 ICCES, vol.3, no.4, pp.169-175, 2007

New Basis Functions and Their Applications to PDEs

Haiyan Tian[1], Sergiy Reustkiy[2] and C.S. Chen[1]

Summary

We introduce a new type of basis functions in this paper to approximate a scattered data set. We test our basis functions on recovering the well-known Franke's function given by scattered data. We then use these basis functions in Kansa's method for solving Helmholtz equations. To demonstrate our proposed approach, we compare the numerical solutions with analytic solutions. The numerical results show that our approach is accurate and efficient.

Introduction

The main goal of our paper is to present a new type of basis functions for solving various types of science and engineering problems. These basis functions can be used for the following two purposes: 1) to approximate scattered data on regular or irregular domains; 2) to be used as basis functions in solving partial differential equations (PDEs). Let us briefly review both approaches.

We consider the boundary value problem,

$$\begin{aligned} &L[u] = f(\mathbf{x}),\ \mathbf{x} = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \\ &B[u] = g(\mathbf{x}),\ \mathbf{x} \in \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a simply connected domain bounded by a simple closed curve $\partial\Omega$, L and B are the differential operators on u over the interior of Ω and the boundary $\partial\Omega$ respectively. We assume that the operator L is of the elliptic type.

In the framework of boundary methods the influence of the solution by the inhomogeneous term f in (1) can be transferred to boundary and the problem then can be solved by standard boundary techniques (see [1], [2], [3] for more detailed information). Following this approach both the accurate approximation of the source term f and the easiness of the derivation of a close form particular solution are important.

Recently, a new approach called Embedded Boundary Method (cf.[10], [11]) is developed for the same purpose. Suppose the following linear differential operator is considered:

$$L = \sum_{k_1,k_2=0}^{k} A_{k_1,k_2} \frac{\partial^{k_1+k_2}}{\partial x_1^{k_1} \partial x_2^{k_2}},$$

where A_{k_1,k_2} are constants. We assume that the solution domain Ω is embedded in a rectangle D. If the inhomogeneous term $f(x_1,x_2)$ can be extended from Ω

[1]Department of Mathematics, University of Southern Mississippi, Hattiesburg, MS 39406

[2]Magnetism Division, Institute of Electrodynamics, National Academy of Science of Ukraine, Industrialnaya St., 19, 61106 Kharkov, Ukraine

to the rectangle D so that the extended function $\widetilde{f}(x_1,x_2)$ is spatially periodic and belongs to the C^∞ class of functions everywhere within and on the rectangle, then a particular solution can be found in the form of the Fourier series. The paper is focused on the approximation problems where f is given at some irregular set of data points $X \subset \Omega$. We have "the extension of the third kind" when f is not known outside Ω, according to Boyd's classification [10].

The basis functions $I_{M,\chi}(\mathbf{x},\xi)$ suggested in [5] have the form of the truncated series $I_{M,\chi}(\mathbf{x},\xi) = \sum_{n=1}^{M} c_n(\xi)\varphi_n(\mathbf{x})$ over an orthogonal system $\varphi_n(\mathbf{x})$ defined on some rectangle D. In particular the trigonometric functions and their products can be used for this purpose. In this way, we can extend f from Ω to D. This is extension of the third kind because we use the data from the initial domain only.

The basis functions $I_{M,\chi}(\mathbf{x},\xi)$ essentially differ from zero only inside some neighborhood of the center point ξ. The size of this neighborhood depends on the parameters M,χ. From this point of view they are similar to the compact supported radial basis functions (CS-RBF) [4] and they can be used for solving PDEs in the framework of Kansa's or straight collocation method [12]. To examine such application of $I_{M,\chi}(\mathbf{x},\xi)$ is the second purpose of our paper. In the second half of this paper, we solve a Helmholtz equation using such approach.

