

## 3D Crack Growth by Considering Re-Entrant Corners

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### Summary

In fracture mechanics, corner and wedge singularities have to be considered for two- and three-dimensional problems in isotropic and layered anisotropic continua. To investigate the behaviour of crack propagation starting from corners and edges the information about stress asymptotics in the vicinity of three-dimensional corner points is needed [1]. Thus, in this paper two aspects are studied: the interface crack in layered anisotropic materials with re-entrant corners and surface cracks in homogeneous isotropic continua. Moreover a strategy is presented to model such surface breaking cracks efficiently within a numerical 3D simulation [2]. To study the effect of geometrical singularities generalized stress intensity factors are defined. Starting with KONDRATIEV's Lemma an elliptic boundary value problem has to be solved with homogeneous DIRICHLET/NEUMANN boundary data that is generating a singular field in the vicinity of corner points. Afterwards the weak form for the described problem is discretized by using the PETROV-GALERKIN finite element method resulting in a quadratic eigenvalue problem. The quadratic eigenvalue problem is solved iteratively by the ARNOLDI method [3], and finite element approximations of corner singularity exponents are computed. These eigenvalues are the basis for the definition of generalized stress intensity factors in the neighbourhood of corner points. For the a-posteriori control of the eigenvalues, an error estimator is developed on the basis of the ZIENKIEWICZ-ZHU algorithm. This approach to determine 3D singularities is tested herein for some typical applications in fracture mechanics. Known 3D singularities are a key input for the formulation of an advanced 3D crack growth criterion [2].

**keywords:** Fracture mechanics, crack growth, 3D stress singularities, eigenvalue problems, corner and edge effects.

### Introduction

This paper deals with the computation of three-dimensional singularities in elasticity. Those singularities are present at non-smooth domains with corners, edges and cracks and in case of jumping material constants from one layer to the next. To simulate 3D fatigue crack growth appropriately to determine e.g. the life-time of components the classical fracture mechanics parameters (K-factors and

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T-stresses) are needed. In case of arbitrary singularities so-called generalized stress intensity factors are introduced. The numerics behind computing singularities will be influenced by the regularity of the solution. On the basis of CÉA's Lemma, the convergence behaviour of those methods depends on the type of trial and test functions. Classical polynomials can lead to problems regarding the convergence behaviour. There are two options available to overcome this drawback, namely an adaptive mesh generation or the consideration of the appropriate singularity in the trial and test spaces. The type of singularities can be described for the displacements in the following form:

$$|\mathbf{x}|^\lambda \sum_{n=0}^N \log^n |\mathbf{x}| \mathbf{U}_n(\mathbf{x}/|\mathbf{x}|) \quad (1)$$

We understand asymptotic solutions from type (1) for that  $\lambda < 1$ .

Given is  $\Omega$  of  $\mathbf{R}^3$  as a bounded domain which cone shaped (see Fig. 1).

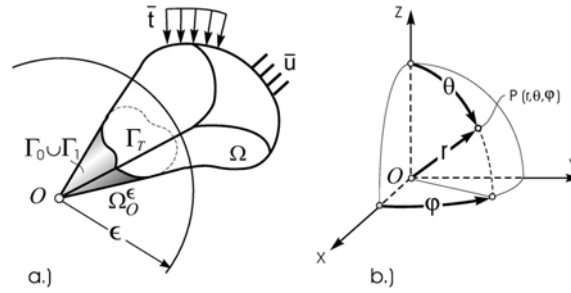


Figure 1: a) Solid Body with Singular, Conical Point  $O$ . b) Cartesian and Spherical Coordinates at  $O$ .

We are defining  $K$ :

$$K := \{\mathbf{x} \in \mathbf{R}^3 : 0 < |\mathbf{x}| < \infty, \mathbf{x}/|\mathbf{x}| \in S\} \quad (2)$$

in an  $\epsilon$ -region  $U_0^\epsilon := \{\mathbf{x} \in \mathbf{R}^3 : 0 < |\mathbf{x}| < \epsilon\}$ , which has the same coordinate point  $O$ , so that we have

$$\Omega_0^\epsilon := K \cap U_0^\epsilon = \Omega \cap U_0^\epsilon. \quad (3)$$

