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Application of Gaussian Approximating Functions to the Solution of the Second Boundary Value Problem of Elasto-Plasticity for 2D Isotropic Bodies

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Summary

In this work Gaussian approximating functions proposed in the works of V. Maz'ya are used for the solution of the integral equations of elasto-plasticity for isotropic bodies. The use of this functions esentially simplify the calculation of the elements of the final matrix of the linear algebraic equations of the discretized problem. The elements of this matrix turn to be a combination of simple elementary functions. The method is applied to a 2D rectangular body that has a cut on a border and is subjected to axial tension. The convergence of the method is studied on this example.

Introduction

A new class of approximating functions of Gaussian type was proposed in the works of Maz'ya [1,2] for the solution of a wide class of integral equations of mathematical physics. The theory of approximation by Gaussian functions was developed in the works of Maz'ya and Schmidt [3,4]. These functions were used for the solution of static and dynamic problems of elasticity for plane bodies with cracks [5,6,7]. The main advantages of using these functions are the following:

- The action of many integral operators of mathematical physics on Gaussian approximating functions are expressed in the form of combinations of some standard functions that can be tabulated and retained in the computer memory to be used afterwards for the solution of a wide class of similar problems. essentially reducing the time required for the calculation of the matrix of the discretized problem.
- If Gaussian functions are used for approximation, the information needed for discretization of the problem is the definition of only coordinates and the normal vectors of equidistant points (nodes) in the boundary region.

In this work we apply the class of Gaussian approximating functions to the solution of the second boundary value problem of elasto-plasticity.

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Integral Equations for the Second Boundary Value Problem of Elasto-Plasticity

Let us consider a body that ocuppies the region Ω in the 3D space with a smooth boundary Γ . The material of the body is homogeneous and isotropic with elastic moduli tensor C. The system of differential equations of elasto-plasticity has the form [8]:

$$\partial_i \sigma_{ij}= 0; \qquad \sigma_{ij}= C_{ijkl}\varepsilon^e_{kl}; \qquad \varepsilon_{kl}= \varepsilon^e_{kl}+\varepsilon^p_{kl}$$
$$\mathrm{Rot}_{ijkl}\ \varepsilon_{kl}= 0; \qquad \varepsilon^p_{ij}= \mathscr{F}_{ij}(\sigma); \qquad \partial_i=\frac{\partial}{\partial x_i} \tag{1}$$

Here, σ is the stress tensor, ε^e is the elastic deformation, ε^p is the plastic deformation, ε is the total deformation, Rot is the Saint-Venant incompatibility operator [8], and $\mathscr{F}(\sigma)$ is a functional that defines plastic deformation ε^p as a function of stress tensor σ_0 and x_i are cartesian coordinates in the space.

The stress tensor that satisfies this system of differential equations may be written as [9]

$$\sigma_{ij}=\int_{\Gamma} S_{ijkl}\left(\mathbf{x}-\mathbf{x}'\right) n_l\left(\mathbf{x}'\right) b_k\left(\mathbf{x}'\right) d\Gamma'+\int_{\Omega} S_{ijkl}\left(\mathbf{x}-\mathbf{x}'\right)\varepsilon^p_{kl}\left(\mathbf{x}'\right) dx'. \tag{2}$$

Here $\mathbf{n}(\mathbf{x})$ is the unit vector normal to the boundary Γ , $\mathbf{b}(\mathbf{x})$ is an arbitrary vector function on Γ , $\mathbf{S}(\mathbf{x})$ is the fourth rank tensor function defined as

$$S_{ijpq}(x) = -C_{ijkl}\nabla_k\nabla_m G_{ls}(x) C_{mspq} - C_{ijpq}\delta(x), \qquad \nabla_i=\frac{\partial}{\partial x_i}, \tag{3}$$

where, $\mathbf{x}$ is a point in the medium, $\delta(\mathbf{x})$ is the Dirac delta-function and $G_{ls}(\mathbf{x})$ is the Green function that satisfies the following equation

$$\nabla_i C_{ijkl}\nabla_k G_{lm}(x) = -\delta_{ij}\delta(x), \tag{4}$$

δ_{ij} is the Kronecker symbol.

