Copyright © 2007 ICCES ICCES, vol.3, no.2, pp.93-99, 2007

A New Collocation Method for Motz's Problem

Chein-Shan Liu[1], Yung-Wei Chen[2] and Jiang-Ren Chang[2]

Summary

A new collocation method is developed here to solve the elliptic boundary value problems with singularities. Specifically, we consider the Motz problem as a test of the performance of the new method, which is found accurate and effective.

keywords: Singularity, Motz problem, Collocation method, Linear interpolation

Introduction

Proves ineffective when one uses the classical numerical methods, like as finite difference, finite element or boundary element, to treat the elliptic boundary value problems with singularities. The singularity often arises in engineering problems when there is a sudden change in the boundary conditions or the boundary itself. In order to achieve a satisfactory solution near to the singular point, some special techniques are usually required as that given by Li (1996, 1998), Georgiou, Olson and Smyrlis (1996), Georgiou, Boudouvis and Poullikkas (1997), Yosibash, Arad, Yakhot and Ben-Dor (1998), Arad, Yosibash, Ben-Dor and Yakhot (1998), Huang and Li (2003, 2006), Dosiyev (2004), and Lu, Hu and Li (2004).

Among the many singular problems, the Motz problem was first studied by Motz (1947) for the relaxation method. Since there is a strong singularity $O(r^{\frac{1}{2}})$ at the original point, the Motz problem has become a benchmark of singularity problems [Li (1998)], which solves the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ \ (x,y)\in S:=\{(x,y)|-1<x<1,0<y<1\}, \tag{1}$$

under the following boundary conditions:

$$\left.\frac{\partial u}{\partial y}\right|_{y=1}=\left.\frac{\partial u}{\partial x}\right|_{x=-1}=0,\ \ u|_{x=1}=500, \tag{2}$$

$$u|_{x<0\wedge y=0}=0,\quad \left.\frac{\partial u}{\partial y}\right|_{x>0\wedge y=0}=0. \tag{3}$$

The collocation method

By utilizing the technique of separation of variables we are easy to write a series expansion of $u(x,y)$ satisfying Eqs. (1) and (2):

$$u(x,y)=500-\sum_{k=1}^{\infty}\frac{4a_k\cosh[(2k-1)\pi(1-y)/4]}{(2k-1)\pi\sinh[(2k-1)\pi/4]}\sin\frac{(2k-1)\pi(1-x)}{4}. \tag{4}$$

[1]Department of Mechanical and Mechatronic Engineering, Taiwan Ocean University, Keelung, Taiwan. Corresponding author, E-mail: csliu@mail.ntou.edu.tw.

[2]Department of Systems and Naval Architecture, Taiwan Ocean University, Keelung, Taiwan.

The above series expansion is well suited to the entire solution domain. Hence, the admissible functions with finite terms

$$u(x,y) = 500 - \sum_{k=1}^{m} \frac{4a_k \cosh[(2k-1)\pi(1-y)/4]}{(2k-1)\pi \sinh[(2k-1)\pi/4]} \sin\frac{(2k-1)\pi(1-x)}{4}, \quad (5)$$

where a_k are unknown coefficients to be determined below, are most efficient numerical solutions of Motz's problem.

By imposing the first condition in Eq. (3) on Eq. (5) it follows that

$$\sum_{k=1}^{m} \frac{4a_k}{(2k-1)\pi \tanh\frac{(2k-1)\pi}{4}} \sin\frac{(2k-1)\pi(1-x)}{4} = 500. \quad (6)$$

Taking the differential of Eq. (5) with respect to y, we obtain

$$\frac{\partial u(x,y)}{\partial y} = \sum_{k=1}^{m} \frac{a_k \sinh[(2k-1)\pi(1-y)/4]}{\sinh[(2k-1)\pi/4]} \sin\frac{(2k-1)\pi(1-x)}{4}. \quad (7)$$

Similarly, by imposing the second condition in Eq. (3) on Eq. (7) one has

$$\sum_{k=1}^{m} a_k \sin\frac{(2k-1)\pi(1-x)}{4} = 0. \quad (8)$$

Eqs. (6) and (8) are imposed at different collocated points on two different intervals with $-1 \le x_i < 0$ and $0 < \bar{x}_j \le 1$:

$$\sum_{k=1}^{m} \frac{4a_k}{(2k-1)\pi \tanh\frac{(2k-1)\pi}{4}} \sin\frac{(2k-1)\pi(1-x_i)}{4} = 500, \quad (9)$$

$$\sum_{k=1}^{m} a_k \sin\frac{(2k-1)\pi(1-\bar{x}_j)}{4} = 0. \quad (10)$$

It can be seen that the basic idea behind the collocation method is rather simple, and it has the great advantages of the flexibility to apply to different geometric shapes and different elliptic equations, and the simplicity of computer programming as shown below.

