 ICCES, vol.3, no.4, pp.189-194, 2007

Boundary Point Method applied for calculating elastic strain and stress in bodies with cracks

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Summary

A new numerical method for the solution of integral equations of the theory of elasticity for bodies with cracks is developed. The method is based on a class of Gaussian approximating functions that simplify essentially the construction of the final matrix of the linear algebraic system of the discretized problem. The results of the application of the method to some plane problems of elasticity were compared with the exact solutions and some other numerical solutions that exist in literature.

Introduction

In this Study we develop a new numerical method for the solution of the integral equations of the second boundary value problem of elasticity for bodies with cracks. The method is based on a class of Gaussian approximating that are Gaussian type functions concentrated in tangent planes to the surface of the body at some number of surface nodes. The idea to use these functions for the solution of a wide class of the integral equations of physics belongs to V. Maz'ya. The theory of approximation Gaussian functions was developed in the works of V. Maz'ya[1,2]. These functions were used for the solution of a static problem of elasticity for an infinite medium with thin inclusions and cracks. In this work we present some numerical results for elasticity problems in bodies with cracks.

The second boundary value problem of elasticity for bodies with cracks

Let us consider the numerical solution of the elasticity problem for a straight crack that occupies an interval $(|x| < l, y = 0)$ in an infinite plane. The plane is subjected to stress field $\sigma_0(x)$ at infinity. The integral equation of this problem takes form:

$$\int_{\Gamma} \mathbf{n}(x) \cdot [\mathbf{S}(x - x') \cdot \mathbf{n}(x')] \cdot \mathbf{b}(x') d\Gamma' = -\mathbf{n}(x) \cdot \sigma_0, \quad for \quad x \in \Gamma \tag{1}$$

where Γ is the crack line.

For a constant stress field directed along y axis $(\sigma_0 = e_2 \otimes e_2)$ the exact solution of this equation takes the form $\mathbf{b}_0(x) = b_0(x)\mathbf{n}$ where $b_0(x) = \sqrt{1 - x^2}$ and $\mathbf{n} = \mathbf{e}_2$ is a normal to the crack line. The error $R(x) = b_n(x) - b_0(x)$ of the numerical solutions for various numbers of the nodes on the crack line is presented in Fig.1. It is seen from these graphs that the maximum of the error is concentrated in

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the vicinities of the crack tips. But the asymptotics of the solution near the crack tips give us important information for engineering applications. It is known [3, 4] that the exact jump $\mathbf{b}(x)$ of the displacement vector on the crack has the following asymptotic near the crack tip.

$$\mathbf{b}(\mathbf{x}) = \beta\sqrt{r} + \mathbf{O}\left(r^{3/2}\right) \tag{2}$$

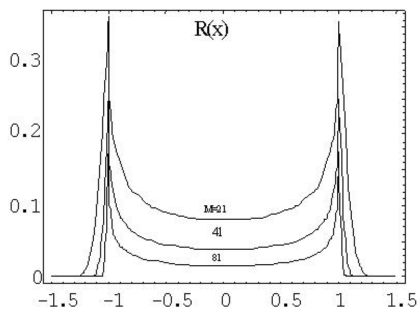

Figure 1: Error $R(x)$ of the numerical solution in the problem for a straight crack in a infinite plane.

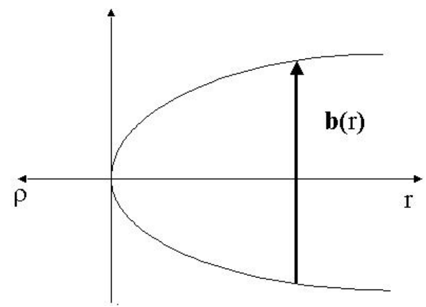

Figure 2: Local coordinate system in the crack tip.