The outline of this paper is as follows: we begin with a brief description of the basis functions $I_{M,\chi}(\mathbf{x},\xi)$ in Section 2. In Section 3, We utilize the basis functions $I_{M,\chi}(\mathbf{x},\xi)$ to approximate some given scattered data on an irregular domain Ω and obtain an extension over a square D. The Franke's function is reconstructed and extended from a star shaped region to a squared region. In Section 4, a set of the basis functions $I_{M,\chi}(\mathbf{x},\xi)$ is used in Kansa's method for solving PDEs. We solve a Helmholtz equation to demonstrate the effectiveness of our proposed approach.

Delta-shaped Basis Functions

For simplicity, let us dwell on the 1D case and assume that all the functions considered are defined in the interval $[-1,+1]$.

Let $\{\varphi_n(x),\mu_n\}$ be a solution of the $Sturm-Liouville$ problem on the interval $[-1,+1]:$

$$-d^2\varphi/dx^2 = \mu\varphi, \quad \varphi(-1) = \varphi(+1) = 0, \tag{2}$$

The solutions $\{\varphi_n(x),\mu_n\}$ of (2) are $\varphi_n(x) = \sin(n\pi(x+1)/2)$ $\mu_n = (n\pi/2)^2$. They satisfy the following conditions: $0 < \mu_1 < \mu_2 < \ldots < \mu_n \to +\infty$; the eigenfunctions $\{\varphi_n(x)\}_{n=1}^{\infty}$ form an orthogonal system on $[-1,+1]$ with a scalar product,

$$\int_{-1}^{+1} \varphi_n(x)\varphi_m(x)dx = \begin{cases} 1, \text{ if } n = m, \\ 0, \text{ if } n \neq m. \end{cases}$$

According to the algorithm presented in [5] we get the basis function:

$$I_{M,\chi}(\mathbf{x},\xi) = \sum_{n=1}^{M}\left[1-\left(\frac{n}{M+1}\right)^2\right]^{\chi}\varphi_n(\xi)\varphi_n(x) \equiv \sum_{n=1}^{M} c_n(\xi)\,\varphi_n(x), \qquad (3)$$

where

$$c_n(\xi) = r_n\varphi_n(\xi),\; r_n = \left[1-\left(\frac{n}{M+1}\right)^2\right]^{\chi}\varphi_n(\xi),$$

and χ is an integer for the purpose of regularization. Notice that $I_{M,\chi}(\mathbf{x},\xi)$ is a Delta-shaped function.

The multi–dimensional Delta-shaped basis functions can be obtained as products of the 1D Delta-shaped functions (3). For example, the 2-D Delta-shaped functions are defined as,

$$I_{M,\chi}(\mathbf{x},\xi) = I_{M,\chi}(x_1,\xi_1)I_{M,\chi}(x_2,\xi_2).$$

In Fig. 1 we show the graphs of the 2D Delta-shaped functions $I_{20,6}(\mathbf{x},\xi)$ and $I_{40,12}(\mathbf{x},\xi)$ with the first centered at $\xi_1 = (-0.25,-0.25)$ and the second centered at $\xi_2 = (0.25,0.25)$. We note that for the basis functions $I_{M,\chi}(\mathbf{x},\xi)$ the parameter M plays the role of the scaling factor. When we increase M, the support of the basis function decreases as is shown in Fig. 1.

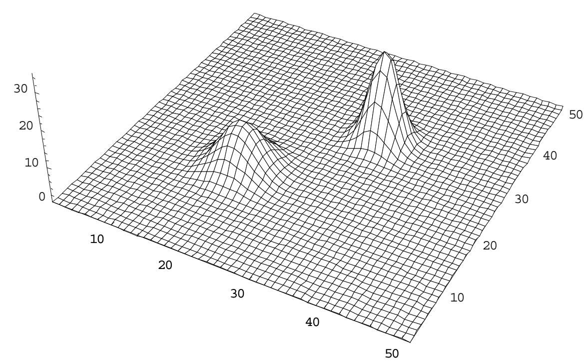

Figure 1: The functions $I_{20,6}(\mathbf{x},\xi)$ and $I_{40,12}(\mathbf{x},\xi)$ centered at $(-0.25,-0.25)$ and $(0.25,0.25)$ respectively

We note that $I_{M,\chi}(x,\xi)$ satisfy the same boundary conditions as the eigenfunctions $\{\varphi_n(x)\}$ which form it,

$$I_{M,\chi}(\pm 1,\xi) = I_{M,\chi}(x,\pm 1) = 0. \qquad (4)$$

The regularizing method described above can be applied to any solution of a $Sturm-Liouville$ problem.