Let

$$x_i = -1 + (i-1)\Delta x, \;\; \bar{x}_j = 1 - (j-1)\Delta x, \;\; i, j = 1, \ldots, n, \quad (11)$$

where $\Delta x = 1/n$ and $m = 2n$. We call x_i the collocated points on the left-hand side, and conversely, $\bar{x}_j$ the collocated points on the right-hand side. When the indices

i, j in Eqs. (9) and (10) run from 1 to n we obtain a linear equations system with dimensions $m = 2n$:

$$\begin{bmatrix} \frac{4\sin\frac{\pi(1-x_1)}{4}}{\pi\tanh\frac{\pi}{4}} & \frac{4\sin\frac{3\pi(1-x_1)}{4}}{3\pi\tanh\frac{3\pi}{4}} & \cdots & \frac{4\sin\frac{(2m-3)\pi(1-x_1)}{4}}{(2m-3)\pi\tanh\frac{(2m-3)\pi}{4}} & \frac{4\sin\frac{(2m-1)\pi(1-x_1)}{4}}{(2m-1)\pi\tanh\frac{(2m-1)\pi}{4}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{4\sin\frac{\pi(1-x_n)}{4}}{\pi\tanh\frac{\pi}{4}} & \frac{4\sin\frac{3\pi(1-x_n)}{4}}{3\pi\tanh\frac{3\pi}{4}} & \cdots & \frac{4\sin\frac{(2m-3)\pi(1-x_n)}{4}}{(2m-3)\pi\tanh\frac{(2m-3)\pi}{4}} & \frac{4\sin\frac{(2m-1)\pi(1-x_n)}{4}}{(2m-1)\pi\tanh\frac{(2m-1)\pi}{4}} \\ \sin\frac{\pi(1-\bar{x}_1)}{4} & \sin\frac{3\pi(1-\bar{x}_1)}{4} & \cdots & \sin\frac{(2m-3)\pi(1-\bar{x}_1)}{4} & \sin\frac{(2m-1)\pi(1-\bar{x}_1)}{4} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sin\frac{\pi(1-\bar{x}_n)}{4} & \sin\frac{3\pi(1-\bar{x}_n)}{4} & \cdots & \sin\frac{(2m-3)\pi(1-\bar{x}_n)}{4} & \sin\frac{(2m-1)\pi(1-\bar{x}_n)}{4} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ a_{n+1} \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} 500 \\ \vdots \\ 500 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{12}$$

We denote the above equation by

$$\mathbf{R}\mathbf{a} = \mathbf{b}_1,$$

where $\mathbf{a} = [a_1, a_2, \cdots, a_m]^{\mathrm{T}}$ is the vector of unknown coefficients. The conjugate gradient method can be used to solve the following normal equation:

$$\mathbf{A}\mathbf{a} = \mathbf{b}, \tag{13}$$

where

$$\mathbf{A} := \mathbf{R}^{\mathrm{T}}\mathbf{R}, \;\; \mathbf{b} := \mathbf{R}^{\mathrm{T}}\mathbf{b}_1. \tag{14}$$

Inserting the calculated $\mathbf{a}$ into Eq. (5) we thus have a semi-analytical solution of $u(x,y)$.

Linear interpolation

The present computation based on the equation which automatically satisfies the governing equation (1) and the boundary conditions in Eq. (2). But the boundary conditions in Eq. (3) are achieved by imposing them on the collocated points. In Figs. 1(a) and 1(b) we display the data of $u(x,y)$ and $u_y(x,y)$ at $y=0$ on all collocated points by using $n = 200$ and thus $m = 400$. It can be seen that both $u(x,0) = 0, \; -1 \le x < 0$ and $u_y(x,0) = 0, \; 0 < x \le 1$ are satisfied very well. However, this is not true for the points which are not the collocated points. In order to show this phenomena, we have calculated $u(x,0)$ on the points $x_i = -1 + i(1-0.05)/260$ and $u_y(x,0)$ on the points $x_i = 1 - i(1-0.05)/260$. The numerical results as shown in Figs. 1(c) and 1(d) cannot satisfy both the two conditions in Eq. (3) obviously. Moreover, when the points are near to the singular point $x = 0$, the errors are largely increased.