On  $\Omega$  we have the mixed boundary value problem in elasticity

$$\begin{aligned} \mathbf{L}\mathbf{u} &:= \mathbf{D}^T \mathbf{C} \mathbf{D}\mathbf{u} = \mathbf{f} && \text{on } \Omega \\ \mathbf{u} &= \bar{\mathbf{u}} && \text{on } \partial\Omega_0 \\ \mathbf{T}\mathbf{u} &:= \mathbf{t}(\mathbf{u}) = \bar{\mathbf{t}} && \text{on } \partial\Omega_1 \end{aligned} \quad (4)$$

We are looking for a solution of an equivalent mixed boundary value problem of  $\Omega_0^\epsilon$  for which the transmission boundary  $|\mathbf{x}| = \epsilon$  has such DIRICHLET boundary

conditions  $\hat{\mathbf{u}}$  that the solution of (4) is identical with the local solution around the tip. Additionally, we define for  $|\mathbf{x}| < \varepsilon$  only homogeneous boundary conditions. We are looking for  $\mathbf{u}$  of the LAMÉ-System:

$$\begin{aligned} \mathbf{L}\mathbf{u} &:= \mathbf{f} && \text{on } \Omega_O^\varepsilon, \\ \mathbf{u} &= \hat{\mathbf{u}} && \text{on } \Gamma_T, \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_0, \\ \mathbf{T}\mathbf{u} &:= \mathbf{0} && \text{on } \Gamma_1. \end{aligned} \tag{5}$$

For the boundaries for DIRICHLET-, NEUMANN and transmission parts we have the following:

$$\begin{aligned} \Gamma_0 &:= \{\mathbf{x} : 0 < |\mathbf{x}| < \varepsilon, \mathbf{x}/|\mathbf{x}| \in \gamma_0\}, \\ \Gamma_1 &:= \{\mathbf{x} : 0 < |\mathbf{x}| < \varepsilon, \mathbf{x}/|\mathbf{x}| \in \gamma_1\}, \\ \Gamma_T &:= \partial\Omega_O^\varepsilon \setminus \{\Gamma_0 \cup \Gamma_1\}, \end{aligned} \tag{6}$$

where  $\gamma_0 \cup \gamma_1 = \partial S$ ,  $\gamma_0 \cap \gamma_1 = \emptyset$  defining  $\partial S$ . We are using the disturbance theory [4], so that we are introducing the scaled coordinates  $\mathbf{y} = \mathbf{x}/\varepsilon$  and after  $\varepsilon \rightarrow 0$ , the domain  $\Omega_O^\varepsilon$  goes to the unbounded domain  $K$ . We can transform the LAMÉ system to the following:

$$\begin{aligned} \mathbf{L}\mathbf{u} &= \mathbf{0} && \text{on } K, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial K_0, \\ \mathbf{T}\mathbf{u} &= \mathbf{0} && \text{on } \partial K_1, \end{aligned} \tag{7}$$

where  $\partial K_0 := \{\mathbf{x} \in \partial K : \mathbf{x}/|\mathbf{x}| \in \gamma_0\}$ ,  $\partial K_1 := \{\mathbf{x} \in \partial K : \mathbf{x}/|\mathbf{x}| \in \gamma_1\}$  defines the DIRICHLET- and NEUMANN-part of the boundary  $\partial K$ . For the spectral problem we come now to the following formulation:

$$\mathbf{u}(r, \theta, \phi) = r^\lambda \mathbf{U}(\theta, \phi) \tag{8}$$

for which we have to consider the following equation set:

$$\begin{aligned} \hat{\mathbf{L}}(\lambda)\mathbf{U} &= \mathbf{0} && \text{on } S \\ \mathbf{U} &= \mathbf{0} && \text{on } \gamma_0 \\ \hat{\mathbf{T}}(\lambda)\mathbf{U} &= \mathbf{0} && \text{on } \gamma_1 \end{aligned} \tag{9}$$

where  $\gamma_0$  and  $\gamma_1$  are DIRICHLET- NEUMANN-part of  $\partial S$ . The operator in (9) is a so-called “operator pencil”  $\mathbf{A}(\lambda)$  for which we have the following properties (see proof in [5]):

1.  $\mathbf{A}(\lambda)$  is a FREDHOLM-operator for all  $\lambda \in \mathbf{C}$ .
2. The spectrum of  $\mathbf{A}(\lambda)$  consists of isolated eigenvalues with finite algebraic multiplicity.
3. If  $\lambda_0$  is an eigenvalue of  $\mathbf{A}(\lambda)$ , then this is also the case with  $\bar{\lambda}_0, -1 - \lambda_0, -1 - \bar{\lambda}_0$ , where the geometrical and algebraic multiplicity of  $\lambda_0$  and  $-1 - \bar{\lambda}_0$  are identical.

