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Analysis of a crack problem via RKPM and GRKPM and a note on particle volume

Mani Khezri[1], Alireza Hashemian[1], Hossein M. Shodja[1,2,3]

Summary

Meshless methods using kernel approximation like reproducing kernel particle method (RKPM) and gradient RKPM (GRKPM) generally use a set of particles to discretize the subjected domain. One of the major steps in discretization procedure is determination of associated volumes particles. In a non-uniform or irregular configuration of particles, determination of these volumes comprises some difficulties. This paper presents a straightforward numerical method for determination of related volumes and conducts a survey on influence of different assumption about computing the volume for each particle. Stress intensity factor (SIF) as a major representing parameter in fracture of solids is calculated by employing meshless methods for an edge-cracked plate under first mode loading condition which is one of the benchmark problems in fracture mechanics. The obtained results are compared using an analysis in terms of dilation parameter.

keywords: Reproducing kernel particle method; Gradient reproducing kernel particle method; Particle area; crack; Stress intensity factor

Introduction

For a long period, finite element method has been considered as the major and dominant method in the field of computational mechanics. Despite being helpful in different kinds of problems, this method has its own drawbacks which limit or complicate its application to certain problems. Its mesh relied attribute makes the application of this method in such cases as large deformations, complex geometry, singular fields or discontinuities, expensive or inefficient. During the past couple of decades, elimination or reduction of these costs and obstacles have become the main motivation for paying increasingly more attention to meshless methods which only consider the geometry of the domain of problems and a set of particles for their discretization.

One such a meshless technique is the so-called "reproducing kernel particle method" (RKPM) introduced by Liu, Jun, and Zhang (1995). This method has evolved by adding correction coefficients to the reproducing kernel function involved in an early meshless approach named smooth particle hydrodynamics (SPH)

[1]Center of Excellence in Structures and Earthquake Engineering, Department of Civil Engineering, Sharif University of Technology, P.O. Box 11155-9313, Tehran, Iran

[2]Institute for Nanoscience and Nanotechnology, Sharif University of Technology, P.O. Box 11155-9161, Tehran, Iran

[3]Corresponding author; Tel: (+98-21) 6616-4209; Fax: (+98-21) 6607-2555. E-mail addresses: man.khez@gmail.com (M. Khezri), al.hash@gmail.com (A. Hashemian), shodja@sharif.edu (H. M. Shodja).

[Lucy (1977)]. Due to its ease and clarity, this method has been used widespread as an effective numerical method. Shodja and Hashemian when dealt with beam-column problems found it inconvenient to enforce the derivative type essential boundary conditions via conventional RKPM. To circumvent this deficiency, they included the gradient of the field variable in the reproducing equation; the new equation along with the generalization of the corrected collocation method has led to extensive reformulation of RKPM. They called the new approach the gradient RKPM (GRKPM), and demonstrated its efficiency through examination of several beam-columns and plate problems [Shodja and Hashemian(2007); Hashemian and Shodja (2008a)] and nonlinear evolutionary partial differential equations with moving shock-like fronts [Hashemian and Shodja (2008b); Shodja and Hashemian (2008)].

In the literature, several other meshless methods have been proposed; for example, diffuse element method by Nayroles, Touzot, and Villon (1992), element free Galerkin method (EFGM) by Belytschko, Lu, and Gu (1994), hp-clouds method by Durate and Oden (1996), partition of unity method by Babuška and Melenk (1997), local boundary integral equation method by Zhu, Zhang, and Atluri (1998a, b), meshless local Petrov-Galerkin method (MLPG) by Atluri and Zhu (1998). Various MLPG methods are discussed comprehensively by Atluri and Shen (2002a, b).

One of the tasks associated with the application of RKPM and GRKPM is the determination of the volume associated with each particle. Different definitions of the volume (area for two dimensional problems) as a characteristic of each particle are available; these definitions will be discussed in section 2.2, and a computational methodology will be proposed.

In section 4, the stress intensity factor (SIF) as the main explanatory parameter of the stress field near a crack tip is calculated for an edge-cracked plate under the first mode loading condition via RKPM and GRKPM. Also a brief survey has been conducted on impact of different assumptions about determination of associated area of particles.

