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An Inverse Problem for the General Kinetic Equation and a Numerical Method

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Summary

This paper has two purposes. The first is to prove the existence and uniqueness of the solution of an inverse problem for the general linear kinetic equation with a scattering term. The second one is to develop a numerical approximation method for the solution of this inverse problem for two dimensional case using finite difference method.

keywords: Inverse Problem; Kinetic Equation; Solvability of the Problem; Finite Difference Method

Formulation of the Problem

We consider the linear kinetic equation

$$Lu \equiv \{u,H\} + I_1(u) = \lambda(x,v), \tag{1}$$

$$\{u,H\} = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial v_i}\frac{\partial u}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial u}{\partial v_i}\right), I_1(u) = \int_G K\left(x,v,v'\right) u\left(x,v'\right) dv'$$

in the domain

$$\Omega = \left\{(x,v) : x \in D \subset \mathbb{R}^n,\ v \in G \subset \mathbb{R}^n,\ n \geq 1\right\}$$

where the boundaries ∂D, $\partial G \in C^2$, $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, $\Gamma_1 = \partial D \times G$, $\Gamma_2 = D \times \partial G$ and $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ are the closures of Γ_1, Γ_2, respectively. $H(x,v)$ is the Hamiltonian, $K(x,v,v')$ is a given function called scattering kernel and $\lambda(x,v)$ is a source function satisfying the equation

$$\left\langle \lambda, \widehat{L}\eta \right\rangle = 0, \widehat{L} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial v_i} \tag{2}$$

for any $\eta \in H_{1,2}(\Omega)$ whose trace on $\partial\Omega$ is zero. Here $\langle .,. \rangle$ is the scalar product in $L_2(\Omega)$ and $H_{1,2}(\Omega)$ is the set of all real-valued functions $u(x,v) \in L_2(\Omega)$ that have generalized derivatives u_{x_i}, u_{v_i}, $u_{x_i v_j}$, $u_{v_i v_j}$ $(i,j = 1,2,...,n)$, which belong to $L_2(\Omega)$. The standard spaces $C^m(G)$, $C_0^\infty(G)$ and $H^k(G)$ are described in detail, for example, in [9, 10].

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Problem 1. *Determine the functions* $u(x,v)$ *and* $\lambda(x,v)$ *defined in* Ω *that satisfy equation* (1)*, assuming that the Hamiltonian* $H(x,v) \in C^2(\Omega)$*,* $K(x,v,v') \in C^1\left(\overline{\Omega}\right)$ *are given and the trace of the solution of equation* (1) *on the boundary* $\partial\Omega$ *is known:* $u|_{\partial\Omega} = u_0$.

In this paper, we prove the solvability of Problem 1 and develope a numerical approximation method for the solution of two dimensional inverse problem. To demonstrate the feasibility of the given method, some numerical experiments are performed in the last section of the paper.

Kinetic equations are widely used for qualitative and quantitative description of physical, chemical, biological, and other kinds of processes on a microscopic scale. They are often referred to as master equations since they play an important role in the theory of substance motion under the action of forces, in particular, irreversible processes. Equation (1) is extensively used in plasma physics and astrophysics [1, 8]. In applications, u represents the number (or the mass) of particles in the unit volume element of the phase space in the neighbourhood of the point (x,v), $\nabla_x H$ is the force acting on a particle. Inverse problems for kinetic equations are important both from theoretical and practical points of view. Interesting results in this field are presented in [2, 6].

Remark 1. *If condition* (2) *is not imposed on* λ*, Problem* 1 *will have infinitely many solutions. In many classical cases, the main difficulty in studying the inverse problems for kinetic equations lies in their over-determinacy. This over-determinacy is due to the dependence of* λ *only on* x *(see [2, 7, 12]). In [2], a genereal scheme is presented for proving the solvability of these problems: It's assumed that the unknown function in the problem depends not only on the space variables* x *but also on the direction* v *in a specific way, that is,* $\widehat{L}\lambda = 0$.