Approximation

We approximate the scattered data $\{\mathbf{x}_i, f_i\}$ by the linear combination

$$\widetilde{f}(\mathbf{x}) = \sum_{j=1}^{K} p_j I_{M,\chi}(\mathbf{x}, \xi_j)$$

of basis functions $I_{M,\chi}$ with fixed value for each of the parameters M and χ. We get the following system to determine the unknowns $\left\{p_j\right\}_{j=1}^{K}$.

$$f_i = \sum_{j=1}^{K} p_j I_{M,\chi}(\mathbf{x}_i, \xi_j),\ \mathbf{x}_i \in X \subset \mathbb{R}^d,\ \ 1 \le i \le N, \tag{5}$$

where d is either 1 or 2.

Due to the condition (4) the functions $I_{M,\chi}(\mathbf{x}, \xi)$ vanish on the boundary of the square $\Omega_0 = [-1,1] \times [-1,1]$. Neither the data points $\{\mathbf{x}_i\}$ nor the centers $\left\{\xi_j\right\}$ can be placed close to $\partial\Omega_0$. Thus we assume that the set of the data points $\{\mathbf{x}_i\}_{i=1}^{N}$ is embedded in the square $D = [-0.5, 0.5] \times [-0.5, 0.5]$. If this is not the case originally, appropriate translation and scaling operations may be performed to make it so.

Solving the linear system we let the number of centers K be approximately twice as small as the number of collocation points N and use the least squares method to solve the resulting overdetermined system. In particular the algorithm of Housenholder transformation is used to transform the matrix of Equation (5) to upper triangular form. See [6] for more details.

We would like to draw the reader's attention to the following property of the approximation with the basis functions described. When the unknown coefficients $\left\{p_j\right\}$ in (5) are determined, we can re-write the approximation in the form of series over the trigonometric functions $\{\varphi_n(x)\}$:

$$\widetilde{f}(\mathbf{x}) = \sum_{n_1,n_2=1}^{M} F_{n_1,n_2} \varphi_{n_1}(x_1) \varphi_{n_2}(x_2),$$

where

$$F_{n_1,n_2} = \sum_{j=1}^{K} p_j \left[1 - \left(\frac{n_1}{M+1}\right)^2\right]^{\chi} \left[1 - \left(\frac{n_2}{M+1}\right)^2\right]^{\chi} \varphi_{n_1}(\xi_{1,j}) \varphi_{n_2}(\xi_{2,j}).$$

EXAMPLE 1. In this example, we reconstruct the Franke's function which is widely used as a test function for surface reconstruction in radial basis functions [7]. The Franke's function was defined initially on the rectangle $[0,1] \times [0,1]$. However, for our purposes it should be re-scaled to $D=[-0.5,0.5]\times[-0.5,0.5]$ which can be expressed as follows:

$$F(\mathbf{x})=\frac{3}{4}\exp\left(-\frac{(9x_1+2.5)^2+(9x_2+2.5)^2}{4}\right)+\frac{3}{4}\exp\left(-\frac{(9x_1+5.5)^2+(9x_2+5.5)^2}{49}\right)$$
$$+\frac{1}{2}\exp\left(-\frac{(9x_1-2.5)^2+(9x_2+1.5)^2}{4}\right)-\frac{1}{5}\exp\left(-(9x_1+0.5)^2-(9x_2-2.5)^2\right).$$