Second integral in the right hand side of Eq.(2) represents the stresses due to plastic deformation ε^p , so

$$\sigma^p=\int_{\Omega} S\left(\mathbf{x}-\mathbf{x}'\right)\varepsilon^p\left(\mathbf{x}'\right) dx'. \tag{5}$$

For the stress-strain analysis in 2D-case equation similar to Eq.(2) takes the form

$$\varepsilon_{33}(\mathbf{x})=\frac{1}{E}\left[-\frac{2\mu}{1-\nu}\varepsilon^p_{33}(\mathbf{x})+\hat{\sigma}_{33}(\mathbf{x})-\nu\left(\sigma_{11}(\mathbf{x})+\sigma_{22}(\mathbf{x})\right)\right]+\varepsilon^p_{33}(\mathbf{x}).$$

a) Plane deformation state. For this situation the strain component $\varepsilon_{33}=0$, so $\hat{\sigma}_{33}(\mathbf{x})$ takes the form

$$\hat{\sigma}_{33}(\mathbf{x})=\nu\left[\sigma_{11}(\mathbf{x})+\sigma_{22}(\mathbf{x})\right]+\frac{\nu^2E}{1-\nu^2}\varepsilon^p_{33}(\mathbf{x}); \qquad x=(x_1,x_2) \tag{6}$$

b) Plane stress state. For this situation stress component $\sigma_{33}=0$, so $\hat{\sigma}_{33}(\mathbf{x})$ takes the following representation

$$\hat{\sigma}_{33}(\mathbf{x})=\frac{E}{1-\nu^2}\varepsilon^p_{33}(\mathbf{x}). \tag{7}$$

The stress tensor in Eq.(2) should satisfy the following boundary conditions

$$\sigma_{ij}(\mathbf{x})n_j(\mathbf{x})\big|_\Gamma=f_i(x), \tag{8}$$

where $f(\mathbf{x})$ is the vector of forces applied at the boundary of the body. After substituting Eq.(2) in Eq.(8) we obtain the equation for the density $\mathbf{b}(\mathbf{x})$ in the form

$$\int_\Gamma T_{ij}(\mathbf{x},\mathbf{x}')b_j(\mathbf{x}')d\Gamma'=f_i(\mathbf{x})-n_j\sigma^p_{ji}(\mathbf{x}), \tag{9}$$

where

$$T_{ij}\left(\mathbf{x},\mathbf{x}'\right)=n_k(\mathbf{x})S_{kijl}\left(\mathbf{x}-\mathbf{x}'\right)n_l(\mathbf{x}). \tag{10}$$

The kernel of the integral operator in Eq.(9) has a strong singularity

$$\mathbf{T}(\mathbf{x},\mathbf{x}')\sim\left|\mathbf{x}-\mathbf{x}'\right|^{-3} \text{ when } \mathbf{x}'\to\mathbf{x},$$

so a regularization procedure for the calculation of this integral should be defined. In [5,8] was demonstrated that integral in Eq.(9) can be understood in the following sense

$$\int_\Gamma T_{ij}(\mathbf{x},\mathbf{x}')b_j(\mathbf{x}')d\Gamma'=\text{ p.v.}\int_\Gamma T_{ij}(\mathbf{x},\mathbf{x}')\left[b_j(\mathbf{x}')-b_j(\mathbf{x})\right]d\Gamma'. \tag{11}$$

Here Γ is a smooth closed boundary, the integral in the right hand side is understood as its Cauchi principal value (p.v.), and $\mathbf{b}(\mathbf{x})$ is a smooth finite function. The same regularization formula holds for an open boundary if $\mathbf{b}(\mathbf{x})\to\mathbf{0}$ when $\mathbf{x}\to\Gamma$.

For calculation of the right hand side of Eq.(9), one has to know the plastic deformation ε^p inside the body. This deformation depends on the stress field $(\varepsilon^p=\mathscr{F}(\sigma))$ via the law of plasticity.