Remark 2. *The solvability of Problem* 1 *depends essentially on the geometry of the domain* Ω*. More precisely, it is important that* Ω *can be represented in the form of the direct product of two domains* D *and* G *(see [2], p. 41).*

Solvability of the Problem

Let $\{w_1, w_2, \ldots\} \subset \widetilde{C}_0^3(\Omega) = \left\{\varphi : \varphi \in C^3(\Omega), \varphi = 0 \text{ on } \partial\Omega\right\}$ be an orthonormal set in $L_2(\Omega)$ and we suppose that the linear span of this set is everywhere dense in $L_2(\Omega)$. We denote the orthogonal projector of $L_2(\Omega)$ onto $\mathscr{M}_n$ by $\mathscr{P}_n$, where $\mathscr{M}_n$ is the linear span of $\{w_1, w_2, \ldots w_n\}$. The set of all functions u with the following two properties is denoted by $\Gamma(A)$:

i. For any $u \in L_2(\Omega)$ there exists a function $\mathscr{F} \in L_2(\Omega)$ such that for all $\varphi \in C_0^\infty(\Omega)$, $\langle u, A^*\varphi\rangle = \langle \mathscr{F}, \varphi\rangle$ and $Au = \mathscr{F}$, where $Au = \widehat{L}Lu$ and A^* is the operator which is conjugate to A in the sense of Lagrange.

ii. There exists a sequence $\{u_k\} \subset \widetilde{C}_0^3$ such that $u_k \to u$ in $L_2(\Omega)$ and $\langle Au_k, u_k\rangle \to \langle Au, u\rangle$ as $k \to \infty$.

Theorem 1. *Suppose that $H \in C^2\left(\overline{\Omega}\right)$ and the inequalities:*

$$\sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial v_i \partial v_j}\xi^i\xi^j \geq \alpha\,|\xi|^2, \ \sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial x_i \partial x_j}\xi^i\xi^j \leq 0, \ \left(\alpha - \frac{1}{2}(1+L_0)\right) > 0 \tag{3}$$

hold for all $\xi \in \mathbb{R}^n$, $(x,v) \in \overline{\Omega}$. In (3), α is a positive number, $L_0 = K_0\,(mesG)^2\,C_0$ where mesG is Lebesgue measure of G, $K_0 = \max\limits_{(x,v)\in\overline{\Omega}}\left\{K_{v_j}^2\right\}$ and C_0 is a constant occurred by virtue of Steklov inequality. Then Problem 1 has at most one solution (u,λ) such that $u \in \Gamma(A)$ and $\lambda \in L_2(\Omega)$.

Proof. The proof of Theorem 1 is similar to Theorem 2.2.1 on p. 60 from [2]. But, due to the scattered term, this proof requires non-trivial modifications. Let (u,λ) be a solution to Problem 1 such that $u = 0$ on $\partial\Omega$ and $u \in \Gamma(A)$. Equation (1) and condition (2) imply $Au = 0$. Since $u \in \Gamma(A)$, there exists a sequence $\{u_k\} \subset \widetilde{C}_0^3$ such that $u_k \to u$ in $L_2(\Omega)$ and $\langle Au_k, u_k\rangle \to 0$ as $k \to \infty$. Observing that $u_k = 0$ on $\partial\Omega$, we have

$$-\langle Au_k, u_k\rangle = \sum_{i=1}^{n}\left\langle \frac{\partial}{\partial v_i}(Lu_k), u_{k_{x_i}}\right\rangle. \tag{4}$$

For $lu \equiv \{u,H\}$,

$$\sum_{j=1}^{n} \frac{\partial u_k}{\partial x_j}\frac{\partial}{\partial v_j}(lu_k) = \frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial^2 H}{\partial v_i \partial v_j}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} - \frac{\partial^2 H}{\partial x_i \partial x_j}\frac{\partial u_k}{\partial v_i}\frac{\partial u_k}{\partial v_j}\right)$$

$$+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial v_j}\left[\frac{\partial u_k}{\partial x_j}\left(\frac{\partial u_k}{\partial x_i}\frac{\partial H}{\partial v_i} - \frac{\partial u_k}{\partial v_i}\frac{\partial H}{\partial x_i}\right)\right] + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial v_i}\frac{\partial u_k}{\partial x_j}\frac{\partial u_k}{\partial v_j}\right)$$
$$-\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left[\frac{\partial u_k}{\partial v_j}\left(\frac{\partial u_k}{\partial x_i}\frac{\partial H}{\partial v_i} - \frac{\partial u_k}{\partial v_i}\frac{\partial H}{\partial x_i}\right)\right] - \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial v_i}\left(\frac{\partial H}{\partial x_i}\frac{\partial u_k}{\partial x_j}\frac{\partial u_k}{\partial v_j}\right) \tag{5}$$

