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Implicit Formulation of Homogenization Method for Periodic Inelastic Solids

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Summary

In this study, to determine incremental, perturbed displacement fields in periodic inelastic solids, an incremental homogenization problem is fully implicitly formulated, and an algorithm is developed to solve the homogenization problem. It is shown that the homogenization problem can be iteratively solved with quadratic convergences by successively updating strain increments in unit cells, and that the present formulation allows versatility in the initial setting of strain increments in contrast to previous studies. The homogenization algorithm developed is then examined by analyzing a holed plate, with an elastoplastic micro-structure, subjected to tensile loading. It is thus demonstrated that the convergence in iteratively solving the homogenization problem strongly depends on the initial setting of strain increments in unit cells, and that quick convergences can be attained if the initial setting of strain increments is appropriate.

Introduction

There are macro-structures made of such inhomogeneous solids, as composite materials and cellular solids, which have explicit micro-structures. This kind of macro-structures, in which macro-strain and macro-stress in general distribute non-uniformly, can be analyzed as those made of homogeneous solids, if the macro-properties of micro-structures are known. If micro-structures are periodic, their macro-properties can be evaluated using the mathematical homogenization method [1], which will be referred to as the homogenization method hereafter. However, if micro-structures are inelastic, this method forces the inelastic finite element analysis of unit cells to be performed at all integration points in each increment in analyzing that kind of macro-structures, resulting in very high computational loads. It is therefore worthwhile to develop efficient computational algorithms so that the homogenization method can be really effective for analyzing macro-structures with periodic, inelastic micro-structures [2,3].

In this study, an incremental homogenization problem is fully implicitly formulated so that macro-structures with periodic, elastoplastic micro-structures can be effectively analyzed. To this end, the virtual work equation mentioned above is employed, along with a linearized constitutive relation and a micro/macro-kinematic relation. An incremental boundary value problem based on the backward Euler method is thus built to efficiently analyze elastoplastic unit cells. An algorithm is then developed to iteratively solve the boundary value problem: this algorithm is

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shown to allow versatility in the initial setting of strain increments. The algorithm developed is examined by analyzing a holed plate, with an elastoplastic micro-structure, subjected to tensile loading.

Basic Equations

Let us consider a macro-structure B with a periodic micro-structure, which has a unit cell Y. Let us assume that small deformation occurs in B. Then, by denoting macro-strain in B as E, micro-strain in Y is expressed as

$$\varepsilon = \mathbf{E} + \tilde{\varepsilon}, \tag{1}$$

where $\tilde{\varepsilon}$ represents the perturbed strain in Y.

Since $\tilde{\varepsilon}$ satisfies the Y-periodicity, the following virtual work equation postulated by Hill [4] is identically satisfied [5]:

$$\Sigma : \delta\mathbf{E} = \langle \sigma : \delta\varepsilon \rangle, \tag{2}$$

where Σ and σ signify macro-stress and micro-stress, respectively, δ indicates any variation, and $\langle\ \rangle$ represents the volume average in Y defined as

$$\langle \# \rangle = |Y|^{-1} \int_Y \# \mathrm{d}Y. \tag{3}$$

Here $|Y|$ denotes the volume of Y. Then, substituting Eq. 1 into Eq. 2, we obtain

$$\Sigma = \langle \sigma \rangle, \tag{4}$$

$$\langle \sigma : \delta\tilde{\varepsilon} \rangle = 0. \tag{5}$$

Eq. 4 is the relation between macro-stress and micro-stress, and Eq. 5 is regarded as the virtual work equation concerning perturbed displacement. It is shown that Eq. 5 represents the micro-stress balance in Y in the absence of body forces [5,6].

Let us suppose that elastoplastic deformation occurs in Y. Then, micro-stress is expressed as

$$\sigma = D^e : (\varepsilon - \varepsilon^p), \tag{6}$$

where D^e indicates elastic stiffness, and ε^p denotes micro-plastic strain.

