 ICCES, vol.3, no.4, pp.217-223, 2007

B-Spline Wavelet Galerkin Method for the Problems of Elastostatics

S. Tanaka[1] and H. Okada[1]

Summary

It has been recognized that the bottle-neck in solid/structural analyses using the finite element method is in their model generation phase. Methodologies that eliminate the needs for "elements" have been proposed by many researchers. They can be categorized into "meshless" and "virtually meshless" finite element methods. The "meshless" method may be represented by moving least square Ptrov-Galerkin (MLPG) method [1] and element free Galerkin Method (EFGM) [2]. The free-mesh method [3] and voxel finite element method [4], etc. are classified to be the "virtually meshless" approaches. The "meshless" methods eliminated needs for element connectivity information in their input data and interpolation functions are based on nodal points only. The "virtually meshless" methods do not require element connectivity data as input, but they do use elements in their computations. The free-mesh method generates triangular (2D) or tetrahedral (3D) elements when stiffness matrices are evaluated. The voxel approach is a kind of fixed grid approach.

The authors have developed an advanced fixed grid finite element method that is based on B-spline scaling function/wavelet functions [5, 6]. Proposed method is considered to be a "virtually" meshless approach. In this paper, we present equation formulations and numerical implementations. Then, some numerical examples are presented.

Equation formulations of B-spline wavelet Galerkin method

B-spline scaling function/wavelet approximation can be written, as:

$$\begin{aligned} f_j(x) &= \sum_k a_{j,k}\phi(2^j x - k) = \sum_k a_{j,k}\phi_{j,k}(x) \\ g_j(x) &= \sum_k b_{j,k}\psi(2^j x - k) = \sum_k b_{j,k}\psi_{j,k}(x) \end{aligned} \tag{1}$$

In equation (1), $\phi_{j,k}(x)$ and $\psi_{j,k}(x)$ are called the scaling function and wavelet of level "j". $f_j(x)$ and $g_j(x)$ are called the approximate functions of level "j". $a_{j,k}$ and $b_{j,k}$ are the coefficients of level "j" that defines the spatial resolution of the approximations. When the coefficients $b_{j,k}$ are appropriately chosen, we can write the following relationship.

$$\begin{aligned} f_{j+1}(x) &= f_j(x) + g_j(x) \\ f_{j+1}(x) &= \sum_k a_{j+1,k}\phi_{j+1,k}(x) = \sum_k a_{j,k}\phi_{j,k}(x) + \sum_k b_{j,k}\psi_{j,k}(x) \end{aligned} \tag{2}$$

[1]Department of Nano-Structure and Advanced Materials, Kagoshima University, 1-21-40 Korimoto, Kagoshima 890-0065, Japan

We can start with an approximation with the level "j_o" scaling function and the approximation of level "j+1" can be written to be:

$$f_{j+1}(x) = \sum_k a_{j_0,k}\phi_{j_0,k}(x) + \sum_{i=j_0}^{j}\sum_k b_{i,k}\psi_{i,k}(x) \tag{3}$$

This is the so-called "multi-resolution property" of wavelet analysis. The m-th order B-spline function is represented by piecewise (m-1) –th order polynomial functions and its derivatives up to (m-2)-the order are continuous. For example, the scaling functions $\phi^{(m)}(x)$ and the wavelet for m=3 are given, as:

$$\phi^{(3)}(x) = \begin{cases} x^2/2, & 0 \le x < 1 \\ (-2x^2+6x-3)/2, & 1 \le x < 2 \\ (x^2-6x+9)/2, & 2 \le x \le 3 \\ 0, & x < 0, 3 < x \end{cases}, \quad \begin{aligned} \psi^{(3)}(x) = & \tfrac{1}{480}\phi^{(3)}(2x) - \tfrac{29}{480}\phi^{(3)}(2x-1) \\ & + \tfrac{147}{480}\phi^{(3)}(2x-2) - \tfrac{303}{480}\phi^{(3)}(2x-3) \\ & + \tfrac{303}{480}\phi^{(3)}(2x-4) - \tfrac{147}{480}\phi^{(3)}(2x-5) \\ & + \tfrac{29}{480}\phi^{(3)}(2x-6) - \tfrac{1}{480}\phi^{(3)}(2x-7) \end{aligned} \tag{4}$$