If the geometry of the domain Ω and the condition $u_k = 0$ on $\partial\Omega$ are taken into account, then from (5) we obtain

$$-\langle Au_k, u_k\rangle = J(u_k) + \sum_{j=1}^{n}\left\langle \frac{\partial}{\partial v_j}(I_1 u_k), u_{k_{x_j}}\right\rangle \tag{6}$$

where

$$J(u_k) \equiv \frac{1}{2}\sum_{i,j=1}^{n}\int_{\Omega}\left(\frac{\partial^2 H}{\partial v_i \partial v_j}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} - \frac{\partial^2 H}{\partial x_i \partial x_j}\frac{\partial u_k}{\partial v_i}\frac{\partial u_k}{\partial v_j}\right)d\Omega.$$

We now estimate the second term on the right hand side of (6). Using the Cauchy-Schwarz inequality and the condition $u_k|_{\partial\Omega} = 0$, we have

$$\sum_{j=1}^{n}\left\langle \frac{\partial}{\partial v_j}(I_1 u_k), u_{k_{x_j}} \right\rangle \leq \frac{1}{2}\sum_{j=1}^{n}\int_{\Omega} u_{k_{x_j}}^2 d\Omega + \sum_{j=1}^{n}\frac{K_0}{2}(mesG)^2 C_0 \int_{\Omega} u_{k_{x_j}}^2 d\Omega \quad (7)$$

where K_0, L_0, C_0 and $mesG$ are given in the satement of the theorem. Thus from (3), (6) and (7), we obtain the following inequality

$$\begin{aligned} J(u_k) + \sum_{j=1}^{n}\left\langle \frac{\partial}{\partial v_j}(I_1 u_k), u_{k_{x_j}} \right\rangle &\geq \alpha \sum_{j=1}^{n}\int_{\Omega} u_{k_{x_j}}^2 d\Omega + \sum_{j=1}^{n}\left\langle \frac{\partial}{\partial v_j}(I_1 u_k), u_{k_{x_j}} \right\rangle \\ &\geq \alpha \sum_{j=1}^{n}\int_{\Omega} u_{k_{x_j}}^2 d\Omega - \frac{1}{2}(1+L_0)\int_{\Omega} u_{k_{x_j}}^2 d\Omega \\ &= \left(\alpha - \frac{1}{2}(1+L_0)\right)\sum_{j=1}^{n}\int_{\Omega} u_{k_{x_j}}^2 d\Omega \end{aligned} \quad (8)$$

and using definition of $\Gamma(A)$ we have $\int_{\Omega} |\nabla_x u|^2 d\Omega \leq 0$ where $\nabla_x u = (u_{x_1}, \ldots, u_{x_n})$. Since $u = 0$ on $\partial\Omega$, it follows that $u = 0$ in Ω. Then (1) implies $\lambda(x, v) = 0$. Hence uniqueness of the solution is proven. □

Problem 2. *Determine the pair* (u,λ) *from the equation*

$$Lu = \lambda + F \quad (9)$$

provided that $F \in H^2(\Omega)$*, the trace of the solution u on the boundary* $\partial\Omega$ *is zero and* λ *satisfies condition* (2).

Problem 1 can be reduced to Problem 2, (see [2], p. 65).

