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# Constitutive Relation for Friction Describing Transition from Static to Kinetic Friction and Vice Versa

K. Hashiguchi[1] and S. Ozaki[2]

## Summary

The *subloading-friction model* with a smooth elastic-plastic sliding transition is extended so as to describe the reduction from the static to kinetic friction and the recovery of static friction. The reduction is formulated as the plastic softening due to the separations of the adhesions of surface asperities induced by the sliding and the recovery is formulated as the creep hardening due to the reconstructions of the adhesions of surface asperities during the elapse of time under a quite high actual contact pressure between edges of asperities.

## Introduction

Description of the friction phenomenon as a constitutive equation has been attained first as a rigid-plasticity, and further it has been extended to an elastoplasticity in which the penalty concept, i.e. the elastic springs between contact surfaces is incorporated[1]. However, the interior of the sliding-yield surface has been assumed to be an elastic domain and thus the plastic sliding velocity due to the rate of traction inside the sliding-yield surface is not described. Therefore, the accumulation of plastic-sliding due to the cyclic loading of contact traction cannot be described by these models. On the other hand, the first author of the present article has proposed the *subloading surface model*[2] within the framework of unconventional plasticity, which is capable of describing the plastic strain rate by the rate of stress inside the yield surface. Based on the concept of subloading surface, the authors proposed the *subloading-friction model*[4, 5], which describes the smooth transition from the elastic to plastic sliding state and the accumulation of sliding displacement during a cyclic loading of tangential contact traction.

A high friction coefficient is first observed as a sliding between bodies commences, which is called the *static friction*. Then, the friction coefficient decreases approaching the lowest stationary value, which is called the *kinetic friction*. Thereafter, if the sliding stops for a while and then it starts again, the friction coefficient recovers and a similar behavior as that in the first sliding is reproduced. These are fundamental characteristics in the friction phenomenon, which have been widely recognized for a long time.

In this article, the subloading-friction model[2, 3] is extended so as to describe the reduction of friction coefficient from the static to kinetic friction as the plastic softening due to the sliding and the recovery of friction coefficient as the viscoplas-

[1]Dept. Civil Environ. Eng., Daiichi Univ., Kirishima-shi, Kagoshima-ken 899-4395, Japan
[2]Dept. Mech. Eng., Tokyo Univ. of Sci., Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

tic hardening due to the creep phenomenon induced with the elapse of time under a high contact pressure between surface asperities.

## Formulation of the Constitutive Equation for Friction

The subloading-friction model[2, 3] is extended below so as to describe the *static-kinetic friction transition*, i.e. the transition from static and kinetic friction, and vice versa.

### Decomposition of sliding velocity

The sliding velocity $\bar{\mathbf{v}}$ is defined as the relative velocity to the other body and is additively decomposed into the normal component $\bar{\mathbf{v}}_n$ and the tangential component $\bar{\mathbf{v}}_t$ as follows:

$$\bar{\mathbf{v}} = \bar{\mathbf{v}}_n + \bar{\mathbf{v}}_t \tag{1}$$

which are given by $\bar{\mathbf{v}}$ as

$$\bar{\mathbf{v}}_n = (\bar{\mathbf{v}} \bullet \mathbf{n})\mathbf{n} = (n \otimes \mathbf{n}) \bullet \bar{\mathbf{v}}, \quad \bar{\mathbf{v}}_t = \bar{\mathbf{v}} - \bar{\mathbf{v}}_n = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \bullet \bar{\mathbf{v}} \tag{2}$$

where $\mathbf{n}$ is the unit outward-normal vector at the contact surface, $(\bullet)$ and $\otimes$ denote the scalar and the tensor products, respectively, and $\mathbf{I}$ is the second-order identity tensor having the components of Kronecker's delta $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$. On the other hand, it is assumed that $\bar{\mathbf{v}}$ is additively decomposed into the elastic sliding velocity $\bar{\mathbf{v}}^e$ and the plastic sliding velocity $\bar{\mathbf{v}}^p$, i.e.

