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General Corotational Rate Tensor and Replacement to Corotational Derivative of Yield Function

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Summary

General corotational rate of tensors in arbitrary order having the objectivity is shown first, and then it is verified that the material-derivative of yield condition can be replaced generally to the corotational derivative, i.e. the consistency condition.

Introduction

Mechanical property of materials are observed to be identical independent of states of observers. Then, it should be described by a unique equation independent of mutual configuration and/or rotation between material and observers. This fact is called the *principle of material-frame indifference* [1]. On account of this principle, all of physical quantities used in constitutive equations have to be described by the tensor quantities obeying the common translation rule, called the *objective transformation*, between coordinate systems. In particular, constitutive equation of inelastic deformation has to be formulated as the relation between rate variables through the stress and internal variables. Whilst all state quantities obey the objective transformation, pertinent tensors obeying the objective transformation independent of material rotation have to be adopted for their rate variables. In addition, they have to be physical quantities capable of describing rates of mechanical state appropriately evaluating a rotation of material. They can be given by the *corotational rate tensor*s which have components obtained by the objective inverse-transformation from the components observed by the coordinate system rotating with material to the fixed coordinate system describing the constitutive relation.

Besides, in the formulation of plastic constitutive equation the consistency condition is obtained first by material-time differentiation of yield condition. In order to use it as a constitutive relation one has to translate the stress rate and rates of internal variables to their corotational rate.

In this note a general corotational rate for tensors in arbitrary order having the objectivity is shown first. Further, it is verified that the material-derivative of yield condition involving arbitrary tensors can be replaced to the corotational derivative, i.e. the consistency condition which can be used as a constitutive relation.

General Corotational Rate Tensor

Consider the normalized-orthogonal coordinate systems $\{O-x_i\}$ (i=1, 2, 3) with the base $\{\mathbf{e}_i\}$ and $\{O'-x'_i(t)\}$ (t: time) with the base $\{\mathbf{e}'_i(t)\}$. Here, let $\{\mathbf{e}_i\}$ be the fixed standard base and $\{e'_i(t)\}$ the movable base, provided that the latter has coincided with the former in the initial state ($t=0$). Let it be assumed that the

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material particle P which had the position vector $\mathbf{X}$ at $t = 0$ has the components $x_i(\mathbf{X},t)$ and $x_i^*(\mathbf{X},t)$ in the coordinate systems $\{O - x_i\}$ and $\{O' - x_i'(t)\}$, respectively. Further, let the position vector of the origin O' of the coordinate system $\{O' - x_i'(t)\}$ have the components $c_i(t)$ in the coordinate system$\{O - x_i\}$. Then, the following relations holds between these components.

$$x_i^*(\mathbf{X},t) = Q_{ir}(t)(x_r(\mathbf{X},t) - c_r(t)),\ x_i(\mathbf{X},t) = Q_{ri}(t)x_r^*(\mathbf{X},t) + c_i(t) \quad (1)$$

where the Einstein's summation convention is used throughout this note. Eq. (1) is rewritten in the symbolic notation as

$$\mathbf{x}^*(\mathbf{X},t) = \mathbf{Q}(t)(\mathbf{x}(\mathbf{X},t) - c(t)), \quad \mathbf{x}(\mathbf{X},t) = \mathbf{Q}^T(t)x^*(\mathbf{X},t) + c(t) \quad (2)$$

where the notation $()^T$ stands for the transpose. Hereafter, the superscript $*$ is added to the components for the movable base $\{\mathbf{e}_i'(t)\}$. $\mathbf{Q}(t)$ is the orthogonal tensor of the base $\{\mathbf{e}_i'(t)\}$ with respect to the standard base $\{\mathbf{e}_i\}$ and has the components

$$Q_{ij}(t) \equiv \mathbf{e}_i'(t) \bullet \mathbf{e}_j \quad (3)$$

where the dot $\bullet$ denotes the scalar product. The symbol (t) describing the time dependence is omitted hereafter.