Theorem 2. *Assume* $H \in C^2(\overline{\Omega})$ *and the following inequalities hold for all* $(x, v) \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$:

$$\sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial v_i \partial v_j}\xi^i\xi^j \geq \alpha_1 |\xi|^2, \quad \sum_{i,j=1}^{n}\frac{\partial^2 H}{\partial x_i \partial x_j}\xi^i\xi^j \leq -\alpha_2 |\xi|^2 \quad (10)$$

where α_1 *and* α_2 *are some positive numbers and* $F \in H^2(\Omega)$*. Then there exists a solution* (u,λ) *of Problem 2 such that* $u \in \Gamma(A)$, $u \in H^1(\Omega)$, $\lambda \in L_2(\Omega)$.

Proof. We will utilize the proof of Theorem 2.2.2 on p. 60 from [2] and take into account the scattering term. Applying the operator $\widehat{L}$ on both sides of (9), the following auxiliary problem

$$Au = \mathscr{F} \tag{11}$$

$$u|_{\partial\Omega} = 0 \tag{12}$$

is obtained where $\mathscr{F} = \widehat{L}F$. An approximate solution to the Problem $(11)-(12)$ is sought in the form $u_N = \sum_{i=1}^{N} \alpha_{N_i} w_i$, $\alpha_N = (\alpha_{N_1}, \alpha_{N_2}, ..., \alpha_{N_N})$ with the help of the following relations:

$$\langle Au_N - \mathscr{F}, w_i \rangle = 0, \qquad i = 1,2,...,N. \tag{13}$$

Equalities (13) form a system of linear algebraic equations for the vectors α_N. Let's multiply ith equation of the homogeneous system ($\mathscr{F} = 0$) by $-2\alpha_{N_i}$ and sum from 1 to N with respect to i, then $-2\langle Au_N, u_N \rangle = 0$ is obtained. If the following identity is considered

$$-\langle Au_N, u_N \rangle = J(u_N) + \sum_{j=1}^{n} \left\langle \frac{\partial}{\partial v_j}(I_1 u_N), u_{N_{x_j}} \right\rangle$$

then the assumptions of Theorem 2 imply $\nabla u_N = 0$, $\nabla u_N = \left(u_{N_{x_1}}, ..., u_{N_{x_n}}, u_{N_{v_1}}, ..., u_{N_{v_n}}\right)$ and due to the conditions $u_N = 0$ on Γ_1 and $u_N \in \widetilde{C}_0^3(\Omega)$, we have $u_N = 0$ in Ω. Since the system $\{w_i\}$ is linearly independent, we obtain $\alpha_{N_i} = 0$, $i = 1,2,...,N$. The homogeneous version of system (13) has only trivial solution and thus, system (13) has a unique solution $\alpha_N = (\alpha_{N_i})$, $i = 1,...,N$ for any function $F \in H^2(\Omega)$.

Now we estimate the solution u_N in terms of F. We multiply the ith equation of the system by $-2\alpha_{N_i}$ and sum from 1 to N with respect to i. Since $\mathscr{F} = \widehat{L}F$,

$$-2\langle Au_N, u_N \rangle = -2\left\langle \widehat{L}F, u_N \right\rangle \tag{14}$$

is obtaned. Observing that $u_N = 0$ on $\partial\Omega$ and transferring derivatives with respect to x_i on the function u_N, the right-hand side of (14) can be estimated as

$$-2\left\langle \widehat{L}F, u_N \right\rangle \leq \beta \int_{\Omega} |\nabla_v F|^2 \, d\Omega + \beta^{-1} \int_{\Omega} |\nabla_x u_N|^2 \, d\Omega$$

where $\beta^{-1} < \alpha_1$ and $\nabla_v F = (F_{v_1}, ..., F_{v_n})$. In the proof of Theorem 1, we showed that $\langle Au_N, u_N \rangle$ is equal to

$$J(u_N) + \sum_{j=1}^{n} \int_{\Omega} u_{N_{x_j}} \int_{G} K_{v_j} u_N dv' d\Omega.$$