$$\bar{\mathbf{v}} = \bar{v}^e + \bar{v}^p \tag{3}$$

$$\bar{\mathbf{v}}_n = \bar{\mathbf{v}}_n^e + \bar{\mathbf{v}}_n^p, \quad \bar{\mathbf{v}}_t = \bar{\mathbf{v}}_t^e + \bar{\mathbf{v}}_t^p \tag{4}$$

$$\bar{\mathbf{v}}_n^e = (\bar{\mathbf{v}}^e \bullet \mathbf{n})\mathbf{n} = (\mathbf{n} \otimes \mathbf{n}) \bullet \bar{\mathbf{v}}^e, \quad \bar{\mathbf{v}}_t^e = \bar{\mathbf{v}}^e - \bar{\mathbf{v}}_n^e = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \bullet \bar{\mathbf{v}}^e \tag{5}$$

$$\bar{\mathbf{v}}_n^p = (\bar{\mathbf{v}}^p \bullet \mathbf{n})\mathbf{n} = (\mathbf{n} \otimes \mathbf{n}) \bullet \bar{\mathbf{v}}^p, \quad \bar{\mathbf{v}}_t^p = \bar{\mathbf{v}}^p - \bar{\mathbf{v}}_n^p = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \bullet \bar{\mathbf{v}}^p \tag{6}$$

The contact traction $\mathbf{f}$ acting on the other body is decomposed into the normal part $\mathbf{f}_n$ and the tangential part $\mathbf{f}_t$ as follows:

$$\mathbf{f} = \mathbf{f}_n + \mathbf{f}_t \tag{7}$$

where the above-mentioned $\mathbf{n}$ is also the normalized direction vectors of $\mathbf{f}_n$ and $\mathbf{t}$ is the normalized direction vectors of $\mathbf{f}_t$, respectively, i.e.

$$\mathbf{n} \equiv \mathbf{f}_n / ||\mathbf{f}_n||, \quad \mathbf{t} \equiv \mathbf{f}_t / ||\mathbf{f}_t|| \tag{8}$$

and $f_n$ and $f_t$ are the normal and the tangential components of the contact traction $\mathbf{f}$, i.e.

$$f_n \equiv -||\mathbf{f}_n|| = \mathbf{n} \bullet (-\mathbf{f}), \quad f_t \equiv ||\mathbf{f}_t|| = \mathbf{t} \bullet \mathbf{f} \tag{9}$$

where the sign of $f_n$ is selected to be plus for compression.

Now, let the elastic-sliding velocity be given by the following hypo-elastic relation, whilst the elastic sliding velocity is usually far small compared with the plastic sliding velocity in the friction phenomenon.

$$\bar{\mathbf{v}}_n^e = \frac{1}{\alpha_n}\mathring{\mathbf{f}}_n, \quad \bar{\mathbf{v}}_t^e = \frac{1}{\alpha_t}\mathring{\mathbf{f}}_t \tag{10}$$

where $\mathring{\mathbf{f}}_n$ and $\mathring{\mathbf{f}}_t$ are the normal component and tangential component, respectively, of $\mathring{\mathbf{f}}$, $(\mathring{\ })$ denoting the corotational rate, which are related to the material-time derivative denoted by $(\dot{\ })$ as follows:

$$\mathring{\mathbf{f}} = \dot{\mathbf{f}} - \Omega\mathbf{f}, \quad \mathring{\mathbf{f}}_n = \dot{\mathbf{f}}_n - \Omega\mathbf{f}_n, \quad \mathring{\mathbf{f}}_t = \dot{\mathbf{f}}_t - \Omega\mathbf{f}_t \tag{11}$$

where the skew-symmetric tensor $\Omega$ is the spin describing the rigid-body rotation of the contact surface. $\alpha_n$ and $\alpha_t$ are the contact elastic moduli in the normal and the tangential directions to the contact surface. On the other hand, the sliding velocity $\bar{\mathbf{v}}$ is not an absolute velocity but the relative velocity, and thus it can be adopted to the constitutive relation as it is since it has the objectivity. It follows from Eq. (10) that

$$\mathring{\mathbf{f}} = \mathring{\mathbf{f}}_n + \mathring{\mathbf{f}}_t = \mathbf{C}^e\bar{\mathbf{v}}^e \tag{12}$$

where the second-order tensor $\mathbf{C}^e$ is the fictitious contact elastic modulus tensor given by

$$\mathbf{C}^e = \alpha_n\mathbf{n}\otimes\mathbf{n} + \alpha_t(\mathbf{I} - \mathbf{n}\otimes\mathbf{n}), \quad \mathbf{C}^{e-1} = \frac{1}{\alpha_n}\mathbf{n}\otimes\mathbf{n} + \frac{1}{\alpha_t}(\mathbf{I} - \mathbf{n}\otimes\mathbf{n}) \tag{13}$$

**Normal-sliding and sliding-subloading surfaces**

Assume the following isotropic *sliding-yield surface* with the isotropic hardening/softening, which describes the *sliding-yield condition*.

$$f(\mathbf{f}) = F \tag{14}$$

where $F$ is the isotropic hardening/softening function denoting the variation of the size of sliding-yield surface.