From Eq. (3) one has the relation

$$\mathbf{e}_i (= (\mathbf{e}_i \bullet \mathbf{e}_r')\mathbf{e}_r') = Q_{ri}\mathbf{e}_r', \quad \mathbf{e}_i' (= (\mathbf{e}_i' \bullet \mathbf{e}_r)\mathbf{e}_r) = Q_{ir}\mathbf{e}_r \quad (4)$$

between these bases. It holds that $\mathbf{QQ^T} = \mathbf{Q^TQ} = \mathbf{I}$, where $\mathbf{I}$ is the second-order identity tensor having the components of Kronecker's delta $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$ and thus $\mathbf{I} \equiv \mathbf{e}_i \otimes \mathbf{e}_i = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$. The tensor $\mathbf{Q}$ is described in the following form with the bases.

$$\mathbf{Q} = Q_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = Q_{ij}\mathbf{e}_i' \otimes e_j' \quad \text{or} \quad \mathbf{Q} = \mathbf{e}_r \otimes \mathbf{e}_r' \quad (5)$$

due to Eq. (4). The relation between these bases is also described from Eq. (5) as follows:

$$\mathbf{e}_i (= \mathbf{e}_r \otimes \mathbf{e}_r'\mathbf{e}_i') = \mathbf{Q}\mathbf{e}_i', \quad \mathbf{e}_i' (= \mathbf{e}_r' \otimes \mathbf{e}_r\mathbf{e}_i) = \mathbf{Q}^T\mathbf{e}_i \quad (6)$$

Introduce the second-order tensor

$$\Omega \equiv \dot{\mathbf{e}}_r' \otimes \mathbf{e}'_r \quad (7)$$

where $(^\bullet)$ denotes the material-time derivative. Eq. (7) is rewritten as

$$\Omega \equiv \dot{Q}_{ri}Q_{rj}\mathbf{e}_i \otimes \mathbf{e}_j, \quad \Omega \equiv \dot{\mathbf{Q}}^T\mathbf{Q} \quad (8)$$

due to Eq. (4). It is known that Ω is the skew-symmetric tensor from Eq. (8) and means the spin of the base $\{\mathbf{e}'_i\}$ from the equation

$$\dot{\mathbf{e}}'_r = \Omega \mathbf{e}'_r \tag{9}$$

obtained from Eq. (7).

The following equation is obtained from Eq. (2).

$$\mathbf{v}^* = \dot{\mathbf{x}}^* = \mathbf{Q}\mathbf{v} + \dot{\mathbf{Q}}\mathbf{x} - \mathbf{Q}\dot{\mathbf{c}} - \dot{\mathbf{Q}}\mathbf{c} = \mathbf{Q}\mathbf{v} + \bar{\Omega}\mathbf{x}^* - \mathbf{Q}\dot{\mathbf{c}} - \dot{\mathbf{Q}}\mathbf{c} \tag{10}$$

where $\mathbf{v}$ is the velocity of material particle P in the movable base $\{\mathbf{e}'_i\}$ and $\bar{\Omega}$ is the second-order skew-symmetric tensor given by

$$\bar{\Omega} \equiv \dot{\mathbf{Q}}\mathbf{Q}^T = -\mathbf{Q}\Omega\mathbf{Q}^T \tag{11}$$

The transformation rule of m-th order tensor $\mathbf{T}$ describing the mechanical state of material is given by

$$\left.\begin{array}{l} T^*_{p_1p_2\cdots p_m} = Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}T_{q_1q_2\cdots q_m} \\ T_{p_1p_2\cdots p_m} = Q_{q_1p_1}Q_{q_2p_2}\cdots Q_{q_mp_m}T^*_{q_1q_2\cdots q_m} \end{array}\right\} \tag{12}$$

The material-time derivative of Eq. (12) is given as

$$\begin{aligned} \dot{T}^*_{p_1p_2\cdots p_m} &= \dot{Q}_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}T_{q_1q_2\cdots q_m} + Q_{p_1q_1}\dot{Q}_{p_2q_2}\cdots Q_{p_mq_m}T_{q_1q_2\cdots q_m} + \cdots \\ &+ Q_{p_1q_1}Q_{p_2q_2}\cdots \dot{Q}_{p_mq_m}T_{q_1q_2\cdots q_m} + Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}\dot{T}_{q_1q_2\cdots q_m} \end{aligned} \tag{13}$$

or inversely

$$\begin{aligned} \dot{T}_{p_1p_2\cdots p_m} &= \dot{Q}_{q_1p_1}Q_{q_2p_2}\cdots Q_{q_mp_m}T_{q_1q_2\cdots q_m} + Q_{q_1p_1}\dot{Q}_{q_2p_2}\cdots Q_{q_mp_m}T^*_{q_1q_2\cdots q_m} + \cdots \\ &+ Q_{q_1p_1}\dot{Q}_{q_2p_2}\cdots Q_{q_mp_m}T^*_{q_1q_2\cdots q_m} + \cdots + Q_{q_1p_1}Q_{q_2p_2}\cdots \dot{Q}_{q_mp_m}T^*_{q_1q_2\cdots q_m} \\ &+ Q_{q_1p_1}Q_{q_2p_2}\cdots Q_{q_mp_m}\dot{T}^*_{q_1q_2\cdots q_m} \end{aligned} \tag{14}$$