Here r is the distance between point $x \in S$ and the crack tip (Fig. 2). The components β_s, β_n of the vector β in Eq. (2) are connected with stress intensity factors K_I and K_{II} by the equations [3,4] $K_I = \sqrt{2\pi}\beta_n\frac{\mu_0}{4}$; $K_{II} = \sqrt{2\pi}\beta_s\frac{\mu_0}{4(1-\nu_0)}$ and the asympotics of components σ_{22} and σ_{12} of the stress tensor near the crack tip have the following well known forms $\sigma_{22} = \frac{K_I}{\sqrt{2\pi\rho}} + O(1)$; $\sigma_{12} = \frac{K_{II}}{\sqrt{2\pi\rho}} + O(1)$.

In order to calculate vector coefficient β in Eq. (2) and the stress intensity factors with high precision we have to modify the above developed method. This modification is based in the theorem of polynominal conservativity [4]. This theorem allows us to find the solution of the integral equation of the crack problem in the form:

$$\mathbf{b}(x) = \mathbf{B}^m(x)\sqrt{l^2 - x^2}; \quad \mathbf{B}^m(x) = \mathbf{a}_1 + \mathbf{a}_2 x + \mathbf{a}_3 x^2 + \ldots + \mathbf{a}_m x^{m-1}, \tag{3}$$

Let us introduce a set of auxiliary nodes in the crack line $x_a^{(j)}$ $(j = 1, 2, \ldots, M_a)$ and use these nodes for gaussian approximation of the function $\mathbf{b}(x)$ in the Eq. (3).

$$\mathbf{b}(x) = \sum_{k=1}^{m} \mathbf{a}_k \sum_{j=1}^{M_a} (x_a^{(j)})^{k-1} B_0(x_a^{(j)}) \frac{1}{\sqrt{\pi D}} \exp\left[-\frac{\left(x - x_a^{(j)}\right)^2}{D h_a^2}\right] \tag{4}$$

where $B_0(x) = \sqrt{l^2 - x^2}$ and h_a is the distance between auxiliary nodes. For a good approximation of $\mathbf{b}(x)$ the number M_a should be sufficiently large $(M_a > 50)$. After substituting Eq. (4) for $\mathbf{b}(x)$ in Eq. (54) in reference [5] it has been found that the components of the stress tensor presented in Eq. (69) [5] depends of $a_s^{(k)}, a_n^{(k)}$ that are the components of the vectors $\mathbf{a}_k$ in Eq. (3) in the local basis $(\mathbf{s}, \mathbf{n})$ $a_k = a_s^{(k)}\mathbf{s} + a_n^{(k)}\mathbf{n}$.

The components of the forces that act on the crack line take the forms

$$f_s(x) = \sigma_{12}^0(x, 0) + \sum_{k=1}^{m} S_{12}^k(x, 0)\, a_s^{(k)}; \quad f_n(x) = \sigma_{22}^0(x, 0) + \sum_{k=1}^{m} S_{22}^k(x, 0)\, a_n^{(k)}, \tag{5}$$

and if the crack sides are free from stress we have

$$f_s(x) = f_n(x) = 0, |x| \le l \tag{6}$$

In order to obtain $2m$ unknowns $a_s^{(k)}$ and $a_n^{(k)}$ in Eq. (3) let us satisfy Eqs. (5) and (6) in m nodes on the crack line (the main nodes). The final system of the equations for $a_s^{(k)}$ and $a_n^{(k)}$ takes the form

$$\sum_{k=1}^{m} S_{12}^k\left(x^{(l)}, 0\right) a_s^{(k)} = -\sigma_{12}^0\left(x^{(l)}, 0\right); \quad \sum_{k=1}^{m} S_{22}^k\left(x^{(l)}, 0\right) a_n^{(k)} = -\sigma_{22}^0\left(x^{(l)}, 0\right), \tag{7}$$

for $l = 1, 2, \ldots, m$. Here $x^{(l)}$ are coordinates of the main nodes on the crack line.