Gaussian Approximating Functions

For the numerical solution of Eq.(9) let us use a special class of Gaussian approximating functions. According to [1,2], unknown vector $\mathbf{b}(\mathbf{x})$ and plastic deformation tensor $\varepsilon^p(\mathbf{x})$ may be pressented as follows

$$\mathbf{b}(\mathbf{x}) \approx \sum_i \mathbf{b}^{(i)} \varphi(\mathbf{x}-\mathbf{x}_i); \qquad \varphi(\mathbf{x}) = \frac{1}{\sqrt{\pi D}} \exp\left(-\frac{|\mathbf{x}|^2}{Dh^2}\right); \tag{12}$$

$$\varepsilon^p(\mathbf{x}) \approx \sum_i \varepsilon^{p(i)} \Psi(\mathbf{x}-\mathbf{y}_i); \qquad \Psi(\mathbf{x}) = \frac{1}{\pi D} \exp\left(-\frac{|\mathbf{x}|^2}{Dh_1^2}\right). \tag{13}$$

Here, x_i $(i = 1,2,\ldots,N)$ is a set of nodes in the boundary Γ, h is the distance between boundary points $\mathbf{x}_i$, $\mathbf{y}_i$ $(i = 1,2,\ldots,N_1)$ is a set of nodes inside the plastic region, h_1 is the distance between nodes $\mathbf{y}_i$ and D is a nondimensional parameter $(D = 2)$.

Using approximation (12) the first integral in the right hand side of Eq.(2) may be written in the form [9]

$$\int_\Gamma \mathbf{S}(\mathbf{x}-\mathbf{x}')\,\mathbf{n}(\mathbf{x}')\,\mathbf{b}(\mathbf{x}')\,d\Gamma' \approx \sum_i \mathbf{S}^\Gamma(\mathbf{x}-\mathbf{x}_i)\,\mathbf{n}^{(i)}\mathbf{b}^{(i)}, \tag{14}$$

where tensor $S^\Gamma(x-x_i)$ is a combination of standard functions, $n^{(i)} = n(x_i)$ is the vector normal to the boundary of the body at point x_i (see details in [9]).

The stresses due to deformation in the plasticity region in eq.(5) are presented in the form

$$\sigma^p(x_1,x_2) = \int_\Omega \mathbf{S}(\mathbf{x}-\mathbf{x}')\,\varepsilon^p(\mathbf{x}')\,d\mathbf{x}' \approx \sum_i \mathbf{S}^p(\mathbf{x}-\mathbf{x}_i)\,\varepsilon^{p(i)}, \tag{15}$$

where

$$\mathbf{S}^p(\mathbf{x}) = \mathbf{S}^p(x_1,x_2) = \iint \mathbf{S}(x_1-x_1', x_2-x_2')\,\varphi_0(x_1',x_2')\,dx_1'dx_2', \tag{16}$$

and

$$\varphi_0(x_1,x_2) = \frac{1}{\pi D} \exp\left(-\frac{r^2}{Dh^2}\right); \qquad r^2 = x_1^2 + x_2^2. \tag{17}$$

Eq.(16) is the convolution integral that can be written and solved using Fourier transforms of $\mathbf{S}(\mathbf{x})$ and $\varphi(\mathbf{x})$.

The Law of Plasticity, $\varepsilon^p = \mathscr{F}(\sigma)$

In this work we use the incremental theory of plasticity. Let us introduce equivalent stress as $\bar{\sigma}$, $(\bar{\sigma} \geq 0)$. In the case of incremental theory of plasticity, the law of plasticity is defined as follows:

$$\bar{\sigma} = \sqrt{\frac{3}{2} S_{ij} S_{ij}}; \qquad S_{ij} = \sigma_{ij} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})\,\delta_{ij} \tag{18}$$

For the plane stress state,

$$\bar{\sigma} = \sqrt{\frac{1}{2}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{11} - \sigma_{33})^2 + (\sigma_{22} - \sigma_{33})^2\right] + 3\sigma_{12}^2}. \tag{19}$$