Formulations for 2D domains

The approximating functions

Consider a two-dimensional domain Ω. In the conventional RKPM a given function $u(\mathbf{x})$ can be expressed by the reproducing formula

$$u^R(\mathbf{x}) = \int_\Omega \tfrac{1}{a(\mathbf{y})} C^{[0]}(\mathbf{x};\mathbf{x}-\mathbf{y})\varphi(\tfrac{\|\mathbf{x}-\mathbf{y}\|}{a(\mathbf{y})})u(\mathbf{y})d\Omega, \tag{1}$$

where $u^R(\mathbf{x})$ is the reproduced function, φ is the kernel function, $\|.\|$ is the Euclidian norm, a is the dilation parameter, and $C^{[0]}$ is the correction function associated with u [Liu, Jun, and Zhang (1995)]. In GRKPM, $u^R(\mathbf{x})$ in terms of the function and its first derivatives is defined as

$$u^R(\mathbf{x}) = \sum_{k=0}^{2} \int_{\Omega} \frac{1}{a(\mathbf{y})} C^{[k]}(\mathbf{x};\mathbf{x}-\mathbf{y})\varphi\left(\frac{\|\mathbf{x}-\mathbf{y}\|}{a(\mathbf{y})}\right) u_{,k}(\mathbf{y})d\Omega, \tag{2}$$

in which $u_{,0} = u$, $u_{,k}(\mathbf{y}) = \partial u(\mathbf{y})/\partial y_k$; $k = 1,2$, and $C^{[k]}$ is the correction function associated with $u_{,k}$ [Shodja and Hashemian (2008)].

For numerical computations, the integrals in Eqs. (1) and (2) should be discretized. Employing the trapezoidal rule, Eqs. (1) and (2) take the following form, respectively

$$u^R(\mathbf{x}) = \sum_{J=1}^{NP} \psi_J^{[0]}(\mathbf{x}) d_J^{[0]}, \tag{3}$$

$$u^R(\mathbf{x}) = \sum_{J=1}^{NP} \sum_{k=0}^{2} \psi_J^{[k]}(\mathbf{x}) d_J^{[k]}, \tag{4}$$

where NP is the number of particles,

$$\begin{aligned} d_J^{[0]} &= u(\mathbf{y})|_{\mathbf{y}=\mathbf{y}_J}, \\ d_J^{[k]} &= \partial u(\mathbf{y})/\partial y_k|_{\mathbf{y}=\mathbf{y}_J}; \quad k = 1,2, \end{aligned} \tag{5}$$

and $\psi_J^{[k]}(\mathbf{x})$ is the kth shape function associated with the Jth particle

$$\psi_J^{[k]}(\mathbf{x}) = C^{[k]}(\mathbf{x};\mathbf{x}-\mathbf{y})\varphi_a(\mathbf{x}-\mathbf{y}_J)\Delta S_J; \quad k = 0,1,2, \tag{6}$$

in which ΔS_J is the area pertinent to the Jth particle. From Eq. (4), it is observed that there are 3 types of shape function $\psi_J^{[k]}(\mathbf{x})$, $k = 0,1,2$ in 2D GRKPM. Whereas, in RKPM only one type of shape function $\psi_J^{[0]}(\mathbf{x})$ is encountered, Eq. (3).

In a random distribution of particles, the computation of ΔS_J is problematic. Hence for simplicity $\Delta S_J = 1$ has been mainly incorporated in the literature; for example, see [Jin, Li, and Aluru (2001)] and [Chen, Han, You, and Meng (2003)]. It is noteworthy to mention that for the case when $\Delta S_J = 1$ is used the reproducing kernel approximation with a non-shifted basis would be identical with the EFGM [Aluru and Li (2001)]. In the following section, a straightforward algorithm for calculation of ΔS_J is proposed.

Computation of the area associated with each particle

Consider a random distribution of particles as shown in Fig. 1. It is desired to compute the area belonging to the Ith particle, ΔS_{I}. In the conventional method, which utilizes the concept of Voronoi diagram, the particles are sorted in ascending order according to their distances (Euclidean norm) from particle *I*, that is 1, 2, 3, ... At first, particle 1 (the nearest particle) is chosen and the perpendicular bisector to the line $\overline{I1}$ is drawn. This procedure is repeated for the other particles in ascending order (2, 3, ...) until the perpendicular bisector lines create a closed polygon around particle *I*, Fig. 1. The area of this polygon is ΔS_{I}, [Sukumar, Moran, and Belytschko (1998)]. To perform this trend for each particle, an efficient algorithm which is suitable for a computer program is necessary. One such algorithm is given in the following.

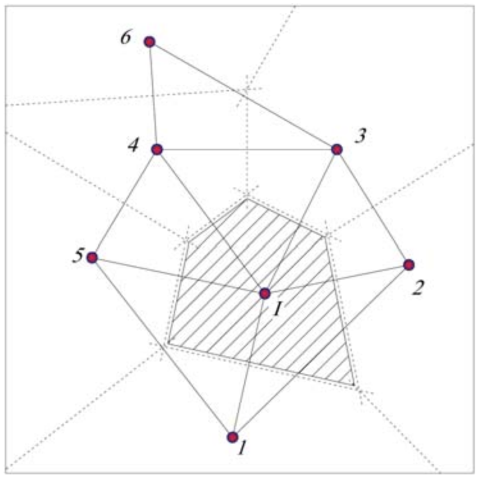

Figure 1: The conventional method for determining the associated area with particle *I*.