Wavelet Galerkin Method (Equation Formulations and Discretizations)

To interpolate the displacements, the interpolation functions consisting of B-spline scaling function/wavelets are used in each coordinate direction. In the case of two-dimensional problems, the displacements are expressed by:

$$\mathbf{u}_j(x_1,x_2) = \sum_k \sum_\ell \mathbf{u}_{j,k,\ell}\boldsymbol{\Phi}_{j,k,\ell}(x_1,x_2), \quad \boldsymbol{\Phi}_{j,k,l}(x_1,x_2) = \phi_{j,k}(x_1)\phi_{j,\ell}(x_2) \tag{5}$$

where $\phi_{j,k}(x_1)$ and $\phi_{j,\ell}(x_2)$ are the scaling functions of level j in x_1 and x_2 directions, respectively. Subscripts k and ℓ indicate the central positions of supports of the scaling functions. $u_{j,k,\ell}$ are the coefficients. When the interpolation functions of level j are expressed by the scaling functions and wavelets of level $j-1$, we have:

$$\mathbf{u}_j(x_1,x_2) = \sum_k \sum_l \mathbf{u}_{j-1,k,l}\boldsymbol{\Phi}_{j-1,k,l}(x_1,x_2) + \sum_{i=1}^{3}\sum_k \sum_l \mathbf{v}^i_{j-1,k,l}\boldsymbol{\Psi}^i_{j-1,k,l}(x_1,x_2) \tag{6}$$

$$\begin{aligned} \boldsymbol{\Phi}_{j-1,k,l}(x_1,x_2) &= \phi_{j-1,k}(x_1)\phi_{j-1,l}(x_2), \quad & \boldsymbol{\Psi}^1_{j-1,k,l}(x_1,x_2) &= \psi_{j-1,k}(x_1)\phi_{j-1,l}(x_2) \\ \boldsymbol{\Psi}^2_{j-1,k,l}(x_1,x_2) &= \phi_{j-1,k}(x_1)\psi_{j-1,l}(x_2), \quad & \boldsymbol{\Psi}^3_{j-1,k,l}(x_1,x_2) &= \psi_{j-1,k}(x_1)\psi_{j-1,l}(x_2) \end{aligned} \tag{7}$$

where $\mathbf{v}_{j-1,k,l}$ are the coefficients.The boundary value problem of our interest is shown in Fig. 1 (a) in which the boundary conditions are given to be:

$$\boldsymbol{\sigma}^T \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{on} \quad \Gamma_t \quad \text{and} \quad \mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad \Gamma_u \tag{8}$$

Since the interpolation functions based B-spline scaling function/wavelet do not have the Kronecker's delta properties, we adopt the penalty method. The statement of virtual work with the penalty term is written, as:

$$\int_{\Omega} \boldsymbol{\varepsilon}(\delta \mathbf{u}) : \mathbf{D} : \boldsymbol{\varepsilon}(\mathbf{u}) d\Omega - \int_{\Gamma_t} \delta \mathbf{u} \cdot \mathbf{t} d\Gamma_t + \alpha \int_{\Gamma_u} \delta \mathbf{u} \cdot (\mathbf{u} - \bar{\mathbf{u}}) d\Gamma_u = 0 \tag{9}$$