Equivalent plastic deformation $\bar{\varepsilon}^p$ is defined as

$$\bar{\varepsilon}^p = \int d\bar{\varepsilon}^p; \qquad d\bar{\varepsilon}^p = \sqrt{\frac{3}{2} d\bar{\varepsilon}^p_{ij} d\bar{\varepsilon}^p_{ij}};$$
$$d\bar{\varepsilon}^p_{ij} d\bar{\varepsilon}^p_{ij} = \left(d\bar{\varepsilon}^p_{11}\right)^2 + \left(d\bar{\varepsilon}^p_{22}\right)^2 + \left(d\bar{\varepsilon}^p_{33}\right)^2 + 2\left(d\bar{\varepsilon}^p_{12}\right)^2 \tag{20}$$

The increrment of equivalent plastic deformation $d\bar{\varepsilon}^p$ is defined by the equations

$$d\bar{\varepsilon}^p_{ij} = \begin{cases} \frac{3}{2} S_{ij} \frac{d\bar{\varepsilon}^p}{\bar{\sigma}_T} & \text{if} \quad \bar{\sigma} > \bar{\sigma}_T \\ 0 & \text{if} \quad \bar{\sigma} < \bar{\sigma}_T \end{cases} \qquad d\bar{\varepsilon}^p = \begin{cases} f(\bar{\sigma}_T)\, d\bar{\sigma} & \text{if} \quad \bar{\sigma} + d\bar{\sigma} > \bar{\sigma}_T \\ 0 & \text{if} \quad \bar{\sigma} + d\bar{\sigma} < \bar{\sigma}_T \end{cases} \tag{21}$$

Calculation of σ_{33} component of stress tensor can be made for two different cases.

- Plane stress state: $\sigma_{33} = 0$.
- Plane deformation state: $\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) - E\varepsilon^p_{33}$.

Algorithm for the Numerical Solution

Eq.(9) can be written as

$$\mathbf{Tb} = \mathbf{f} - \mathbf{g}^p(\varepsilon^p), \tag{22}$$

where $g^p(\varepsilon^p)$ is the vector of loads that depends of plastic deformation tensor ε^p obtained when applied loads produce stresses higher than yield stress σ_Y of material. Ussing the class of Gaussian aproximating functions (Eqs. 12 and 13) to Eq.(29) we obtain the following system of algebraic equations

$$\textstyle\sum_{j=1}^{2N} A_{ij} X_j = F_i; \qquad j = 1, 2, \ldots, 2N \ . \tag{23}$$

$$X_{2i-1}= b_s^{(i)} \quad \text{and} \quad X_{2i}= b_n^{(i)} \quad \text{for} \quad i = 1,2,\ldots,N \tag{24}$$

$$F_{2i-1}= f_s^{(i)} - g_s^{p(i)}\left(\varepsilon^p\right) \quad \text{and} \quad F_{2i}= f_n^{(i)} - g_n^{p(i)}\left(\varepsilon^p\right) \quad \text{for} \quad i = 1,2,\ldots,N \tag{25}$$

A_{ij} are the elements of the matrix of the discretized system and are expressed through standard functions that can be calculated only once and kept in the computer memory to be used in every step of the iterative process, X_j represents the unknown components of potential vector b to be calculated as in Eq.(31) and F_i represents the column vector whose elements are known from the boundary conditions through f , which is the external forces vector applied to the boundary of the body $(n \cdot \sigma|_\Gamma = f)$ and $g^p\left(\varepsilon^p\right)$ the vector obteined due to plastic deformation tensor ε^p . N is the total number of nodes at the boundary.

The system of equations (31) is solved at every step of the process of loading as follows:

1. Load increment $\Delta\sigma^{(k)}$ is defined for every step of loading, and the total tension applied to the plate is $\sigma = \sum_k \Delta\sigma^{(k)}$.
2. The increment of the plastic deformation according to incremental theory of plasticity is defined by equation (28) where S_{ij} is calculated using Eq.(25), $f\left(\bar{\sigma}_T\right)$ is the law of plasticity of the material considered, $\bar{\sigma}^{(k-1)}$ is the equivalent stress in every node of the plastic deformation zone at step $k-1$ calculated using Eq.(26) and $\bar{\sigma}_T^{(k-1)}$ is the stress due to the load applied at step $k-1$. Then total plastic deformation is obtained as $\varepsilon_{ij}^p = \sum_k \Delta\varepsilon_{ij}^{p(k)}$.
3. Then vector $g^P\left(\varepsilon^P\right)$ in Eq.(29) has the form $\mathbf{g}^{p(k)} = \mathbf{g}^{p(k)}\left(\varepsilon^p\right)$
4. Stress $\bar{\sigma}_T^{(k)}$ is updated in every node of plastic deformation mesh using the following condition

