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Free Surfaces Modeling Based on Level Sets

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Summary

We use a finite element formulation of the level set method to model the evolution of the free surface of axi-symmetric spreading flows of highly viscous media on a horizontal plane. We consider specifically the growth of a lava dome as an example however similar problems also occur in flows involving the spreading of molten metals or ceramics. Here we restrict ourselves on constant viscosity fluids for simplicity. In real lavas or melts the viscosity is highly temperature dependent. This manifests itself in the formation of thin predominantly elastic-plastic boundary layers along the free (cold) surfaces of the spreading flows. In our model we follow Iverson [23] who assumes that the thin boundary layer behaves like an ideal plastic membrane shell enclosing the free surface. The effect of the membrane shell is then formally identical to a surface tension-like boundary condition for the normal stress at the free surface.

Introduction

The thermo-mechanical behavior of fluids in the vicinity of interfaces often differs from the behavior in the bulk of the fluid. Typical reasons for this are an imbalance between the various intermolecular forces resulting in surface tension or a strong temperature dependent viscosity leads to the existence of a brittle elastic crust with a thickness dependent on the ratio of the thermal diffusion time to the characteristic time scale of the flow (Peclet number). If the thickness of the crust is small compared to the smaller of the principal curvature radii of the surface then bending effects may be negligible and the crust can be modeled like a brittle-elastic membrane. The thickness of the membrane varies along the surface depending on the exposure time of the surface. Surface tension is modeled in a similar whereby the membrane stresses usually are assumed as constant or dependent on the temperature or solute concentrations only (e.g. *Adamson* [20]). Structural elements such as membranes, shells and plates are relatively easy to model on the basis of the level set method since geometric quantities such the curvature radius are straight forward to calculate from the level set function. It should be mention that models involving bending stiffness such as shells and plates are somewhat more complicated to model since they include fourth order spatial derivatives. Membranes and surface tension models involve only second order derivatives. In the following we illustrate how level sets can be used to characterize not only the free surface of an

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axi-symmetric lava dome but also model in a simplified way the influence of the brittle cold boundary layer.

Lava domes are steep-sided mounds of lava. They form during an eruption when the extruded lava is so viscous that it cannot flow freely away from the vent. Their propensity to collapse in a hazardous manner *Voight* [18] makes them of concern to the surrounding area. Improved models are required to better understand this phenomenon.

Model Formulation

The lava dome is modeled as an axi-symmetric viscous continuum. The governing equations read:

$$\begin{aligned}&\left(r\left(2\eta v_{r,r}-P\right)\right)_{,r}+r\left(\eta\left(v_{r,z}+v_{z,r}\right)\right)_{,z}-2\eta\tfrac{v_r}{r}+P+rf_r=0\\&r\left(2\eta v_{z,z}-P\right)_{,z}+\left(r\eta\left(v_{r,z}+v_{z,r}\right)\right)_{,r}+rf_z=0\end{aligned}\tag{1}$$

, where P is the lava pressure, (r,z) are the coordinates of the cylindrical coordinate system, $(v_r, v_z)$are the corresponding velocities, η is the shear viscosity which we assume as constant for simplicity and $(f_r, f_z)$are volume forces. We also assume that on the time scale of interest elastic volume changes can be neglected so that

$$v_{r,r}+v_{z,z}+\frac{v_r}{r}=0\tag{2}$$

For the modeling of surface effects we also need to solve the heat equation

$$\rho c_p\left(T_{,t}+v_iT_{,i}\right)=\frac{1}{r}\left(rkT_{,i}\right)_{,i}\ \ ,\ i=(r,z)\tag{3}$$

,where ρ is the lava density, $c_p$is the heat capacity at constant pressure and $k$is the thermal conductivity. Boundary conditions for (1)-(3) will be introduced at the end of this section.

The model geometry and the assumed boundary conditions are represented in Figure 1.

The fluid surrounding the lava dome is modeled as an incompressible low-viscosity medium. The pressure P_0 at the inlet is assumed as constant. In our simulations we concentrate on cases where $l >> a$ (compare Figure 1).

For the surface of the lava dome we adopt the assumption made by Iverson [23] in connection with a model for brittle shells enclosing pressurized magma. Iverson assumes that the stress resultants of the membrane are both equal to the tensile strength of the membrane σ_T, which is assumed as constant, times the thickness d of the membrane. This simplifies the treatment of the membrane significantly since no membrane strains need to be calculated. The strength parameter $\sigma_T d$ becomes the effective surface tension acting on the cold boundaries of the lava dome.