which are rewritten as

$$\begin{aligned} \dot{T}^*_{p_1p_2\cdots p_m} &= Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}(\dot{T}_{q_1q_2\cdots q_m} \\ &- \Omega_{q_1r_1}T_{r_1q_2\cdots q_m} - \Omega_{q_1r_2}T_{q_1r_2\cdots q_m} - \Omega_{q_1r_m}T_{q_1q_2\cdots r_m}) \end{aligned} \tag{15}$$

or inversely

$$\begin{aligned} \dot{T}_{p_1p_2\cdots p_m} &= Q_{q_1p_1}Q_{q_2p_2}\cdots Q_{q_mp_m}\{\dot{T}^*_{q_1q_2\cdots q_m} \\ &- \bar{\Omega}_{q_1r_1}T^*_{r_1q_2\cdots q_m} - \bar{\Omega}_{q_2r_2}T^*_{q_1r_2\cdots q_m} + \cdots - \bar{\Omega}_{q_mr_m}T^*_{q_1q_2\cdots r_m}\} \end{aligned} \tag{16}$$

The rate of tensor quantity used for constitutive equations in rate forms has to fulfill the following conditions.

1. It obeys the objective transformation since the material properties are independent of the observers.
2. The components in the standard fixed coordinate system describing constitutive equation changes only when the components observed in the coordinate system moving with material changes.

As known from Eqs. (13)-(16) the material-time derivative does not obey the objective transformation and it changes even if the components observed by the coordinate system rotating with material does not change, provided that the coordinate system $\{\mathbf{e}'_i\}$ rotates with material, selecting the spin Ω as the spin of material. In other words, the material-time derivative violates both conditions 1 and 2.

Then, consider the tensor having the components obtained from the components observed in the coordinate system rotating with material by the objective inverse transformation rule, and let it be called the *corotational rate*, denoting it by $\overset{\bullet}{\mathbf{T}}$, i.e.

$$\begin{aligned}\overset{\circ}{T}_{p_1p_2\cdots p_m} &= Q_{q_1p_1}Q_{q_2p_2}\cdots Q_{q_mp_m}\overset{\bullet}{T}{}^*_{q_1q_2\cdots q_m}\\ &= \overset{\bullet}{T}_{p_1p_2\cdots r_m} - \omega_{p_1r_1}T_{r_1p_2\cdots p_m} - \omega_{p_2r_2}T_{p_1r_2\cdots p_m} - \cdots - \omega_{p_mr_m}T_{p_1p_2\cdots r_m} \qquad (17)\end{aligned}$$

where the *corotational spin* $\boldsymbol{\omega}$ is the second-order skew symmetric tensor describing the spin of material and thus obeys the transformation rule

$$\boldsymbol{\omega}^* = \mathbf{Q}(\boldsymbol{\omega} - \Omega)\mathbf{Q}^T \qquad (18)$$

$\overset{\circ}{\mathbf{T}}$ obeys the objective transformation rule as known from

$$\begin{aligned}\overset{\circ}{T}{}^*_{p_1p_2\cdots p_m} &= \overset{\bullet}{T}{}^*_{p_1p_2\cdots p_m} - \omega^*_{p_1r_1}T^*_{r_1p_2\cdots p_m} - \omega^*_{p_2r_2}T^*_{p_1r_2\cdots p_m} - \cdots - \omega^*_{p_mr_m}T^*_{p_1p_2\cdots r_m}\\ &= \overset{\circ}{Q}_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}\{(\overset{\bullet}{T}_{p_1p_2\cdots p_m} - \Omega_{p_1r_1}T_{r_1p_2\cdots p_m} - \Omega_{p_2r_2}T_{p_1r_2\cdots p_m} - \cdots - \Omega_{p_mr_m}T_{p_1p_2\cdots r_m})\\ &\quad -(\omega_{p_1r_1} - \Omega_{p_1r_1})T_{r_1p_2\cdots p_m} - (\omega_{p_2r_2} - \Omega_{p_2r_2})T_{p_1r_2\cdots p_m} - \cdots - (\omega_{p_mr_m} - \Omega_{p_mr_m})T_{p_1p_2\cdots r_m}\}\\ &= Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}(\overset{\bullet}{T}_{p_1p_2\cdots p_m} - \omega_{p_1r_1}T_{r_1p_2\cdots p_m} - \omega_{p_2r_2}T_{p_1r_2\cdots p_m} - \cdots - \omega_{p_mr_m}T_{p_1p_2\cdots r_m})\\ &= Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}\overset{\circ}{T}_{p_1p_2\cdots p_m} \qquad (19)\end{aligned}$$