Let us consider the numerical solution of the system (7) when the applied field is a constant tension along $y-$axis. $\sigma_{11}^0 = \sigma_{12}^0 = 0$ and $\sigma_{22}^0 = 1$.For $m = 5, M_a = 50$ and homogeneous distribution of the main nodes $(l = 1)$

$x_a^{(j)} = -1 + \left(j - \frac{1}{2}\right) h_a,\ h_a = \frac{2}{M_a}$; $for\ j = 1, 2, \ldots, \mathrm{M}_a$,	$x^{(k)} = -1 + \left(k - \frac{1}{2}\right),\ h = \frac{2}{m}$, $for\ \mathrm{j} = 1, 2\ldots, \mathrm{m}$,

the solution of the system (7) is $a_s^{(k)} = 0$, for $k = 1, 2, \ldots, 5$, and $a_n^{(1)} = 0.9912$, $a_n^{(2)} = 0$, $a_n^{(3)} = 0.0037$, $a_n^{(4)} = 0$, $a_n^{(5)} = 0.0048$. Taking into account that the exact solution of this problem has form $\mathbf{b}_0(x) = b_0(x)\,\mathbf{n}$, therefore $a_s^{(k)} = 0$, for $k = 1, 2, 3, \ldots, 5$, and $a_n^{(1)} = 1$, $a_n^{(k)} = 0$, for $k = 2, 3, .., 5$, we see that the error of the numerical solution is less than 1% in this case.

In the case of a finite rectangular area with a central straight crack subjected to a constant tension in $y-$direction .The solution of this problem can be found in the form

$$\sigma(x,y) = \int_{\Gamma_c} [\mathbf{S}(x-x',y-y')\cdot\mathbf{n}(x',y')]\cdot\mathbf{b}_c(x',y')\,d\Gamma' + \int_{\Gamma_b} [\mathbf{S}(x-x',y-y')\cdot\mathbf{n}(x',y')]\cdot\mathbf{b}_c(x',y')\,d\Gamma', \tag{8}$$

where the first integral is spread over the crack line Γ_c and the second integral is over the external border of the area Γ_b.

Using Eqs. (4) for the presentation of the vector $\mathbf{b}_c(x)$ we have to choose a set of auxiliary nodes $x_a^{(j)}$ $(j = 1, 2, ..., M_a)$ on the crack line as well as the power $m-1$ of the polynominal $\mathbf{B}^m(x)$ in Eq. (3). Then the coordinates of m main nodes on the crack line should be defined and the boundary conditions are satisfied at these nodes. Let us enumerate the main nodes on the crack line from 1 to m . We have also to define the nodes on the external boundary of the body where vector $\mathbf{b}(x)$ is approximated in reference [5]. The numeration of the latter nodes is from $m+1$ to M , where M is the total number of the main nodes. As a result, the stress tensor $\sigma(x,y)$ in the global basis $(\mathbf{e}_1,\mathbf{e}_2)$ is approximated by the Eq. (5) that follows and Eqs. (43)-(46) in reference [5]:

$$\sigma(x, y) = \sigma_{11} e_1 \otimes e_1 + \sigma_{12}(e_1 \otimes e_2 + e_2 \otimes e_1) + \sigma_{22} e_2 \otimes e_2$$

where $\sigma_{11}, \sigma_{12}, \sigma_{22}$ are defined in Eq. (46) in reference [5].

The forces $\mathbf{f}(x) = \mathbf{n}(x)\cdot\sigma(x)$, for$x \in S$ on the surface of the body and crack surfaces are calculated from Eq. (9).