In what follows, we assume that the interior of the sliding-yield surface is not a purely elastic domain but that the plastic sliding velocity is induced by the rate of traction inside that surface. Henceforth, let the surface described by Eq. (14) be renamed the *normal-sliding surface*.

Then, in accordance with the concept of *subloading surface* (Hashiguchi, 1980, 1989), we introduce the *sliding-subloading surface*, which always passes through the current traction point $\mathbf{f}$ and keeps a similar shape and orientation to the normal-sliding surface with respect to the zero traction point $\mathbf{f} = 0$. Then, let the ratio of the size of the sliding-subloading surface to that of the normal-sliding surface be called the *normal-sliding ratio*, denoted by $\bar{R}$ $(0 \leq \bar{R} \leq 1)$. Therefore, the normal-sliding ratio $\bar{R}$ plays the role of three-dimensional measure of the degree of approach to the normal-sliding state. Then, the sliding-subloading surface is described by

$$f(\mathbf{f}) = \bar{R}F \tag{15}$$

The material-time derivative of Eq. (15) leads to

$$\frac{\partial f}{\partial \mathbf{f}} \bullet \mathring{\mathbf{f}} = \bar{R}\dot{F} + \dot{\bar{R}}F \tag{16}$$

while the direct transformation of the material-time derivative to the corotational derivative is verified for the general scalar function[6].

**Evolution rules of the hardening function and the normal-sliding ratio**

It could be stated from experiments that

1. If the sliding commences, the friction coefficient reaches first the maximal value of static-friction and then it reduces to the minimal stationary value of kinetic-friction. Physically, this phenomenon could be interpreted to be caused by the separations of the adhesions of surface asperities between contact bodies due to the sliding. Then, let it be assumed that the reduction is caused by the contraction of the normal-sliding surface, i.e. the plastic softening due to the sliding.
2. If the sliding ceases after the reduction of friction coefficient, the friction coefficient recovers gradually with the elapse of time and the static-friction is reproduced after an elapse of sufficient time. Physically, this phenomenon could be interpreted to be caused by the reconstructions of the adhesions of surface asperities during the elapsed time under a quite high contact pressure between edges of surface asperities. Then, let it be assumed that the recovery is caused by the viscoplastic hardening due to the creep phenomenon.

Taking account of these facts, let the evolution rule of the isotropic hardening/softening function $F$ be postulated as follows:

$$\dot{F} = -\kappa \left\{ \left( \frac{F}{F_k} \right)^m - 1 \right\} ||\bar{\mathbf{v}}^p|| + \xi \left\{ 1 - \left( \frac{F}{F_s} \right)^n \right\} \tag{17}$$

where $F_s$ and $F_k (F_s \geq F \geq F_k)$ are the maximum and minimum values of $F$ for the static and kinetic frictions, respectively. $\kappa$ and $m$ are the material constants influencing the decreasing rate of $F$ due to the plastic-sliding, and $\xi$ and $n$ are the material constants influencing the recovering rate of $F$ due to the elapse of time, while they would be functions of absolute temperature in general.

It is observed in experiments that the tangential traction increases almost elastically with the plastic sliding when it is zero but thereafter it increases gradually approaching the normal-sliding surface and it does not increase any more when it reaches the normal-sliding surface. Then, we assume the evolution rule of the normal-sliding ratio as follows:

$$\dot{\bar{R}} = \bar{U}(\bar{R})||\bar{\mathbf{v}}^p|| \text{ for } \bar{\mathbf{v}}^p \neq \mathbf{0} \tag{18}$$

where $\bar{U}(\bar{R})$ is a monotonically decreasing function of $\bar{R}$ fulfilling the following conditions.

$$\bar{U}(\bar{R}) \to +\infty \text{ for } \bar{R} \to 0, \quad \bar{U}(\bar{R}) = 0 \text{ for } \quad \bar{R} = 1 \quad (\bar{U}(\bar{R}) < 0 \text{ for } \bar{R} > 1) \tag{19}$$

The simplest function $\bar{U}$ fulfilling Eq. (19) is given by

$$\bar{U}(\bar{R}) = -\bar{u} \ln \bar{R} \tag{20}$$

where $\bar{u}$ is the material constant.