Then using (7), (10) and (14) we have

$$\alpha_2 \int_\Omega |\nabla_v u_N|^2 \, d\Omega + \alpha_3 \int_\Omega |\nabla_x u_N|^2 \, d\Omega \le \beta \int_\Omega |\nabla_v F|^2 \, d\Omega + \beta^{-1} \int_\Omega |\nabla_x u_N|^2 \, d\Omega$$

where $\alpha_3 = \alpha_1 - \frac{1}{2}(1+L_0)$. Recalling that Ω is bounded and $u_N = 0$ on Γ_1, the last inequality implies

$$\|u_N\|_{\overset{\circ}{H}^1(\Omega)} \le C \| |\nabla_v F| \|_{L_2(\Omega)} \tag{15}$$

where the constant $C > 0$ does not depend on N. Since $\overset{\circ}{H}^1(\Omega)$ is a Hilbert space, there exists a subsequence in this set, denoted again by $\{u_N\}$, converges weakly to a certain function $u \in \overset{\circ}{H}^1(\Omega)$ and $\|u\|_{\overset{\circ}{H}^1(\Omega)} \le \underline{\lim}_{N\to\infty} \|u_N\|_{\overset{\circ}{H}^1(\Omega)} \le C \| |\nabla_v F| \|_{L_2(\Omega)}$ holds. Transferring the operator $\widehat{L}$ to w_i in (13) and passing to the limit as $N \to \infty$ yield to

$$\left\langle Lu - F, \widehat{L}\eta \right\rangle = 0 \tag{16}$$

for any $\eta \in \overset{\circ}{H}_{1,2}(\Omega)$. Setting $\lambda = Lu - F$, we see that λ satisfies the condition (2) for any $\eta \in \overset{\circ}{H}_{1,2}(\Omega)$ and using (15) we obtain

$$\|\lambda\|_{L_2(\Omega)} \le C \|\nabla_v F\|_{L_2(\Omega)} + \|F\|_{L_2(\Omega)}. \tag{17}$$

In expression (17), C stands for different constants that depend only on the given functions and the measure of the domain D. Thus we have found a solution (u, λ) to Problem 2, where $u \in \overset{\circ}{H}^1(\Omega)$ and $\lambda \in L_2(\Omega)$. Now it will be proven that $u \in \Gamma(A)$. Since $u \in L_2(\Omega)$, $F \in H^2(\Omega)$ and $\mathscr{F} = \widehat{L}F$, from (16) it follows that $\mathscr{F} = Au \in L_2(\Omega)$ in the generalized sense. Indeed, for any $\eta \in C_0^\infty(\Omega)$ we have

$$\langle u, A^*\eta \rangle = \left\langle u, \left(\widehat{L}L\right)^* \eta \right\rangle = \left\langle Lu, \widehat{L}\eta \right\rangle = \left\langle F, \widehat{L}\eta \right\rangle = \langle \mathscr{F}, \eta \rangle.$$

To complete the proof, it remains to show the convergence $\langle Au_N, u_N \rangle \to \langle Au, u \rangle$ as $N \to \infty$. From (13), it follows that $\mathscr{P}_N A u_N = \mathscr{P}_N \mathscr{F}$. Since $\mathscr{P}_N$ is an orthogonal projector onto $\mathscr{M}_n$, $\mathscr{P}_N \mathscr{F}$ strongly converges to $\mathscr{F}$ in $L_2(\Omega)$ as $N \to \infty$, i.e., $\mathscr{P}_N A u_N \to \mathscr{F} = Au$ strongly in $L_2(\Omega)$ as $N \to \infty$. Then, $\langle \mathscr{P}_N A u_N, u_N \rangle \to \langle Au, u \rangle$ as $N \to \infty$ because $\{u_N\}$ weakly converges to u in $L_2(\Omega)$ as $N \to \infty$. By the definition of $\mathscr{P}_N$ and u_N (since the operator $\mathscr{P}_N$ is self adjoint in L_2), $\langle Au_N, u_N \rangle = \langle Au_N, \mathscr{P}_N u_N \rangle = \langle \mathscr{P}_N A u_N, u_N \rangle$. Hence $\langle Au_N, u_N \rangle \to \langle Au, u \rangle$ as $N \to \infty$, which completes the proof. □

The Finite Difference Method (FDM)

