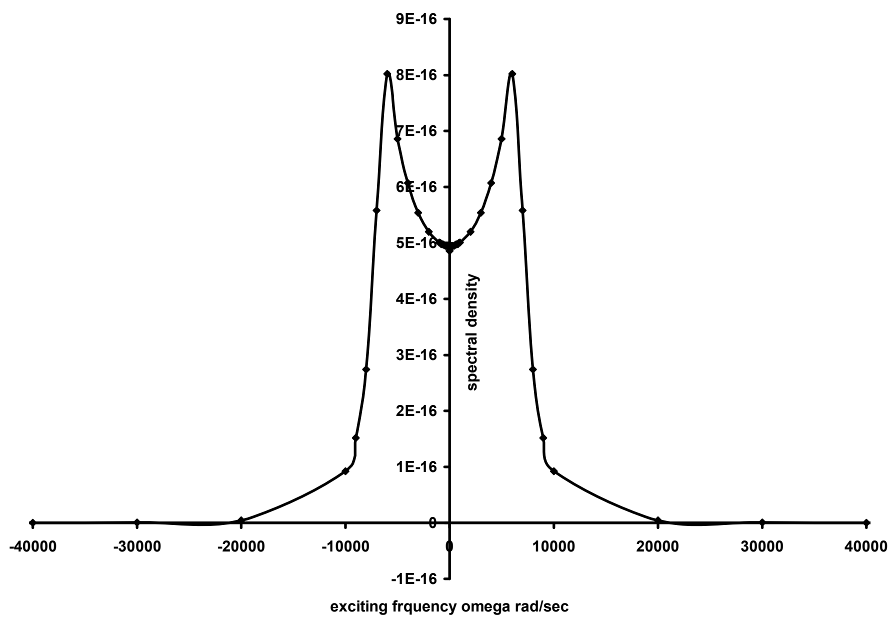

Figure 8: Spectral density of response at node 3 (transverse) (rpm of shaft =100rad/sec)

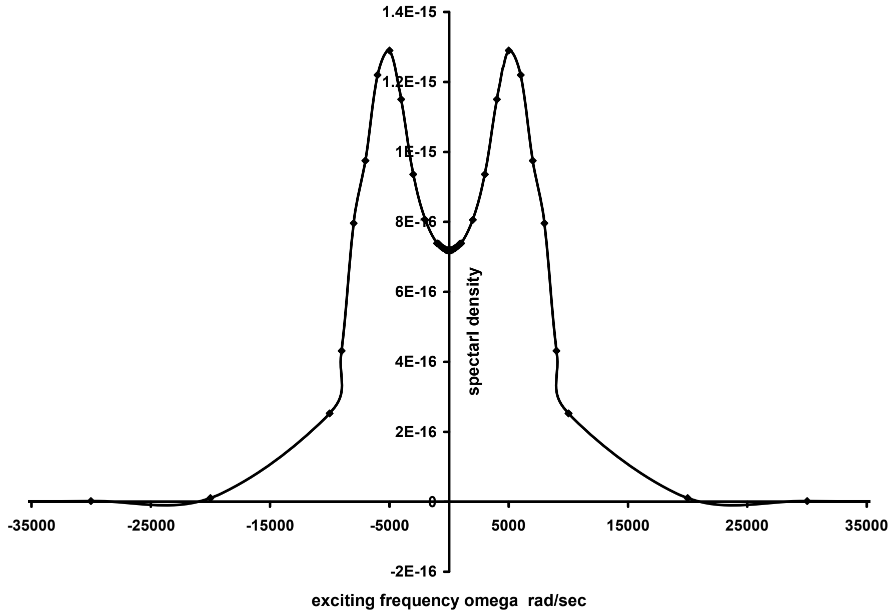

Figure 9: Spectral density of response at node 9 (transverse) (rpm of shaft =100rad/sec)