Incremental Homogenization Problem

Let us consider the incremental step from n to $n+1$. The problem discussed here is then stated as follows: Given $\Delta\mathbf{E}_{n+1} (= \mathbf{E}_{n+1} - \mathbf{E}_n)$ in addition to $\mathbf{u}_n$ and ε_n^p in Y, find macro-stress Σ_{n+1} based on the backward Euler method. Here and from now on, Δ indicates the increments in the step from n to $n+1$.

To evaluate Σ_{n+1} implicitly, we suppose that Eq. 1 and Eqs. 4-6 are satisfied at $n+1$:

$$\varepsilon_{n+1} = \mathbf{E}_{n+1} + \tilde{\varepsilon}_{n+1}, \tag{7}$$

$$\Sigma_{n+1} = \langle \sigma_{n+1} \rangle, \tag{8}$$

$$\langle \sigma_{n+1} : \delta\tilde{\varepsilon} \rangle = 0, \tag{9}$$

$$\sigma_{n+1} = D^e_{n+1} : (\varepsilon_n + \Delta\varepsilon_{n+1} - \varepsilon^p_n - \Delta\varepsilon^p_{n+1}). \tag{10}$$

Here it is noted that $\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_{n+1}$ and $\varepsilon^p_{n+1} = \varepsilon^p_n + \Delta\varepsilon^p_{n+1}$. Let us linearlize Eq. 10 as

$$\sigma^{(i+1)}_{n+1} - \sigma^{(i)}_{n+1} = D^{(i)}_{n+1} : (\Delta\varepsilon^{(i+1)}_{n+1} - \Delta\varepsilon^{(i)}_{n+1}), \tag{11}$$

where the superscript (i) denotes the i th iteration, and $D^{(i)}_{n+1}$ is a consistent tangent modulus

$$D^{(i)}_{n+1} = \partial\sigma^{(i)}_{n+1} \Big/ \partial\Delta\varepsilon^{(i)}_{n+1}. \tag{12}$$

Then, representing $\Delta\varepsilon^{(i+1)}_{n+1}$ in Eq. 11 as

$$\Delta\varepsilon^{(i+1)}_{n+1} = \Delta\mathbf{E}_{n+1} + \Delta\tilde{\varepsilon}^{(i+1)}_{n+1}, \tag{13}$$

and assuming that $\sigma^{(i+1)}_{n+1}$ in Eq. 11 satisfies Eq. 9, we obtain an incremental boundary value problem

$$\left\langle \delta\tilde{\varepsilon} : D^{(i)}_{n+1} : \nabla_y \Delta\tilde{\mathbf{u}}^{(i+1)}_{n+1} \right\rangle = -\left\langle \delta\tilde{\varepsilon} : \sigma^{(i)}_{n+1} \right\rangle - \left\langle \delta\tilde{\varepsilon} : D^{(i)}_{n+1} : (\Delta\mathrm{E}_{n+1} - \Delta\varepsilon^{(i)}_{n+1}) \right\rangle. \tag{14}$$

Computational Algorithm

Eq. 14 can be in general solved for $\Delta\tilde{\mathbf{u}}^{(i+1)}_{n+1}$ using FEM, if $\Delta\mathbf{E}_{n+1}$ is prescribed, and if $D^{(i)}_{n+1}$, $\sigma^{(i)}_{n+1}$ and $\Delta\varepsilon^{(i)}_{n+1}$ are known. Hence, we can consider the computational algorithm, in which micro-strain increment $\Delta\varepsilon^{(i)}_{n+1}$ is updated to $\Delta\varepsilon^{(i+1)}_{n+1}$ after solving Eq. 14 for $\Delta\tilde{\mathbf{u}}^{(i+1)}_{n+1}$. If $\Delta\varepsilon^{(i+1)}_{n+1}$ satisfies a convergence condition, Σ_{n+1} is evaluated using Eq. 8.