We are coming to the kernel theorem formulated by KONDRATIEV [6]. If  $\mathbf{u} \in [H^1(\Omega)]^3$ , we have the following asymptotic series:

$$\mathbf{u} = \sum_{i=0}^I \sum_{k=0}^{k_j} K_{ik} r^{\lambda_i} \ln^k(r) \mathbf{U}_{ik}(\theta, \phi) \quad (10)$$

where  $\lambda_i$  are eigenvalues of the operator pencil and are called “*singular exponents*”,  $\mathbf{U}_{ik}$  are the generalized eigenvectors and  $K_{ik}$  are the amplitudes and are called “*generalized stress intensity factors*”. We have to consider that the strain energy must be finite. In the following we are interested only in the singular part of the solution, thus we restrict ourselves to

$$-0.5 < \Re(\lambda) < 1 \quad (11)$$

where we have the logarithmic power series for the singularities. We understand asymptotic solutions of the type  $|\mathbf{x}|^\lambda \sum_{n=0}^N \log^n |\mathbf{x}| \mathbf{U}_n(\mathbf{x}/|\mathbf{x}|)$ , for that  $\lambda < 1$  and has infinite gradients of the displacements.

### Weak Formulation of the Problem

We search for a solution for  $\mathbf{u} \in [H^1(\Omega_0^\varepsilon)]^3$  so that

$$\mathbf{B}(\mathbf{u}, \mathbf{v}) = 0, \quad \forall \mathbf{v} \in [H_0^1(\Omega_0^\varepsilon)]^3 \quad (12)$$

We introduce now different trial and test functions which are associated with the operator pencil  $\mathbf{A}(\lambda)$ :

$$\begin{aligned} \mathbf{u} &= r^\lambda \mathbf{U}(\theta, \phi) \in [H^1(\Omega_0^\varepsilon)]^3 \\ \mathbf{v} &= \Phi(r) \mathbf{V}(\theta, \phi) \in [H_0^1(\Omega_0^\varepsilon)]^3 \end{aligned} \quad (13)$$

where  $\Phi(r)$  is a scalar function with a compact support. Thus, we can formulate the following:

For  $U \in [H^1(S)]^3$  we get:

$$\hat{\mathbf{B}}(\mathbf{U}, \mathbf{V}; \lambda) = 0, \quad \forall \mathbf{V} \in [H_0^1(S)]^3 \quad (14)$$

where  $\hat{\mathbf{B}}(\mathbf{U}, \mathbf{V}; \lambda)$  depends polynomial on the operator  $\lambda$  and represents the weak form of the operator pencil  $\mathbf{A}(\lambda)$ .  $(\lambda_i, \mathbf{U}_{ik})$  are only eigenpairs of  $\mathbf{A}(\lambda)$ , if they are at the same time weak solutions of (14).

The approximation with finite elements leads to:  $\mathbf{u}^h \in \mathbf{U}_h \subset [H^1(\Omega_0^\varepsilon)]^3$  so that

$$\mathbf{B}(\mathbf{u}^h, \mathbf{v}^h) = 0, \quad \forall \mathbf{v}^h \in \mathbf{V}_h \subset [H_0^1(\Omega_0^\varepsilon)]^3 \quad (15)$$

where we have the situation that  $\mathbf{U}_h \neq \mathbf{V}_h$ . As different spaces are used for trial and test functions, we are in the scheme GALERKIN-PETROV method. The displacements are formulated in the sector  $(r, \theta, \phi) \in [0, \varepsilon] \times \Delta_i$ , where we have the following finite elements initial formulations:

$$\begin{aligned} \mathbf{u}_i^h(r, \theta, \phi) &= r^\lambda \mathbf{N}(\theta, \phi) \mathbf{T}_d^{-1} \mathbf{d}_i, \\ \mathbf{v}_i^h(r, \theta, \phi) &= \Phi(r) \mathbf{N}(\theta, \phi) \mathbf{T}_d^{-1} \mathbf{b}_i. \end{aligned} \quad (16)$$