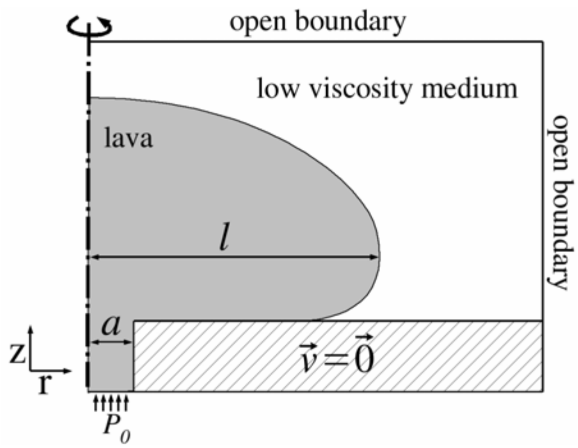

Figure 1: Model geometry and boundary conditions

It should be mentioned that Iverson's assumption is consistent with plastically admissible stress states of a Mohr-Coulomb medium. The relationship between the normal stress exerted by the lava on the membrane and the membrane stresses reads [22]:

$$p_n = \frac{n_{ss}}{R_s} + \frac{n_{\phi\phi}}{R_\phi} \quad \text{where} p_n = -\sigma_{ij} n_i n_j \ , \ i,j = (r,z) \tag{4}$$

In (4) p_n is positive in compression; n_{ss} and $n_{\phi\phi}$ are the stress resultants (integrals of normal stress over the membrane cross-section) in the direction of s and in ring direction (out of plane in Figure 2) respectively; $R_s$is the curvature radius in the plane containing the s-direction and the surface normal vector **n**. The radius R_ϕ is the projection of the radial coordinate onto **n** (Figure 2).

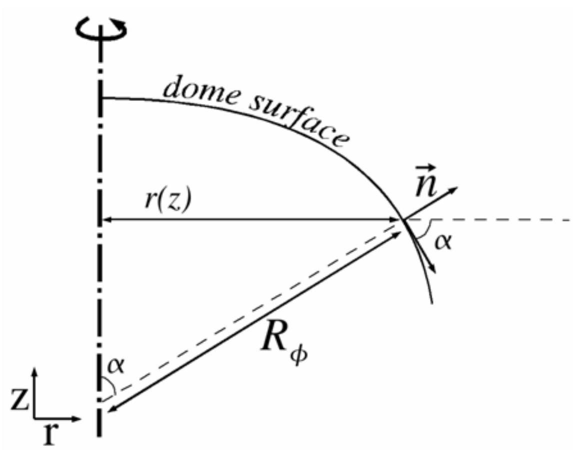

Figure 2: Definition of curvature radius in the $\mathbf{n} - \phi$ plane

Inserting $n_{ss} = n_{\phi\phi} = \sigma_T d$into (4), yields

$$p_n = \sigma_T d(\frac{1}{R_s} + \frac{1}{R_\phi}) \tag{5}$$

We use the half space cooling model [21] as a simple means to estimate the membrane thickness. In this model (using notations as appropriate for the present application) the halfspace is initially at temperature $T = T_{lava}$ everywhere except at the free surface where the temperature is kept at $T = T_{air}$. As time processes the thermal surface gradient decreases from infinite at $t = 0$ to zero at $t = \infty$. Turcotte and Schubert [21] define the thickness of the thermal boundary layer as the distance$d$ from the free surface into the interior of the half space where $(T - T_{lava})/(T_{air} - T_{lava})$has dropped from 1.0 to 0.1. The result reads $d = 2.32\sqrt{\kappa t}$ where $\sqrt{\kappa t}$ is the characteristic thermal diffusion distance. The membrane thickness d is associated with a material point the position of which needs to be traced in a computational simulation. Thickness values can also be assigned in a simplified way to spatial surface locations by using the relationship between the thermal surface flux and the characteristic thermal diffusion time. We have

$$q = -kT_{,i}n_i \tag{6}$$

Where k is the thermal conductivity and n_i is the surface normal vector. From the half space cooling solution we obtain the following relationship:

$$d = 2.32k(T_{lava} - T_{air})/q \tag{7}$$

The Level Set Method

Level set methods are computational techniques for tracking moving interfaces [12-15]. They rely on an implicit representation of the interface whose equation of motion is numerically approximated using schemes built from those for hyperbolic-conservation laws. A scalar function ϕ is initialized over the domain as a “signed” distance function with respect to the interface with a constant gradient of unity, i.e. $|\phi_{,i}| = 1$. The interface is represented as the zero level-set of this function. At each time step, once the velocity **v** is solved, the new ϕ is calculated by solving the advection equation:

$$\phi_{,t} + v \cdot \nabla\phi = 0 \tag{8}$$

The property of ϕ being a distance function is not preserved in general during advection i.e. $|\phi_{,i}| \neq 1$ after some time steps. Therefore a re-initialization procedure is required to restore the $|\phi_{,i}| = 1$ property. If at time t the level set function is to be re-initialized we follow *Sussman et al.* [13] and solve:

$$\psi_{,\tau} = sign(\phi)(1 - |\nabla\psi|), \tag{9}$$

where ô is an artificial time. Solving the above equation to a steady state and using $\psi(\tau = 0) = \phi(t)$ as an initial condition, the solution will have the same zero level set as $\phi(t)$ and will be a real distance function$(|\nabla\psi| = 1)$. It should be noted that equation (9) can also be written as a non-homogeneous advection equation as:

$$\psi_{,\tau} + w \cdot \nabla\psi = sign(\phi) \text{ with } w = sign(\phi)\frac{\nabla\psi}{|\nabla\psi|} \quad (10)$$

Finite Element Implementation

The modeling library *escript* has been developed as a module extension of the scripting language Python to facilitate the rapid development of 3-D parallel simulations on the Altix 3700 [*Davies et al*, 2004]. The finite element kernel library, *Finley*, has been specifically designed for solving large-scale problem and has been incorporated as a differential equation solver into *escript*. In the escript programming model Python scripts orchestrate numerical algorithms which are implicitly parallelised in *escript* module calls, without low-level explicit threading implementation by the user.

Results

Simple benchmark test for level set with surface tension

We consider a closed surface with surface tension embedded in a viscous medium. In Figure 3 we have represented the unfolding process from the initial distorted shape to the energetically preferred spherical shape. The relationship between the pressure jump Δp across the surface and the radius at equilibrium reads $\Delta pR = n_T$ where $n_T$is the surface tension. The latter relationship was satisfied to a maximum relative error of the pressure jump of 5%.

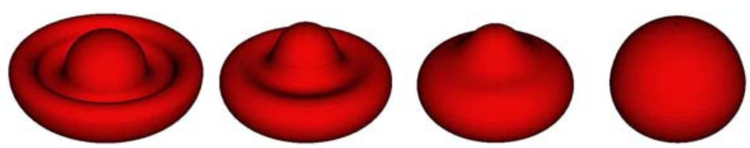

Figure 3: Level set representation of the evolution of an initially distorted surface with surface tension to the energetically preferred spherical shape

Lava dome evolution

In figure 4 we compare the influence of the presence of a stabilizing membrane on the surface of an evolving lava dame on its shape. The shape of the dome with an enclosing membrane (Figure 4 , right) is more rounded; but more significantly and contrary to the situation without a membrane, there exists a steady state where the size and shape of the dome is in equilibrium with the applied pressure.

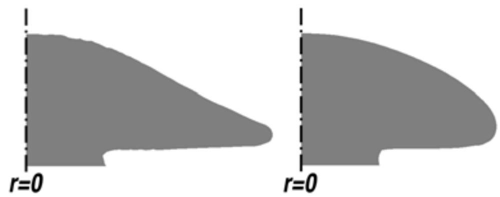

Figure 4: Lava dome without (lefty) and with surface tension (right)

Conclusion

We employed a finite element formulation of the level set method to model the evolution of lava domes. It should be pointed out that similar problems also occur in flows involving the spreading of molten metals or ceramics. The viscosity of solidifying fluids is highly temperature dependent. In spreading flows where the free surface is in contact with a cooling fluid (air) this manifests itself in the formation of a thin, highly viscous or viscous-elastic plastic boundary. From a computational modeling point of view the appropriate resolution of the thin layer either requires an extremely fine computational mesh, dynamic mesh refinement around the cold boundary layer or, and this is what we have chosen to do here, the thin layer is represented as a structurally distinct membrane shell while the lava enclosed by the membrane is modeled as a constant viscosity fluid. Following Iverson [23] we assume that the thin boundary layer behaves like an ideal plastic membrane shell. The effect of the membrane shell is then formally identical to a surface tension-like boundary condition for the normal stress at the “free” surface.

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