where $\boldsymbol{\varepsilon}(\mathbf{u})$ and $\varepsilon(\delta \mathbf{u})$ are the symmetric parts of displacement gradients and $\mathbf{D}$ represents the elastic constants. α is the penalty constant that has a large positive value. To discretize the body by the B-spline scaling function and wavelet, the scaling functions and wavelets are placed periodically along the directions of the coordinate axes. For example, in the case that only the scaling functions are used, we construct a sort of cellular structure as depicted in Fig. 1 (b). The cells are used to perform stiffness integrations. The boundaries of complex shaped structures are modeled by dividing the cells at the boundaries into many sub-cells. Then, the integration for the stiffness is performed based on the sub-cells and those outside the boundary of the body are excluded from the stiffness integration. Degrees of freedoms that have zero stiffness are eliminated to avoid singular matrix. Hence, the final matrix form can be written to be:

$$(\mathbf{K} + \mathbf{K}_\alpha)\mathbf{u} = \mathbf{f} + \mathbf{f}_\alpha \tag{10}$$

The stiffness matrix $\mathbf{K}$ is consisting of a number of sub-matrices, as:

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{\Phi,\Phi} & \mathbf{K}_{\Phi,\Psi^1} & \mathbf{K}_{\Phi,\Psi^2} & \mathbf{K}_{\Phi,\Psi^3} \\ \mathbf{K}_{\Psi^1,\Phi} & \mathbf{K}_{\Psi^1,\Psi^1} & \mathbf{K}_{\Psi^1,\Psi^2} & \mathbf{K}_{\Psi^1,\Psi^3} \\ \mathbf{K}_{\Psi^2,\Phi} & \mathbf{K}_{\Psi^2,\Psi^1} & \mathbf{K}_{\Psi^2,\Psi^2} & \mathbf{K}_{\Psi^2,\Psi^3} \\ \mathbf{K}_{\Psi^3,\Phi} & \mathbf{K}_{\Psi^3,\Psi^1} & \mathbf{K}_{\Psi^3,\Psi^2} & \mathbf{K}_{\Psi^3,\Psi^3} \end{bmatrix} \tag{11}$$

where the subscripts indicate the corresponding components of the interpolation functions, as shown in equation (7). When the computation is based only on the scaling functions, the right hand side of equation (11) contains $K_{\Phi,\Phi}$ only. Force vector f and the penalty term K_α are expresses, as:

$$\mathbf{f} = \int_{\Gamma_t} \mathbf{N}_j^T \cdot \bar{\mathbf{t}} d\Gamma_t, \quad \mathbf{K}_\alpha = \alpha \int_{\Gamma_u} \mathbf{N}_j^T \cdot \mathbf{N}_j d\Gamma_u \tag{12}$$

Discretization Technique at the Boundary of Body

As discussed in previous section, due to the multi-resolution property in the theory of wavelet, we can add interpolation functions based on the wavelets for the purpose of enhancing the spatial resolution of discretization. In previous wavelet Galerkin formulations [7, 8, 9], the so-called fictitious domain approach was adopted. In the fictitious domain approach, the body is extended to its exterior but very small stiffness (Young's modulus) is given to the exterior region. When the fictitious domain approach is used in the penalty formulation [equations (9)-(12)], the stiffness matrix that is given in equations (10) and (11) contains very large and very small elements. Therefore, iterative equation solvers such as CG and ICCG method may suffer from convergence problems. Also, the degrees of freedoms may become large. We think that the reason why the fictitious domain was used in the previous formulations was due to the loss of linear independence of the basis functions.

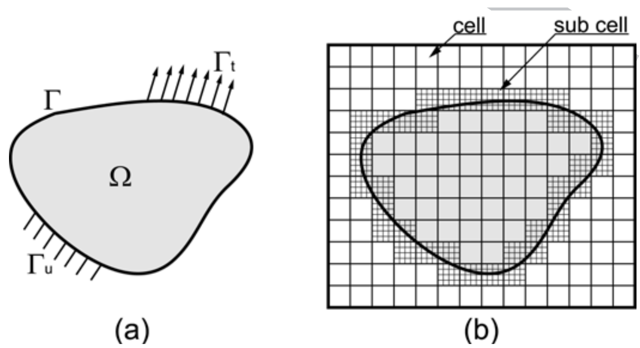

Figure 1: The boundary value problem and the scheme of discretization [(a) BVP and (b) discretization scheme]