$$\bar{\sigma}_T^{(k)} = \begin{cases} \bar{\sigma}^{(k-1)} & \text{if} \quad \bar{\sigma}^{(k-1)} > \bar{\sigma}_T^{(k-1)} \\ \bar{\sigma}_T^{(k-1)} & \text{if} \quad \bar{\sigma}^{(k-1)} < \bar{\sigma}_T^{(k-1)} \end{cases}$$

5. Linear system of equations is solved as defined by Eq. (30) to obtain vector b .
6. Equivalent stresses $\bar{\sigma}^{(k)}$ are calculated in every node of mesh for plastic deformation. Plastic deformation tensor ε_{ij}^p and equivalent stress $\bar{\sigma}^{(k)}$ are kept in the computer memory to be used in the next step.
7. Process is repeated from step 1.

Numerical Results

The numerical method was applied to a rectangular plate with a cut in a side and a square mesh was considered in the plasticity region as shown in Fig. 2. The calculations were made for several square mesh sizes with 21, 41,61 and 81 nodes per side.To optimize the time of calculation process we used only the nodes where the stress condition $\bar{\sigma} > \bar{\sigma}_T$ were fullfilled, so in every step of the iterative process the number of nodes used was growing gradually.

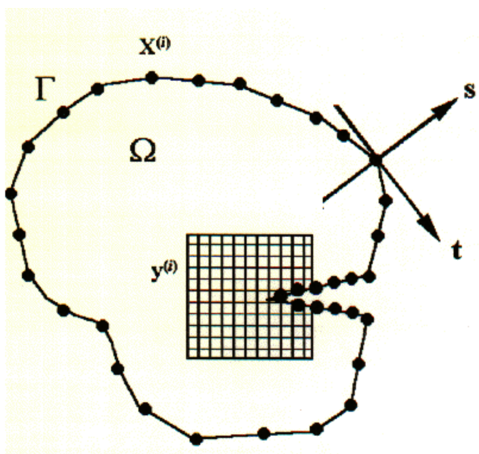

Figure 1: 2D representation of plastic deformation region.

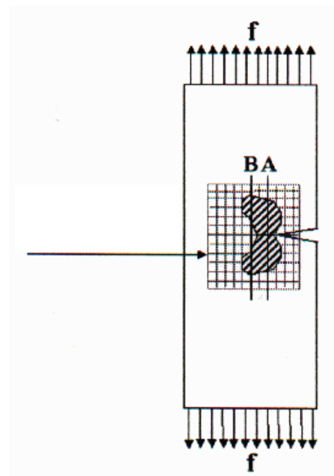

Figure 2: Lines A and B in the deformation region for the comparison of $\bar{\sigma}$ and $\bar{\varepsilon}$.

For the analysis of the results two lines in the region of plastic deformation were considered as can be seen in Fig. 2. Stresses and deformations for different rnesh sizes are compared along each line. Stresses and deformations in line A are compared in Figs. 3 and 4, respectively. It can be seen that the precision of the numerical results grows together with the number of nodes M and it achieves its limit for $M = 81$ nodes per side. Comparissons of stresses and deformation in line B are presented in Figs. 5 and 6, respectively. It can be seen that the precision is not changed when the mesh size increases.

The difference in the behavior of precision in both line is due to the presence of a strong singularity in the tip of the cut so this requires that the distance between nodes in the plastic zone be of the same order of magnitude as the distance between the boundary nodes near this region.

Conclusions

In this work we applied the Gaussian aproximating functions to the calculation of the elastio-plastic stress field in homogeneous plate with stress concentration.