The values of these forces at the main nodes have the following forms in the local bases $\left(\mathbf{s}^{(i)},\mathbf{n}^{(i)}\right)$

$$\mathbf{f}\left(x^{(i)},y^{(i)}\right) = f_s^{(i)}\mathbf{s}^{(i)} + f_n^{(i)}\mathbf{n}^{(i)}, \tag{9}$$

The final system for unknown coefficients $a_s^{(k)}, a_n^{(k)}$ that define the solution $b(x)$ on the crack line and coefficients $b_s^{(i)}, b_n^{(i)}$ in the nodes on the external boundary of the area follows from the boundary conditions and takes the form:

$$\sum_{l=1}^{2M} B_{kl}^c X_l = F_k,\ \ for\ k = 1, 2, ..., 2M \tag{10}$$

The solution of Eq. (10) was obtained for various sizes of the rectangle and the crack lenghts. In Fig. 3 the graphs of the functions $\sigma_{22}(x,0)$ are presented for $a = 2$ (length of plate),$b = 3$ (height of plate), and $l = 1$ (length of crack). The number of auxiliary nodes on the crack line was chosen as $M_a = 70$, the number of the main nodes was $m = 5$ and $m = 27, 47, 67$. The increasing of the number of the auxiliary nodes and the main nodes does not change the solution. The results of the calculation of stress intensity factors in the crack tips are presented in Fig. 4.

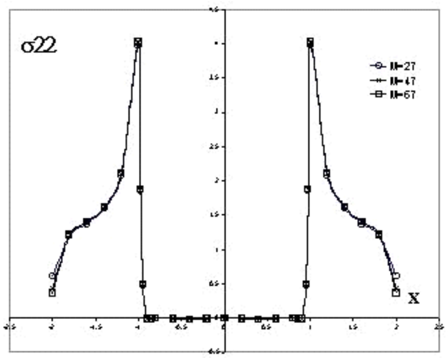

Figure 3: Distributions of the stress components $\sigma_{22}(x,0)$ along the crak line.

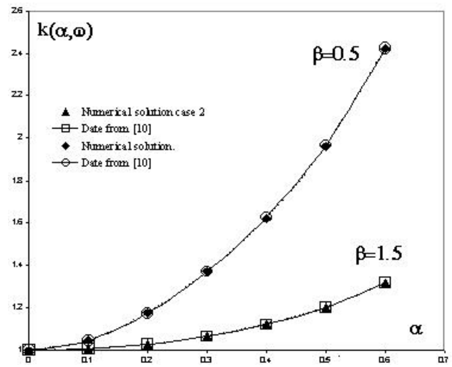

Figure 4: Stress intensity factors for a central crack in rectangular plane area.

In this figure $k(\alpha,\omega) = \frac{K_I}{K_I^\infty} = \frac{\beta_n}{\beta_n^\infty} = \mathbf{B}_n^m(l), \quad \omega = \frac{a}{b}, \quad \alpha = \frac{l}{a}$

where $\mathbf{B}_n^m(l)$ is the normal component of the vector $\mathbf{B}^m(x)$ defined in Eq. (3),

K_I^∞ and β_n^∞ are stress intensity factor K_I and coefficient β_n that correspond to an infinite plane with the crack length l. Solid lines in Fig. 4 are numerical solutions presented in Ref. 6 (the precision is 1%), dashed lines are numerical solutions obtained by the developed method. For $\omega = 0.5$ the following numbers of nodes were used: $m = 5$, $M_a = 70$, and $M = 105$, and for $\omega = 1.5$, $m = 11$, $M_a = 70$, $M = 71$. Numerical results in Fig. 4 were obtained for these coordinates of the main nodes on the crack line.

Conclusions

The numerical method developed in this work is an effective tool for the solution of the second boundary value problem of elasticity. The accuracy of the method depends on the density of he nodes on the body surface. In the developed version of the method the distances between the neighbor nodes were chosen the same. The method allows also to consider non homogeneous distributions of the nodes on the surface of the considered body [5, 6].

The area of possible application of the method is in all the problems of mathematical physics that can be reduced to the solution of integral equations. A future development of the method is the solution of Elasto-Plastic problems for bodies with cracks in 2D and 3D cases.

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