Introduce the following notation for the orthogonal transformation.

$$\left.\begin{aligned}(\mathbf{Q}[\![\mathbf{T}]\!])_{p_1p_2\cdots p_m} &\equiv Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}T_{q_1q_2\cdots q_m}\\ (\mathbf{Q}^T[\![\mathbf{T}]\!])_{p_1p_2\cdots p_m} &\equiv Q_{q_1p_1}Q_{q_2p_2}\cdots Q_{q_mp_m}T_{q_1q_2\cdots q_m}\end{aligned}\right\} \qquad (20)$$

By use of this notation Eqs. (12), (17) and (19) are rewritten as follows:

$$\mathbf{T}^* = \mathbf{Q}[\![\mathbf{T}]\!],\ \mathbf{T} = \mathbf{Q}^T[\![\mathbf{T}^*]\!] \qquad (21)$$

and

$$\overset{\circ}{\mathbf{T}}^{*} = \mathbf{Q}[\![\overset{\circ}{\mathbf{T}}]\!], \quad \overset{\circ}{\mathbf{T}} = \mathbf{Q}^{T}[\![\overset{\circ}{\mathbf{T}}^{*}]\!] \tag{22}$$

$$\overset{\circ}{\mathbf{T}} = \mathbf{Q}^{T}[\![\overset{\bullet}{\mathbf{T}}^{*}]\!] \tag{23}$$

One must select explicitly the physically pertinent spin of material for $\boldsymbol{\omega}$. The velocity gradient $\mathbf{L}$, the strain rate $\mathbf{D}$ and the continuum spin $\mathbf{W}$ are given by

$$\mathbf{L} \equiv \partial \mathbf{v}/\partial \mathbf{x}, \quad \mathbf{D} \equiv (\mathbf{L}+\mathbf{L}^{T}), \quad \mathbf{W} \equiv (\mathbf{L}-\mathbf{L}^{T})/2 \tag{24}$$

the transformations of which are given as

$$\mathbf{L}^{*} = \mathbf{Q}(\mathbf{L}-\Omega)\mathbf{Q}^{T} = \mathbf{Q}\mathbf{L}\mathbf{Q}^{T} + \bar{\Omega} \tag{25}$$

$$\mathbf{D}^{*} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{T}, \mathbf{W}^{*} = \mathbf{Q}(\mathbf{W}-\Omega)\mathbf{Q}^{T} = \mathbf{Q}\mathbf{W}\mathbf{Q}^{T} + \bar{\Omega} \tag{26}$$

It is known that the continuum spin $\mathbf{W}$ obeys the same translation as that in Eq. (18) for $\boldsymbol{\omega}$.

The following corotational rate with the continuum spin $\mathbf{W}$ is regarded as the generalization of *Zaremba-Jaumann rate.*

$$\overset{\square}{T}_{p_1 p_2 \cdots p_m} = \overset{\bullet}{T}_{p_1 p_2 \cdots r_m} - W_{p_1 r_1} T_{r_1 p_2 \cdots p_m} - W_{p_2 r_2} T_{p_1 r_2 \cdots p_m} - \cdots - W_{p_m r_m} T_{p_1 p_2 \cdots r_m} \tag{27}$$

Jaumann rate is determined merely geometrically by an external appearance of body independent of material properties and deformation history.

While the corotational spin would have to reflect the rotation of substructure in material, it is not so large as calculated by the continuum (material) spin $\mathbf{W}$ when a plastic deformation is induced. Then, the corotational rate with the following elastic spin $\mathbf{W}_e$ would be physically pertinent [2].

$$\mathbf{W}_e \equiv \mathbf{W} - \mathbf{W}^p \tag{28}$$

where $\mathbf{W}^p$ is called the *plastic spin* and could be given as follows [3]:

$$\mathbf{W}^p = \mu(\boldsymbol{\sigma}, \mathbf{H}, \mathbf{H})(\mathbf{D}^p\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{D}^p) \tag{29}$$

where $\mathbf{D}^p$ is the plastic strain rare, and μ is the material function of stress $\boldsymbol{\sigma}$ and internal variables of scalar quantity $\mathbf{H}$ for isotropic hardening/softening and tensor-valued quantity $\mathbf{H}$ for inherent and/or induced anisotropy. Obviously $\mathbf{W}_e$ obeys the same translation as that in Eq. (18) of $\boldsymbol{\omega}$, i.e.