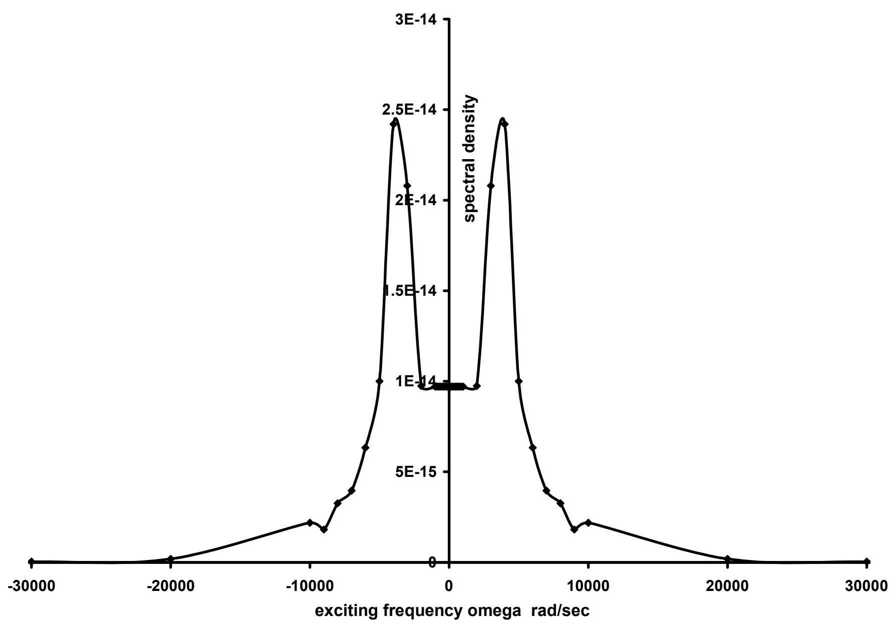

Figure 10: Spectral density of response at node 27 (transverse, middle of the shaft) (rpm of shaft =100rad/sec)

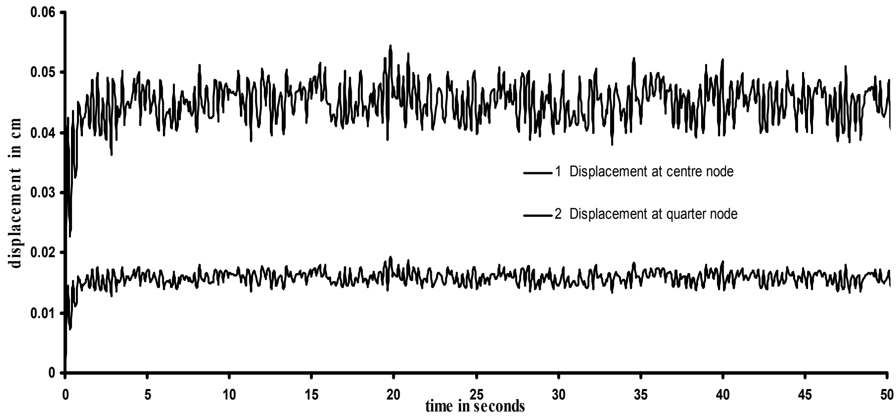

Figure 11: Displacements at middle node and node at quarter length (Time History) for speed 100 rad/s

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Nomenclature

$[C]$	Damping matrix
$[D]$	Transition matrix
$[G]$	Gyroscopic matrix
$[K]$	Stiffness matrix
$[M]$	Mass matrix
$[S]$	Spectral density matrix
$\{Q\}$	Generalised forces
$\langle f, g\rangle$	Inner product of f and g
$A(x)$	Cross sectional area
C	Damping coefficient
E	Young's modulus N/mm^2
$f(x)$	Continuous function
$f[x]$	Discrete function
k	Stiffness
$L(s)$	Laplace transformation
L_e	Evaluation length
m	Mass
N_i, N_j	Interpolation functions
$S(\omega)$	Spectral density function
u	Generalised local displacements
U_i	Global displacements

V Potential energy
x, y Coordinate directions normal to the shaft axis
z Coordinate direction parallel to the shaft axis

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