Let us note that the initial value of micro-strain increment, $\Delta\varepsilon^{(0)}_{n+1}$, is arbitrarily chosen in the algorithm, because $\Delta\varepsilon^{(0)}_{n+1}$ has not been replaced by $\Delta\mathbf{E}_{n+1} + \Delta\tilde{\varepsilon}^{(0)}_{n+1}$ using Eq. 13 in deriving Eq. 14. Examples of $\Delta\varepsilon^{(0)}_{n+1}$ are

$$\Delta\varepsilon^{(0)}_{n+1} = \mathbf{0}, \tag{15}$$

$$\Delta\varepsilon^{(0)}_{n+1} = \Delta\varepsilon_n, \tag{16}$$

$$\Delta\varepsilon^{(0)}_{n+1} = \Delta\mathbf{E}_{n+1}, \tag{17}$$

where $\Delta\varepsilon_n$ indicates the micro-strain increment that converged in the preceding incremental step.

Example of Numerical Analysis

A holed plate with a micro-structure was analyzed using the system mentioned above. The plate was subjected to longitudinal displacement u_L^0 at its ends under the plane strain condition. A quarter of the plate was divided into finite elements, as shown in Fig. 1(a). The displacement u_L^0 was increased from zero to $u_{L,\max}^0$ =0.4 mm in N steps, so that the increment of u_L^0 was taken as $\Delta u_L^0 = u_{L,\max}^0/N$ in the analysis. The micro-structure was assumed to have the unit cell Y shown in Fig. 1(b).

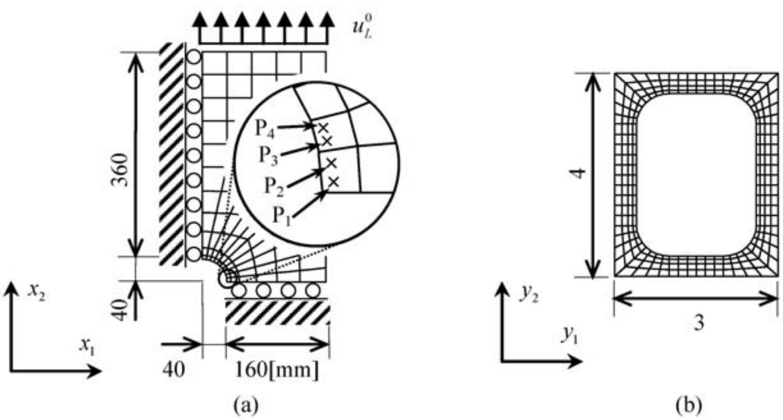

Fig. 1 Example of analysis; (a) finite element mesh of a quarter of holed plate subject to tensile displacement u_L^0, and (b) finite element mesh of unit cell Y

The constituent of Y was an elastoplastic solid, for which the linear isotropic hardening J_2 plasticity model based on the following yield condition was employed to specify $\dot{\varepsilon}^p$:

$$f = \bar{\sigma}^2 - (\sigma_0 + E^p p)^2, \tag{18}$$

where σ_0 and E^p are material parameters, $\bar{\sigma}$ denotes the equivalent micro-stress defined in terms of deviatoric micro-stress $\mathbf{s}$ as $\bar{\sigma} = (\frac{3}{2}\mathbf{s} : \mathbf{s})^{1/2}$, p indicates the accumulated micro-plastic strain obtained by integrating $\dot{p} = (\frac{2}{3}\dot{\varepsilon}^p : \dot{\varepsilon}^p)^{1/2}$. The analysis was done by assuming the material parameters as follows: Young's modulus E =150 GPa, Poisson's ratio ν =0.3, initial yield stress σ_0 =120 MPa, and plastic hardening modulus E^p =5 GPa. The return mapping algorithm was employed to compute $\sigma_{n+1}^{(i)}$ and $D_{n+1}^{(i)}$ in the homogenization algorithm.