The GALERKIN-PETROV approach leads to a non-symmetric stiffness-matrix, which is the source of the fundamental equation for solving the eigenvalue problem:

$$[(\mathbf{K} - \mathbf{D}) + \lambda (\mathbf{D}^T - \mathbf{D} - \mathbf{M}) - \lambda^2 (\mathbf{M})]^T \mathbf{d} = \mathbf{0} \quad (17)$$

### The Solution of the Eigenvalue Problem

We have to solve now the eigenvalue problem:

$$[\mathbf{P} + \bar{\lambda} \mathbf{Q} + \bar{\lambda}^2 \mathbf{R}]^T \mathbf{d} = \mathbf{0} \quad (18)$$

with the definitions  $\bar{\lambda} = \lambda + 1/2$  and

$$\begin{aligned} \mathbf{P} &= \mathbf{K} + \frac{1}{4} \mathbf{M} - \frac{1}{2} (\mathbf{D} + \mathbf{D}^T), \\ \mathbf{Q} &= [\mathbf{D}^T - \mathbf{D}]^T, \\ \mathbf{R} &= -\mathbf{M}. \end{aligned} \quad (19)$$

The matrices  $\mathbf{P}$ ,  $\mathbf{R}$  are symmetric whereas  $\mathbf{Q}$  is skew-symmetric. For applying the ARNOLDI-method [1, 3] we are doing the following transformation:  $\bar{\lambda} \mathbf{x} = \bar{\lambda}^2 \mathbf{R} \mathbf{d}$  so that we get

$$\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \mathbf{x} \end{bmatrix} = \bar{\lambda} \begin{bmatrix} -\mathbf{Q} & -\mathbf{I} \\ \mathbf{R} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \mathbf{x} \end{bmatrix} \quad (20)$$

with  $\mathbf{I}$  as the identity matrix.

Since for our fracture mechanics problems we need eigenvalues in the interval  $0 < \Re \bar{\lambda} < 1.5$ , an additional spectral transformation  $\bar{\lambda} = 1/\theta$  is applied

$$\underbrace{\begin{bmatrix} -\mathbf{Q} & -\mathbf{I} \\ \mathbf{R} & \mathbf{0} \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \mathbf{d} \\ \mathbf{x} \end{bmatrix} = \theta \underbrace{\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}}_{\mathbf{B}} \begin{bmatrix} \mathbf{d} \\ \mathbf{x} \end{bmatrix}. \quad (21)$$

Finally, a standard eigenvalue problem

$$\mathbf{X}\hat{\mathbf{d}} - \theta\hat{\mathbf{d}} = \mathbf{0} \quad (22)$$

with  $\mathbf{X} = \mathbf{B}^{-1}\mathbf{A}$  and  $\hat{\mathbf{d}} = [\mathbf{d}, \mathbf{x}]^T$  is obtained.

### Numerical Tests and Applications for 3D Crack Growth

We will discuss an elasticity problem with edge and corner singularities and this for homogeneous and inhomogeneous material properties. The singular exponents will be discussed in dependence of material data. We are working with an adaptive fine mesh series so that we can have the result for the first nine eigenvalues for a residuum of  $10^{-4}$  within one ARNOLDI-step (see Fig. 2).

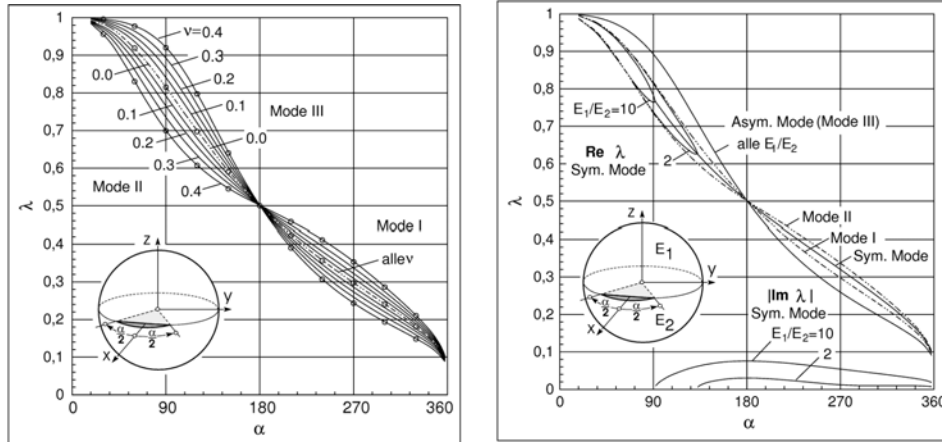


Figure 2: Wedge-shaped crack with homogenous (left) and inhomogeneous material (right). Reference Solution is given in [7]