**Relationships of sliding velocity and contact traction rate**

The substitution of Eqs. (17) and (18) into Eq. (16) gives rise to the *consistency condition* for the sliding-subloading surface:

$$\frac{\partial f}{\partial \mathbf{f}} \bullet \mathring{\mathbf{f}} = \bar{R}\{-\kappa \left\{ \left( \frac{F}{F_k} \right)^m - 1 \right\} ||\bar{\mathbf{v}}^p|| + \xi \left\{ 1 - \left( \frac{F}{F_s} \right)^n \right\} + \bar{U}||\bar{\mathbf{v}}_t^p||F \tag{21}$$

Assume the following *sliding-plastic flow rule*.

$$\bar{\mathbf{v}}^p = \bar{\lambda}\mathbf{M} \quad (||\mathbf{M}|| = 1) \tag{22}$$

where $\bar{\lambda}(> 0)$ is a positive proportionality factor and the unit vector $\mathbf{M}$ is the function of stress and internal variables.

Substituting Eq. (22) into Eq. (21), the proportionality factor $\bar{\lambda}$ is derived as follows:

$$\bar{\lambda} = \frac{\frac{\partial f(\mathbf{f})}{\partial \mathbf{f}} \bullet \mathring{\mathbf{f}} - M^c}{M^f}, \quad \bar{\mathbf{v}}^p = \frac{\frac{\partial f(\mathbf{f})}{\partial \mathbf{f}} \bullet \mathring{\mathbf{f}} - M^c}{M^f}\mathbf{M} \tag{23}$$

where

$$M^f \equiv -\kappa \left\{ \left( \frac{F}{F_k} \right)^m - 1 \right\} \bar{R} + F\bar{U}, \quad M^c \equiv \xi \left\{ 1 - \left( \frac{F}{F_s} \right)^n \right\} \bar{R} \quad (\geq 0) \tag{24}$$

Substituting Eqs. (12) and (23) into Eq. (21), the sliding velocity is given by

$$\bar{\mathbf{v}} = C^{e-1}\overset{\circ}{\mathbf{f}} + \frac{\frac{\partial f(\mathbf{f})}{\partial \mathbf{f}} \bullet \overset{\circ}{\mathbf{f}} - M^c}{M^f}\mathbf{M} \tag{25}$$

The positive proportionality factor in terms of the sliding velocity, denoted by the symbol $\bar{\Lambda}$, is given from Eqs. (25) as

$$\bar{\Lambda} = \frac{\frac{\partial f(\mathbf{f})}{\partial \mathbf{f}} \bullet \mathbf{C}^e \bullet \bar{\mathbf{v}} - M^c}{M^f + \frac{\partial f(\mathbf{f})}{\partial \mathbf{f}} \bullet \mathbf{C}^e \bullet \mathbf{M}} \tag{26}$$

The traction rate is derived from Eqs. (3), (12), (22) and (26) as follows:

$$\overset{\circ}{\mathbf{f}} = \mathbf{C}^e \left\{ \bar{\mathbf{v}} - \left\langle \frac{\frac{\partial f(\mathbf{f})}{\partial \mathbf{f}} \bullet \mathbf{C}^e \bullet \bar{\mathbf{v}} - M^c}{M^f + \frac{\partial f(\mathbf{f})}{\partial \mathbf{f}} \bullet \mathbf{C}^e \bullet \mathbf{M}} \right\rangle \mathbf{M} \right\} \tag{27}$$

The function $f(\mathbf{f})$ for the closed normal-sliding surface is given by

$$f(\mathbf{f}) = f_n g(\chi), \quad \chi \equiv \eta/M, \quad \eta \equiv f_t/f_n \tag{28}$$

The simple example is $g(\chi) = \exp(\chi^2/2)$ having the tear-shape[4].

On the other hand, the normal-sliding and the sliding-subloading surfaces for the circular cone of the *Coulomb friction condition* are given by putting

$$f(f) = f_t/f_n, \quad F = \mu \tag{29}$$

as follows:

$$f_t/f_n = \mu, \quad f_t/f_n = \bar{R}\mu \tag{30}$$

where $\mu$ is the friction coefficient.

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