$$\mathbf{W}^{e^*} = \mathbf{Q}(\mathbf{W}^e - \Omega)\mathbf{Q}^{T} \tag{30}$$

The generalized corotational rate $\overset{*}{\mathbf{T}}$ based on the plastic spin is given as follows.

$$\overset{*}{T}_{p_1 p_2 \cdots p_m} = \overset{\bullet}{T}_{p_1 p_2 \cdots r_m} - W^e_{p_1 r_1} T_{r_1 p_2 \cdots p_m} - W^e_{p_2 r_2} T_{p_1 r_2 \cdots p_m} - \cdots - W^e_{p_m r_m} T_{p_1 p_2 \cdots r_m} \tag{31}$$

The corotational rate has to be adopted for rates of tensor valued state variables, i.e. the stress and tensor-valued internal variables for describing anisotropy.

Transformation to Corotational Tensors in Consistency Condition

In order to obtain the consistency condition from the material-time derivative of yield condition, which is pertinent as a constitutive relation, one has to translate the stress rate and rates of internal variables to their corotational rate.

Yield condition is described generally as

$$f(\mathbf{A},\mathbf{B},\cdots)=0 \tag{32}$$

where $\mathbf{A},\mathbf{B},\cdots$ are the arbitrary tensors. The material-time derivative of Eq. (32) is described as

$$\dot{f}(\mathbf{A},\mathbf{B},\cdots)=\mathrm{tr}\left(\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial\mathbf{A}}\dot{\mathbf{A}}^T\right)+\mathrm{tr}\left(\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial\mathbf{B}}\dot{\mathbf{B}}^T\right)+\cdots=0 \tag{33}$$

Here, f is a scalar-valued function and then it holds that

$$f(\mathbf{A},\mathbf{B},\cdots)=f(\mathbf{A}^*,\mathbf{B}^*,\cdots),\quad \dot{f}(\mathbf{A},\mathbf{B},\cdots)=\dot{f}(\mathbf{A}^*,\mathbf{B}^*,\cdots) \tag{34}$$

where

$$\begin{aligned}\dot{f}(\mathbf{A}^*,\mathbf{B}^*,\cdots)&=\mathrm{tr}\left(\frac{\partial f(\mathbf{A}^*,\mathbf{B}^*,\cdots)}{\partial\mathbf{A}^*}\dot{\mathbf{A}}^{*T}\right)+\mathrm{tr}\left(\frac{\partial f(\mathbf{A}^*,\mathbf{B}^*,\cdots)}{\partial\mathbf{B}^*}\right.+\cdots\\
&=\mathrm{tr}\left(Q\left[\!\left[\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial\mathbf{A}}\right]\!\right]\dot{\mathbf{A}}^{*T}\right)+\mathrm{tr}\left(\left[\!\left[\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial\mathbf{B}}\right]\!\right]\dot{\mathbf{B}}^{*T}\right)+\cdots\\
&=\mathrm{tr}\left(\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial\mathbf{A}}\mathring{\mathbf{A}}^T\right)+\mathrm{tr}\left(\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial\mathbf{B}}\mathring{\mathbf{B}}^T\right)+\cdots\end{aligned} \tag{35}$$

using Eq. (23) and the following relation for two arbitrary tensors T and S.

$$f(\mathbf{Q}[\![\mathbf{T}]\!]\mathbf{S}^T)=Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}T_{q_1q_2\cdots q_m}\overset{=}{S}_{p_1p_2\cdots p_m}f(\mathbf{T}(\mathbf{Q}^T[\![\mathbf{S}]\!])^T) \tag{36}$$

From Eqs. (33), (34) and (35) we have

$$\mathrm{tr}\left(\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial\mathbf{A}}\mathring{\mathbf{A}}^T\right)+\mathrm{tr}\left(\frac{\partial f(\mathbf{A},\mathbf{B},\cdots)}{\partial\mathbf{B}}\mathring{\mathbf{B}}^T\right)+\cdots=0 \tag{37}$$

Then, it is concluded that the material-time derivative of the yield function can be directly replaced to the corotational derivative, while the notation of transpose can be omitted for stress and anisotropic hardening variables which are symmetric tensors.

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