The region of the problem is discretized with a very fine grid as displayed in Fig. 2. It is assumed that every cell in the grid represents a point in the region. As a direct result of the conventional approach, every point within the area associated with particle *I* are closer to this particle than any other particles in the region. Hence, each cell should be assigned to the area of the nearest particle, see Fig. 2. An algorithm for the proposed methodology can be summarized as below:

```
Loop over the constructed cells
          Loop over the particles
                    Calculate the distances between the particles and the considered cell

                    Determine the nearest particle to the considered cell
          End particle loop

          Assign the subjected cell to the area of its nearest particle
End cell loop
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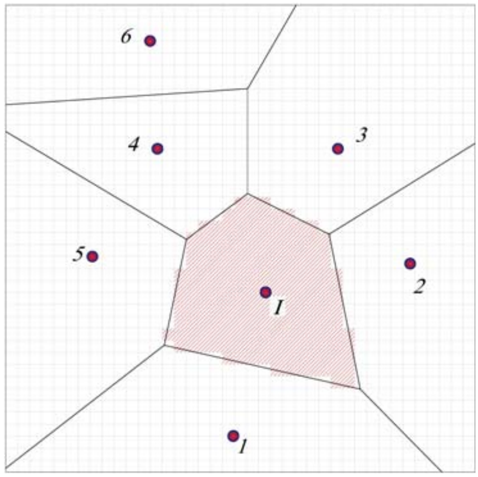

Figure 2: The proposed grid-based approach with determining the associated area for particle I.

Crack problem

Consider an edge-cracked plate under uniform tension as shown in Fig. 3. The width and the length of the plate are $w = 1$ units and $L = 2$ units, respectively. Assume crack length $a = 0.4$ units, far-field tension $\sigma_{22} = 1$ units and small deformations. A plane stress condition is supposed with $E = 207,000$ units and $\nu = 0.3$. The mid point of the left edge coincides with the origin of the Cartesian coordinate system. This is a benchmark example which has been considered by many authors; for example, see [Belytschko, Lu, and Gu (1994)] and [Rao and Rahman (2000)]. In the absence of body forces, the equations of equilibrium for this plate are

$$\sigma_{ij} = 0, \quad i, j = 1, 2, \tag{7}$$

in which σ_{ij} is the component of the stress field. Due to symmetry, only half of the plate is modeled. Hence, the boundary conditions consist of

$$\text{on } x_2 = 1 \text{ for } x_1 >= 0.4 \ : \ \sigma_{12} = 0 \text{ and } \sigma_{22} = 1, \tag{8}$$

$$\text{on } x_2 = 0 : \ u_2 = 0, \tag{9}$$

where u_2 is the displacement in x_2- direction. Moreover for the static stability a constraint in x_1 direction, like

$$u_1(1,0) = 0, \tag{10}$$

should be enforced, where u_1 is the horizontal component of the displacement. Note that Eq. (8) is a natural boundary condition and Eqs. (9) and (10) represent the essential boundary conditions.

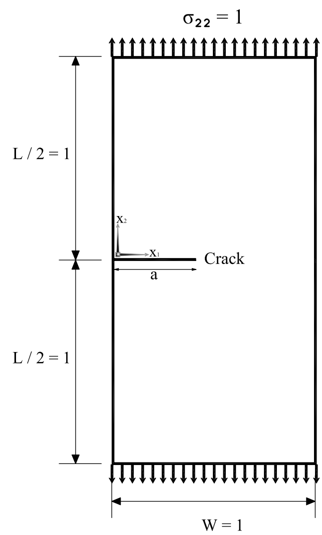

Figure 3: Edge-Cracked Plate under mode-I loading.

The weak form of the Eq. (7) with the mentioned boundary conditions takes on the form

$$\int_0^1 \int_0^1 \sigma_{ij} \delta u_{i,j} dx_1 dx_2 = \int_0^1 \delta u_2 dx_1 |_{x_2=1}, \tag{11}$$

where δ denotes the variation operator. The stress components are related to the displacement gradients by