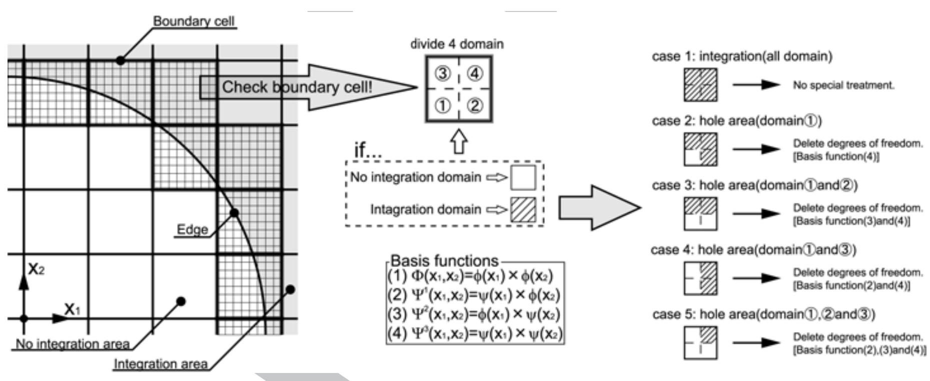

Figure 2: The treatment of the boundary of body [interpolation functions are eliminated appropriately]

We developed a scheme to eliminate the basis functions appropriately so that remaining ones are linearly independent from each other. The scheme is schematically presented in Fig. 2.

Numerical Examples

First, analyses using the scaling function only are presented. A plate with a small hole as depicted in Fig. 3 (a) is solved and the numerical results are compared with that of analytical solution for a hole in an infinite plate subject to remote tension. The domain of analysis is divided into 20 by 20 cells as shown in Fig. 3 (b). The central positions of linear and cubic scaling functions are at the grid points and that of quadratic scaling functions are at the centers of the cells. Cells that contain the boundary of the hole are divided into 10 by 10 sub-cells. The distributions of stress σ_{22} along the bottom edge are shown in Fig. 4. It is seen that stress computed with the linear scaling function slightly deviates from the analytical solution. The solutions of the quadratic and cubic scaling functions are on the curve of analytical solution. However, they overestimate the maximum stress at the edge of the hole.

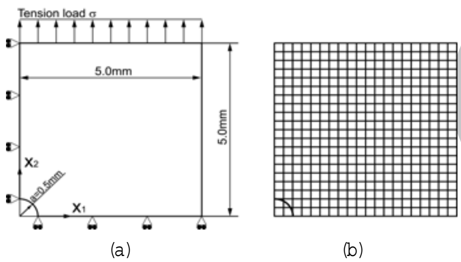

Figure 3: Quarter model of a plate with a hole (a=0.5) [(a) Analysis model and (b) wavelet FE model]

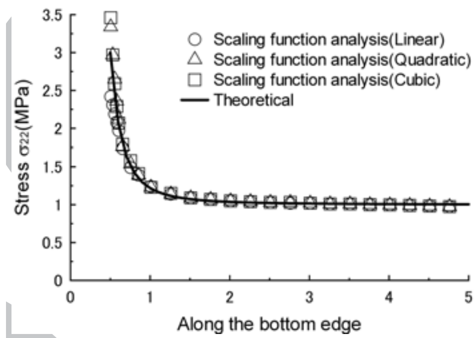

Figure 4: Stress (σ_{22}) distribution along the bottom edge.

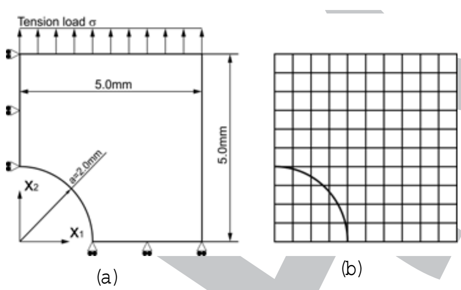

Figure 5: Quarter model of a plate with a hole (a=2.0) [(a) Analysis model and (b) wavelet FE model].