Now we concern with the construction of finite difference approximation for the following 2-dimensional inverse problem: Find (u,λ) from the relations

$$\begin{aligned} H_v(x,v)\,u_x(x,v) - H_x(x,v)\,u_v(x,v) &= \lambda(x,v) \\ u(x,v)|_{\partial\Omega} &= u_0(x,v) \\ \widehat{L}\lambda &= 0 \end{aligned} \tag{18}$$

where $\Omega = \{(x,v)|\ x\in(a,b)\subset\mathbb{R},\ v\in(c,d)\subset\mathbb{R}\}$. By applying operator $\widehat{L}$ to both sides of the equation (18), the following auxiliary Dirichlet boundary value problem for third order partial differential equation is obtained:

$$Au \equiv u_{xvx}H_v - u_{vvx}H_x + u_{xx}H_{vv} - u_{vv}H_{xx} + u_{xv}H_{vx} - u_{vx}H_{xv} + u_xH_{vvx} - u_vH_{xvx} = 0 \tag{19}$$

$$u|_{\partial\Omega} = u_0. \tag{20}$$

Using the central finite difference formulas in (19), we obtain the following system of simultaneous algebraic nodal equations:

$$\begin{aligned} & (-k_1+k_2)\,\tilde{u}_{i-1,j-1} + (2k_1-k_4+k_6)\,\tilde{u}_{i,j-1} + (-k_1-k_2)\,\tilde{u}_{i+1,j-1} \\ & + (-2k_2+k_3-k_5)\,\tilde{u}_{i-1,j} + (-2k_3+2k_4)\,\tilde{u}_{i,j} + (2k_2+k_3+k_5)\,\tilde{u}_{i+1,j} \\ & + (k_1+k_2)\,\tilde{u}_{i-1,j+1} + (-2k_1-k_4-k_6)\,\tilde{u}_{i,j+1} + (k_1-k_2)\,\tilde{u}_{i+1,j+1} \\ = \ & 0,\ i=1,\dots,I,\ j=1,\dots,J \end{aligned} \tag{21}$$

where $I,\ J$ are positive integers, $\Delta x = (b-a)/(I+1)$ and $\Delta v = (d-c)/(J+1)$ are step sizes in the directions $x,\ v$, respectively and $\tilde{u}_{i,j}$ is the finite difference approximation for the solution $u(x_i,v_j) = u(a+i\Delta x, c+j\Delta v)$,

$$\begin{aligned} k_1 &= \frac{h_{i,j+1}-h_{i,j-1}}{4(\Delta x)^2(\Delta v)^2},\ k_2 = \frac{h_{i+1,j}-h_{i-1,j}}{4(\Delta x)^2(\Delta v)^2}, \\ k_3 &= \frac{h_{i,j+1}-2h_{i,j}+h_{i,j-1}}{(\Delta x)^2(\Delta v)^2},\ k_4 = \frac{h_{i+1,j}-2h_{i,j}+h_{i-1,j}}{(\Delta x)^2(\Delta v)^2}, \\ k_5 &= \frac{h_{i+1,j+1}-2h_{i+1,j}+h_{i+1,j-1}-h_{i-1,j+1}+2h_{i-1,j}-h_{i-1,j-1}}{4(\Delta x)^2(\Delta v)^2}, \\ k_6 &= \frac{h_{i+1,j+1}-2h_{i,j+1}+h_{i-1,j+1}-h_{i+1,j-1}+2h_{i,j-1}-h_{i-1,j-1}}{4(\Delta x)^2(\Delta v)^2}. \end{aligned}$$

Taking into account (20), we have the following discrete boundary conditions

$$\begin{gathered} \tilde{u}_{0,j} = u(a,v_j),\ \tilde{u}_{I+1,j} = u(b,v_j) \\ \tilde{u}_{i,0} = u(x_i,c),\ \tilde{u}_{i,J+1} = u(v_i,d) \\ (i=0,1,\dots,I+1,\ j=0,1,\dots,J+1). \end{gathered}$$