The function $F(\mathbf{x})$ is approximated in a star shaped region Ω. The parametric equation of boundary curve $\partial\Omega$ is given by

$$x_1=0.25(\cos t)\left(1+\cos^2 4t\right),\ \ x_2=0.25(\sin t)\left(1+\cos^2 4t\right),\ \ t\in[0,2\pi].$$

The data points $\{\mathbf{x}_i\}_{i=1}^N$,the centers $\{\xi_j\}_{j=1}^K$, and the test points $\{\mathbf{t}_k\}_{k=1}^{N_t}$ are chosen with the help of SOBSEC{} routine [8] which generates pseudorandom numbers. We use the center points of the type $I_{30,9}(x,\xi)$, $I_{40,12}(x,\xi)$ and $I_{50,14}(x,\xi)$ for $N=500$, $N=1000$ and $N=2000$ data points correspondingly. The squared error

$$E_{sq}=\sqrt{\frac{1}{N_t}\sum_{k=1}^{N_t}\left[F(\mathbf{t}_k)-\widetilde{f}(\mathbf{t}_k)\right]^2} \tag{6}$$

is computed with the total number of test points $N_t=500$. The results are placed in Table 1.

N	K	F
500	250	$1.0\cdot10^{-4}$
1000	500	$6.5\cdot10^{-7}$
2000	1000	$2.1\cdot10^{-8}$

Table 1: The squared error E_{sq} in the approximation of Franke's function.

Straight Collocation (Kansa's Method)

The basis functions $I_{M,\chi}(\mathbf{x},\xi)$ can be used for solving PDEs in the framework of the Kansa's approach [12]. We consider using a set of center points $\{\xi_j\}_{j=1}^K$ that corresponds to the basis functions $\{I_{M,\chi}(x,\xi_j)\}_{j=1}^K$. Let $\{\mathbf{x}_i\}_{i=1}^N$ be a set of collocation points in $\overline{\Omega}$ of which $\{\mathbf{x}_i\}_{i=1}^{N_1}$ are interior points and $\{\mathbf{x}_i\}_{i=N_1+1}^N$ are boundary points. An approximate solution $\widetilde{u}(\mathbf{x})$ of the boundary value problem (1) is looked for in the form:

$$\widetilde{u}(\mathbf{x})=\sum_{j=1}^K p_j I_{M,\chi}(\mathbf{x},\xi_j). \tag{7}$$

K	E_{sq}	EX_{sq}	EY_{sq}
50	2.8E-03	3.7E-03	4.3E-03
100	1.5E-05	6.4E-05	7.2E-05
200	1.3E-05	6.2E-05	6.4E-05
400	1.1E-05	5.3E-05	5.2E-05
500	9.0E-06	4.4E-05	4.3E-05
600	6.8E-06	3.2E-05	3.2E-05

Table 2: The squared error of the computed solution and its first order derivatives

The collocation system arises when satisfying the governed equation and the boundary conditions at the collocation points:

$$\sum_{j=1}^{K} p_j L_{(x)} \left[I_{M,\chi}(\mathbf{x}_i, \xi_j) \right] = f(\mathbf{x}_i), 1 \le i \le N_1 \tag{8}$$

$$\sum_{j=1}^{K} p_j B_{(x)} \left[I_{M,\chi}(\mathbf{x}_i, \xi_j) \right] = g(\mathbf{x}_i), N_1 + 1 \le i \le N. \tag{9}$$

We denote by $L_{(x)}$ and $B_{(x)}$ the operators acting on $I_{M,\chi}$ viewed as a function of the first argument. In general $K \le N$ and the least squares method could be used to solve it. Thus the basis functions are just $\left\{ B_{(x)} \left(I_{M,\chi}(\mathbf{x}, \xi_j) \right) \right\}_{j=1}^{K}$ for boundary points and $\left\{ L_{(x)} \left(I_{M,\chi}(\mathbf{x}, \xi_j) \right) \right\}_{j=1}^{K}$ for interior points from (8) and (9).