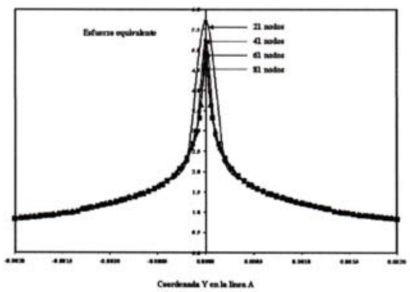

Figure 3: Comparison of $\bar{\sigma}$ in line *A* for meshes with 2l, 41, 61 and 81 nodes.

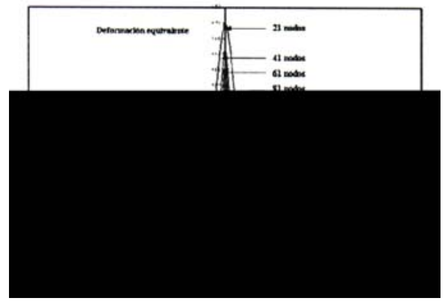

Figure 4: Comparison of $\bar{\varepsilon}$ in line *A* for meshes with 2l, 41, 61 and 81 nodes.

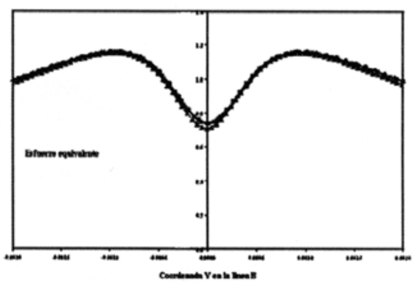

Figure 5: Comparison of $\bar{\sigma}$ in line *B* for meshes with 2l, 41, 61 and 81 nodes.

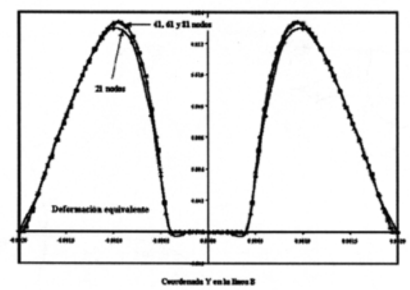

Figure 6: Comparison of $\bar{\varepsilon}$ in line *B* for meshes with 2l, 41, 61 and 81 nodes.

We have shown that integrals in the right hand side of the stress integral equation can be presented by some standard functions that can be calculated and tabulated easily to be kept in the computer memory to be used lately to obtain the elements of coefficient matrix of the linear equation system.

The numerical results obtained showed that their precision depend on the mesh nodes density near the boundary where the tip indentation is present and that the distance between nodes in the plasticity region must be the same order of magnitude as the distance between boundary points near the indentation. Figures 4 to 7 show that the method developed is effective.

References

1. Maz'ya V., "A new approximation method and its applications to the calculation of volume potentials". Boundary point method, in "3. DPG-Koloqium des DFS-Forschungs schwerpunctes 'Randelementmethoden'",19,1991.
2. Maz'ya V., Approximate approximation, in The Mathematics of Finite Elements and Applications. Highlights 1993, edited by J. R. Whitman, 77-104,

Wiley, Chichester, 1994.

3. V. Maz'ya and G. Shmidt, IMA J. Num Anal.16 (1996) 13.
4. V. Maz'ya and G. Shmidt, Applied and Computational Harmonic analysis 6 (1999) 287.
5. Vilchevskaya E. N. and S. K. Kanaun, "Integral equation for thin inclusions in homogeneous elastic media", Applied Mathematics and Mechanics (PMM), V. 56, N. 2, 235-243, 1992.
6. S. Kanaun and V. Romero,"A numerical method for the solution of the second boundary value problem of elasticity for solids with cracks", International Journal of Fracture, 107, N1, L15-L20, 2001.
7. Kanaun, S., y Levin, V., " Effective field method in the mechanics of composites", Petrozavodsk,
8. Kunin, I. A., "Elastic media with microstructure II", Springer-Verlag, Berlin 1983, V. 2. 1993.
9. S. Kanaun, V. Romero, J. Bernal, Revista Mexicana de Física, 47 (4), pp. 309-323 (2001).