The approximate solution $\tilde{u}_{i,j}$ is obtained at $I \times J$ mesh points of Ω by solving the matrix equation

$$\mathbf{T}\,\widetilde{\mathbf{U}} = \mathbf{V}$$

where $\mathbf{T}$ is a block tridiagonal matrix

$$\mathbf{T} = \begin{bmatrix} A^{(1)} & B^{(1)} & 0 & \cdots & 0 \\ C^{(2)} & A^{(2)} & B^{(2)} & \ddots & \vdots \\ 0 & C^{(3)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B^{(J-1)} \\ 0 & \cdots & 0 & C^{(J)} & A^{(J)} \end{bmatrix}_{IJ \times IJ}$$

and $A^{(j)}$, $B^{(j)}$, $C^{(j)}$ are given by

$$A^{(j)} = \begin{bmatrix} a_1^{(1,j)} & a_2^{(1,j)} & 0 & \cdots & 0 \\ a_3^{(2,j)} & a_1^{(2,j)} & a_2^{(2,j)} & \ddots & \vdots \\ 0 & a_3^{(3,j)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_2^{(I-1,j)} \\ 0 & \cdots & 0 & a_3^{(I,j)} & a_1^{(I,j)} \end{bmatrix}_{I \times I}$$

$$B^{(j)} = \begin{bmatrix} b_1^{(1,j)} & b_2^{(1,j)} & 0 & \cdots & 0 \\ b_3^{(2,j)} & b_1^{(2,j)} & b_2^{(2,j)} & \ddots & \vdots \\ 0 & b_3^{(3,j)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_2^{(I-1,j)} \\ 0 & \cdots & 0 & b_3^{(I,j)} & b_1^{(I,j)} \end{bmatrix}_{I \times I}$$

$$C^{(j)} = \begin{bmatrix} c_1^{(1,j)} & c_2^{(1,j)} & 0 & \cdots & 0 \\ c_3^{(2,j)} & c_1^{(2,j)} & c_2^{(2,j)} & \ddots & \vdots \\ 0 & c_3^{(3,j)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & c_2^{(I-1,j)} \\ 0 & \cdots & 0 & c_3^{(I,j)} & c_1^{(I,j)} \end{bmatrix}_{I \times I}$$

where $a_1 = -2k_3 + 2k_4$, $a_2 = 2k_2 + k_3 + k_5$, $a_3 = -2k_2 + k_3 - k_5$, $b_1 = -2k_1 - k_4 - k_6$, $b_2 = k_1 - k_2$, $b_3 = k_1 + k_2$, $c_1 = 2k_1 - k_4 - k_6$, $c_2 = -k_1 - k_2$, $c_3 = -k_1 + k_2$.

$\mathbf{V}$ is a column matrix, which consists of boundary values $\tilde{u}_{0,j}$, $\tilde{u}_{I+1,j}$, $\tilde{u}_{i,0}$ and $\tilde{u}_{i,J+1}$ $(i = 0, 1, ..., I+1,\ j = 0, 1, ..., J+1)$ and $\widetilde{\mathbf{U}}$ is the solution vector:

$$\widetilde{\mathbf{U}} = [\tilde{u}_{1,1}, \tilde{u}_{2,1},, ..., \tilde{u}_{I,1}, \tilde{u}_{1,2}, \tilde{u}_{2,2}, ..., \tilde{u}_{I,2}, ..., \tilde{u}_{1,J}, \tilde{u}_{2,J}, ..., \tilde{u}_{I,J}]^T .$$

To calculate λ numerically, the central-difference formulas are used in (18) and the following difference equation is solved:

$$\Delta x \Delta v \left[k_1 \tilde{u}_{i+1,j} - k_1 \tilde{u}_{i-1,j} - k_2 \tilde{u}_{i,j+1} + k_2 \tilde{u}_{i,j-1}\right] = \tilde{\lambda}_{i,j}$$

$i = 1,2,...,I,\ \ j = 1,2,...,J,$ where $\tilde{\lambda}_{i,j}$ is the approximation to the function $\lambda(x_i, v_j) = \lambda(a + i\Delta x, c + j\Delta v)$.