With the known singular behaviour at arbitrary points one can easily switch to a well-known formulation for smooth crack fronts [8]

$$\sigma_{ij}(r, \phi, P) = \sum_{M=I}^{III} \frac{K_M(P)}{\sqrt{2\pi r}} f_{ij}^M(\phi) + T_{ij}(P) + o(1), r \rightarrow 0 \quad (23)$$

to identify the classical stress intensity factors (SIFs)  $K_M(P)$  ( $M=I,II,III$ ) and the corresponding  $T$ -stresses  $T_{ij}(P)$ . The present wedge singularity has the known value of  $\lambda=0.5$  with a multiplicity of three, cf. Fig. 2 (left). A similar formulation for corner points is given in Eq. (24).

$$\sigma_{ij}(\rho, \theta, \phi, O) = \sum_{L=1}^{N_0} K_L^*(O) \rho^{\lambda_L-1} g_{ij}^L(\theta, \phi, O) + T_{ij}^*(O) + o(1), \rho \rightarrow 0. \quad (24)$$

The stresses are primarily characterized by the asymptotic exponents  $\lambda_L$  and generalized intensity factors  $K_L^*$ . The angular functions  $g_{ij}^L(\theta, \phi, O)$  are defined with respect to the spherical co-ordinate system  $(\rho, \theta, \phi)$ , which is centered at the singular point  $O$ . As the focus lies on the asymptotical behaviour the interval  $-0.5 < \lambda_L < 1$  with  $L = 1, 2, \dots, N_0$  is considered excluding the rigid body motion modes.

The classical SIF-concept only fails at some special points with  $\lambda_L$  not equal to 0.5. But to be still able to apply this concept for the description of the behavior in the crack near-field the SIFs are numerically defined at these particular points. If  $\lambda_L$  is greater than 0.5 and less than 1.0 the stresses are still singular but weaker compared to the square-root singularity. Hence,  $K_M(P)$  tends to zero as  $P$  tends to  $O$ . If  $\lambda_L$  is less than 0.5,  $K_M(P)$  tends to infinity. Therefore, the asymptotic exponents have to be known to determine even the classical SIF asymptotically.

Based on experimental evidence the propagating crack front is shaped that a valid square-root singularity is present along the whole crack front providing a unique crack front angle  $\gamma$  [8,9]. This guarantees a bounded energy release rate that is additionally constantly distributed along the crack front. Consequently, this strategy is part of a 3D crack growth criterion and realized in [2].

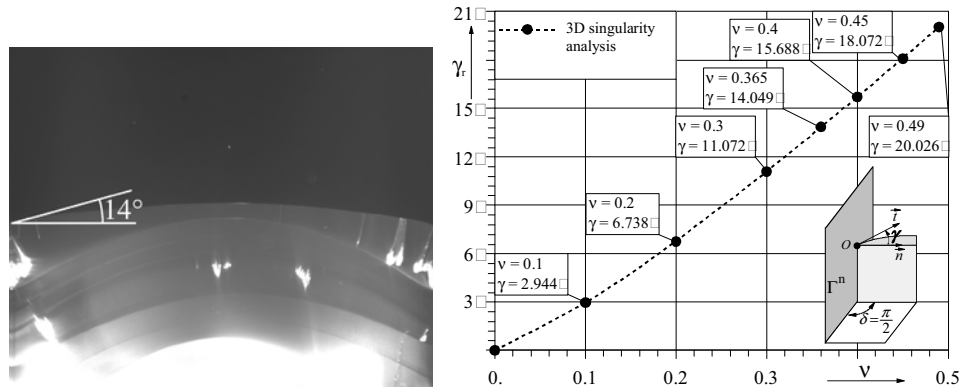


Figure 3: Crack front shape of a single edge cracked beam under bending and general intersection angles between the crack front and outer surface [9]

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