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \mathbf{D} \begin{bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \end{bmatrix}, \quad \mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \tag{12}$$

The matrix form of Eq. (11) is obtained by spatial discretization of the functions $\mathbf{u}$ and $\delta\mathbf{u}$. To this end, 121 particles are distributed uniformly in the region. For refinement, 15 additional particles are positioned around the crack-tip in a fashion shown in Fig. 4. Eqs. (3) and (4) are employed for discretizations in RKPM and GRKPM, respectively. The essential boundary conditions, Eqs. (9) and (10), are applied by utilizing the corrected collocation method [Wagner and Liu (2000)] for RKPM and the modified corrected collocation method [Hashemian and Shodja (2008a)] for GRKPM. The numerical integrations in Eq. (11) are performed by applying the standard Gaussian quadrature rule. To this end, a background mesh is constructed by drawing imaginary grid lines through the 121 uniformly distributed particles. For each hypothetic cell a 6×6 quadrature is considered. The SIF is computed by calculating the J integral along the path displayed in Fig. 4 [Portela, Aliabadi, and Rooke (1992)].

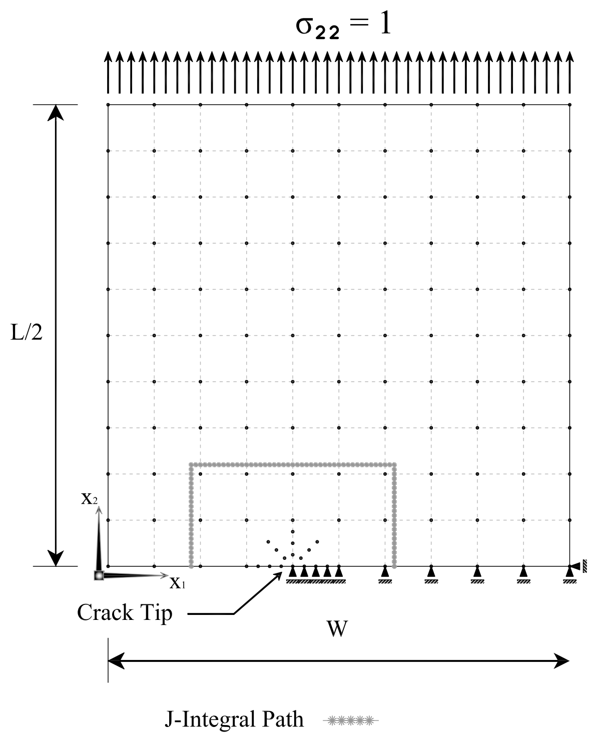

Figure 4: Meshless model for the edge-cracked plate; see Fig. 3.

Numerical results

In the numerical calculations, different dilation parameters are considered. The SIF has been computed using RKPM and GRKPM and plotted versus the dilation parameter. The effect of the particle area is investigated through comparison of the results obtained using $\Delta S_I = 1$ and the variable ΔS_I evaluated by the proposed methodology.

The accuracy of the results is verified by the reference solution $K_I = 2.358$ [Tada, Paris, and Irwin (1973)]. It is observed that the results of GRKPM are more accurate than RKPM's outcomes. Moreover, GRKPM's solutions for $0.165 \leq a \leq 0.18$ give rise to the reference solution in excellent estimation. It is remarkable that the assumption of $\Delta S_I = 1$ degrades the accuracy of both RKPM and GRKPM; this adverse effect is more severe for GRKPM than for RKPM.

Conclusions

The mode-I SIF of an edge-cracked plate was analyzed by GRKPM for the first time and the result was compared with that obtained via conventional RKPM. Consideration of the proposed approach for computing the particle area (ΔS_I) as a characteristic of each particle, instead of using $\Delta S_I = 1$, led to some interesting observations. An important implication was that, both RKPM and GRKPM yielded

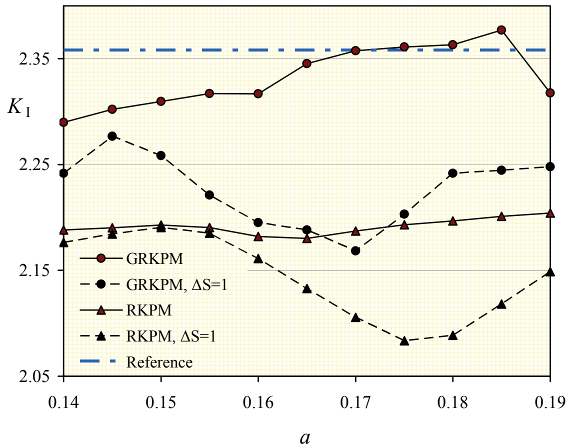

Figure 5: The stress intensity factor resulted from different approximations.

more accurate results. The other attractive outcome was that, the sensitivity of both RKPM and GRKPM to dilation parameter was reduced significantly.

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