Numerical Experiments

Example 1. *Let's consider the problem of finding* (u,λ) *in* $\Omega = (-1,1)\times(1,2)$ *from equation* (18) *provided that* $H(x,v) = \frac{1}{2}v^2$ *and the boundary conditions*

$$\begin{aligned} u(-1,v) &= \frac{1}{2v}(2-v)^2,\ u(1,v) = \frac{1}{2v}(2-v)^2 \\ u(x,1) &= \frac{1}{2v}x^2,\ u(x,2) = \frac{1}{4}\left(x^2-1\right) \end{aligned}$$

are given. The exact solution of the problem is $u(x,v) = \frac{1}{2v}\left(x^2 + (2-v)^2 - 1\right)$, $\lambda(x,v) = x$. *In the following figures, a comparison between the exact solution (purple surface) and the finite difference solution (black points) of the inverse problem for* $I = J = 39$ *is presented. The computations are performed using MATLAB 7.0 program on a PC with Intel Core 2 T7200 2.00 GHz CPU, 1 Gb memory, running under Windows Vista.*

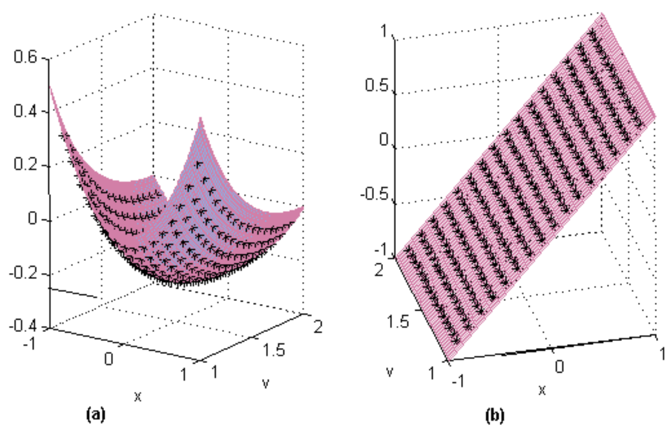

Figure 1: (a) Exact and approximate values of u (b) Exact and approximate values of λ.

The obtained numerical results for $u(x,v)$ and $\lambda(x,v)$ on some points of the domain Ω for the different values of I and J are shown in Table 1 and Table 2, respectively. In the calculation of $u(x,v)$, the maximum error occured for $I = J = 7$ is -0.00373585544664928330 and -0.00000095343455730479 for $I = 7, J = 511$.

Consequently, the computational experiments show that the proposed method gives efficient and reliable results.

Table 1: Exact $u(x,v)$ and the finite difference solution for $I = J = 7$ and $I = 7, J = 511$.

(x,v)	Exact $u(x,v)$	FDM $I = J = 7$	FDM $I = 7, J = 511$
$(-0.75, 1.25)$	0.05000000000000003	0.049713684855366862	0.049999926776574929
$(-0.75, 1.75)$	-0.107142857142857140	-0.107291337599467250	-0.107142894775529030
$(0, 1.25)$	-0.175000000000000020	-0.175654434616304120	-0.175000167367829080
$(0, 1.50)$	-0.2500000000000000000	-0.250607618136903350	-0.250000154528931650
$(0, 1.75)$	-0.2678571428571428500	-0.268196526757965900	-0.267857228874678780
$(0.75, 1.25)$	0.0500000000000000030	0.049713684855366917	0.049999926776574846
$(0.75, 1.75)$	-0.107142857142857140	-0.107291337599467210	-0.107142894775529270

Table 2: Exact $\lambda(x,v)$ and the finite difference solution for $I = J = 7$ and $I = 7, J = 511$.

(x,v)	Exact $\lambda(x,v)$	FDM $I = J = 7$	FDM $I = 7, J = 511$
$(-0.75, 1.25)$	-0.750000000000000000	-0.751227064905570670	-0.7500003138146791300
$(-0.75, 1.75)$	-0.750000000000000000	-0.750890882739660650	-0.7500002257960318700
$(0, 1.25)$	0.0000000000000000000	0.0000000000000002220	0.000000000000021094
$(0, 1.50)$	0.0000000000000000000	0.0000000000000001110	0.000000000000032196
$(0, 1.75)$	0.0000000000000000000	0.0000000000000000000	0.000000000000029976
$(0.75, 1.25)$	0.7500000000000000000	0.7512270649055701200	0.7500003138146779100
$(0.75, 1.75)$	0.7500000000000000000	0.7508908827396604300	0.7500002257960316500

*

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