We demonstrate the effectiveness of the proposed algorithm by carrying out a numerical test with the following modified Helmholtz equaiton.

EXAMPLE 2. We consider the modified Helmholtz equation

$$\begin{aligned} \Delta u - pu &= f(\mathbf{x}), \ \mathbf{x} \in \Omega \\ u(\mathbf{x}) &= g(\mathbf{x}), \ \mathbf{x} \in \partial\Omega \end{aligned}$$

with $\Omega = [-0.5, 0.5] \times [-0.5, 0.5]$. Let $p = 10$, the functions $f(\mathbf{x})$ and $g(\mathbf{x})$ be chosen such that the exact solution of the problem is $u = \frac{1}{3}\exp(-81\left(x^2 + y^2\right)/16)$.

We use the set of center points corresponding to $I_{M,\chi} = I_{10,4}$. We randomly choose K center points of this type inside the square Ω. We also choose $N_1 = 600$ collocation points randomly inside the square and $N_b = 100$ collocation points equally spaced on the boundary, with a total of collocation points $N = 700$ not less than the total K of the center points of the basis functions.

The squared error (6) is computed with the total of $N_t = 200$ randomly chosen test points inside Ω. The results are shown in Table 2. Note that in this Table, E_{sq} is the squared error of the approximate solution $\widetilde{u}$, EX_{sq} is the squared error of $\widetilde{u}_x$, and EY_{sq} is the squared error of $\widetilde{u}_y$.

References

1. Fairweather, G. and Karageorghis, A. (1998): "The method of fundamental solutions for elliptic boundary value problems", *Advances in Comp. Math.*, Vol. 9, pp. 69–95.
2. Golberg, M.A. and Chen, C.S. (1997): *Discrete Projection Methods for Integral Equations*, Southampton: Computational Mechanics Publications.
3. Golberg M.A. and Chen, C.S. (1998): *The Method of Fundamental Solutions for Potential, Helmholtz and Diffusion Problems*, in: Boundary Inetgral Methods - Numerical and Mathematical Aspects, ed. MA. Golberg, Computational Mechanics Publications, pp. 103–176.
4. Chen, C.S. and Golberg, M.A. and Schaback, R.S. (2002): *Recent developments of the dual reciprocity method using compactly supported radial basis functions*. In: Rashed YF, editor. Transformation of Domain Effects to the Boundary. Boston: WIT Press.
5. Reutskiy, S.Yu. (2005): "A boundary method of Trefftz type for PDEs with scattered data", *Engineering Analysis with Boundary Elements*, Vol. 29, pp. 713–724.
6. Golub, G. (1965): "Numerical methods for solving linear least squares problems", *Numerische Mathematik*, Vol. 7/3, pp. 206–216.
7. Franke, R. (1982): "Scattered data interpolation: test of some methods", *Math. Comput.* Vol. 48, pp. 181–200.
8. (1996): "Numerical Recipes in Fortran", *Cambridge University Press, Second Edition*
9. Floater, M.S. and Iske A. (1996): "Multistep scattered data interpolation using compactly supported radial basis functions", *J. Comp. Appl. Math.*, Vol. 73. pp. 65–78.
10. Boyd, J.P. (2002): "A comparison of numerical algorithms for Fourier extensuion of the first, second, and third kinds", *J. of Comp. Phys.*, Vol. 178, pp. 118–160.
11. Boyd, J.P. (2005): "Fourier embedded domain methods: extending a function defined on an irregular region to a rectangle so that the extension is spatially periodic and C^∞", *Appl. Math. and Comp.*, Vol. 161, pp. 591–597.
12. Kansa, E.J. (1990): "Multiquadrics - A scattered data approximation scheme with applications to computational fluid-dynamics–II. Solutions to parabolic, hyperbolic and elliptic partial differential equations", *Computers and Mathematics with Applications*, Vol. 19(8/9), pp. 147–161.