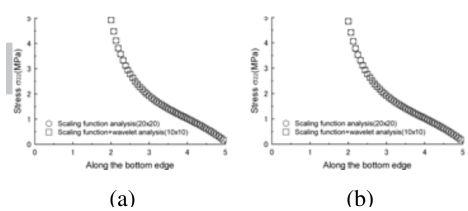

Figure 6: Stress (σ_{22}) distributions along the bottom edge evaluated by using the scaling function only and scaling function/wavelet [(a) quadratic and (b) cubic functions].

Next, we compare the solutions with the scaling functions only and with the scaling functions and the wavelets. In Fig. 5 (a), a plate with relatively large size hole subject to tension is presented. Analyses based on the scaling functions and wavelets are performed using 10 by 10 cells as depicted in Fig. 5 (b). According to the multi-resolution property, their solutions must be identical with those based on the scaling functions only with 20 by 20 cells. In this problem, cells that contain the boundary of circular hole are divided into 32 by 32 sub-cells. In Figs. 6 (a) and (b), stress distributions based on the quadratic and cubic B-spline scaling functions/wavelets are depicted, respectively. The solutions based on the scaling functions only and on the scaling functions and the wavelets perfectly agree. Therefore, the validity of the multi-resolution property in present wavelet Galerkin method is confirmed.

Concluding Remarks

The multi-resolution property is confirmed to be valid in present wavelet Galerkin method. This means that the resolution of spatial discretization can be enhanced by adding wavelets. Wavelets may be added to parts of domain of analysis where higher spatial resolutions are required. Furthermore, wavelets can be superposed repeatedly according to their multi-resolution properties. These open a possibility to develop an adaptive strategy

References

1. Atluri, S.N. (2004): The Meshless Method (MLPG) for Domain & BIE Discretizations, Tech Science Press (2004).

2. Belytschko, T., Lu, Y.Y. and Gu, L. (1994): "Element-Free Galerkin Methods", Int. J. Numer. Meth. Engng., Vol. 37, pp. 229-256.

3. Yagawa, G. and Furukawa, T. (2000): "Recent Developments of Free Mesh Method", Int. J. Numer. Meth. Engng., Vol. 47, pp. 1419-1443.

4. Hollister, S. J. and Riemer, B. A. (1994): "Digital Image Based Finite Element Analysis for Bone Microstructure using Conjugate Gradient and Gaussian Filter Techniques", Mathematical Methods in Medical Imaging II, Vol. 2035, pp. 95-106.

5. Tanaka, S. and Okada, H. (2006): "On The Analysis of Stress Concentration Problems Using Wavelet Galerkin Method (1st Report, Mathematical Formulations and treatments on external boundaries)", Trans JSME Series A, Vol. 72-718, pp.856-863.

6. Tanaka, S. and Okada, H. (2006): "On The Analysis of Stress Concentration Problems Using Wavelet Galerkin Method (2nd Report, Using Higher Degree B-splines as the Basis Functions), Trans JSME Series A, Trans JSME Series A, 72-719, pp.982-989.

7. Kim, J. E., Jang, G.-W. and Kim, Y. Y. (2003): "Adaptive Multiscale Wavelet-Galerkin Analysis for Plate Elasticity Problems and its Application to Multiscale Topology Design Optimization, Int. J. of Solids Struct, Vol. 40, pp. 6473-6496.

8. Jang, G.-W., Kim, J. E. and Kim, Y. Y. (2004): "Multiscale Galerkin Method

using Interpolation Wavelets for Two-Dimensional Elliptic Problems in General Domains", Int. J. Numer. Meth. Engng. 59, pp. 225-253.

9. Diaz, A.R. (1999): "A Wavelet-Galerkin Scheme for Analysis of Large-Scale Problems on Simple Domains", Int. J. Numer. Meth. Engng, Vol. 44, pp. 1599-1616.