The three initial settings of $\Delta\varepsilon_{n+1}^{(0)}$, Eqs. 15-17, were examined by performing the analysis mentioned above. Then, $\Delta\varepsilon_{n+1}^{(0)}$ was found to have significant influences (Table 1); $\Delta\varepsilon_{n+1}^{(0)} = \Delta\varepsilon_n$ was the best among them, $\Delta\varepsilon_{n+1}^{(0)} = \mathbf{0}$ was fairly successful,

Table 1 Influence of $\Delta\boldsymbol{\varepsilon}_{n+1}^{(0)}$ on the convergence of micro-iteration under N step loading; S represents success in the convergence of micro-iteration, while F(n) indicates no convergence of micro-iteration in the nth step of macro-analysis

Number of steps, N	$\Delta\boldsymbol{\varepsilon}_{n+1}^{(0)} = \mathbf{0}$	$\Delta\boldsymbol{\varepsilon}_{n+1}^{(0)} = \Delta\boldsymbol{\varepsilon}_n$	$\Delta\boldsymbol{\varepsilon}_{n+1}^{(0)} = \Delta\mathbf{E}_{n+1}$
5	F (2)	F (2)	F (1)
10	F (3)	S	F (3)
20	S	S	F (6)
40	S	S	F (18)
70	S	S	F (45)
100	S	S	F (78)

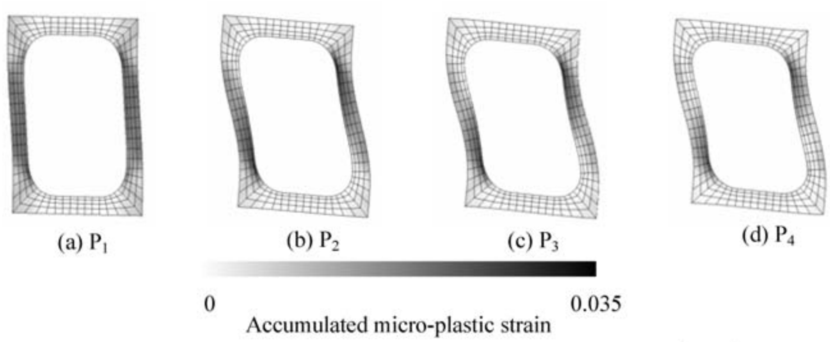

Fig. 2 Configuration of unit cell Y at macro-points P_1 - P_4 at $u_L^0 = u_{L,\max}^0$ under 10 step loading in the analysis of $\Delta\boldsymbol{\varepsilon}_{n+1}^{(0)} = \Delta\boldsymbol{\varepsilon}_n$; displacement $\times 10$

but $\Delta\varepsilon_{n+1}^{(0)} = \Delta\mathbf{E}_{n+1}$ resulted in very bad convergences in iteratively solving the incremental homogenization problem. Figs. 2(a)-(d) depict the configurations of Y at macro-integration points P_1 to P_4 indicated in Fig. 1(a). It is seen from the figures that the longitudinal cell walls were considerably bent at P_2 to P_4. It is however noted that the initial setting of $\Delta\varepsilon_{n+1}^{(0)} = \Delta\mathbf{E}_{n+1}$ completely ignores the bending of cell walls. This may account for the very bad convergences brought about by $\Delta\varepsilon_{n+1}^{(0)} = \Delta\mathbf{E}_{n+1}$.

Conclusions

In this study, an incremental homogenization problem was fully implicitly formulated to determine perturbed displacement fields in periodic elastoplastic solids, and an algorithm was developed to solve the homogenization problem. It was pointed out that the present formulation allows versatility in the initial setting of micro-strain increment $\Delta\varepsilon_{n+1}^{(0)}$ in contrast to those by Terada and Kikuchi [2] and Miehe [3]. Three initial settings of $\Delta\varepsilon_{n+1}^{(0)}$, i.e., $\Delta\varepsilon_{n+1}^{(0)} = \mathbf{0}$, $\Delta\varepsilon_{n+1}^{(0)} = \Delta\varepsilon_n$ and

$\Delta\varepsilon_{n+1}^{(0)} = \Delta\mathbf{E}_{n+1}$, were considered. A holed plate with a micro-structure was then analyzed as a numerical example, resulting in the following finding: $\Delta\varepsilon_{n+1}^{(0)} = \Delta\varepsilon_n$ was the best among the three, $\Delta\varepsilon_{n+1}^{(0)} = \mathbf{0}$ was fairly successful, but $\Delta\varepsilon_{n+1}^{(0)} = \Delta\mathbf{E}_{n+1}$ gave very bad convergences in iteratively solving the incremental homogenization problem.

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