In order to overcome this trouble, we can employ the linear interpolation method to calculate $u(x,y)$ and $u_y(x,y)$. For each given y_0, we use Eq. (5) to generate a sequence of $u(i)$ on $x_i, \; i = 1, \ldots, m+1$, which including $x_{n+1} = 0$, are the collocated

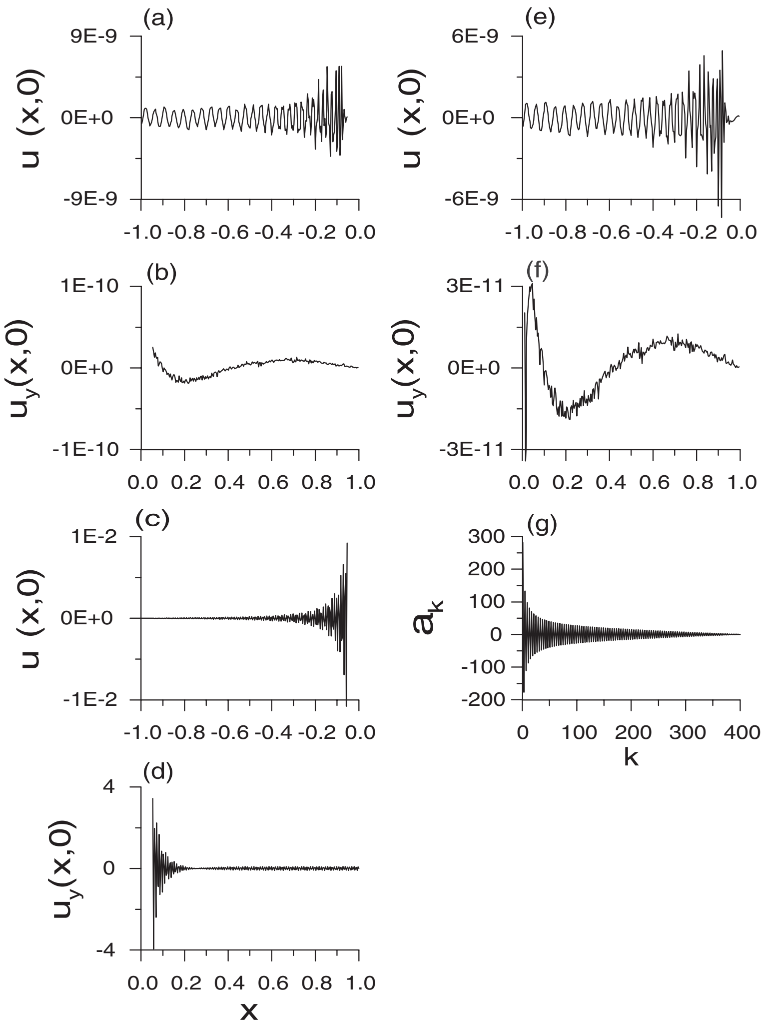

Figure 1: By applying the collocation method on the Motz problem we displaying the numerical results of u at y=0, which is zero on the boundary and u_y at y=0, which is zero on the boundary. In (g) we plotting the a_k.

points. Similarly, we use Eq. (7) to generate a sequence of $u_y(i)$ on the collocated points x_i. Then, we can calculate $u(x_0,y_0)$ and $u_y(x_0,y_0)$ at any point (x_0,y_0) in the problem domain by

$$u(x_0,y_0) = u(j) + \frac{x_0 - x_{j-1}}{x_j - x_{j-1}}[u(j) - u(j-1)], \tag{15}$$

$$u_y(x_0,y_0) = u_y(j) + \frac{x_0 - x_{j-1}}{x_j - x_{j-1}}[u_y(j) - u_y(j-1)], \tag{16}$$

where j is determined by $x_{j-1} < x_0 \leq x_j$.

In order to show the improvement of this technique, we have calculated $u(x,0)$ on the points $x_i = -1 + i/260$ and $u_y(x,0)$ on the points $x_i = 1 - i/260$. The numerical results are shown in Figs. 1(e) and 1(f). Obviously, the data satisfy the condition (3) very well. They are also better than that on the collocated points as shown in Figs. 1(a) and 1(b). Especially, when comparing with Figs. 1(c) and 1(d), at the points which are near to the singular point $x = 0$, the errors are largely reduced. It can be seen that the first condition in Eq. (3) is fulfilled within the error smaller than 6×10^{-9}, while the second condition in Eq. (3) is fulfilled within the error smaller than 3×10^{-11}.

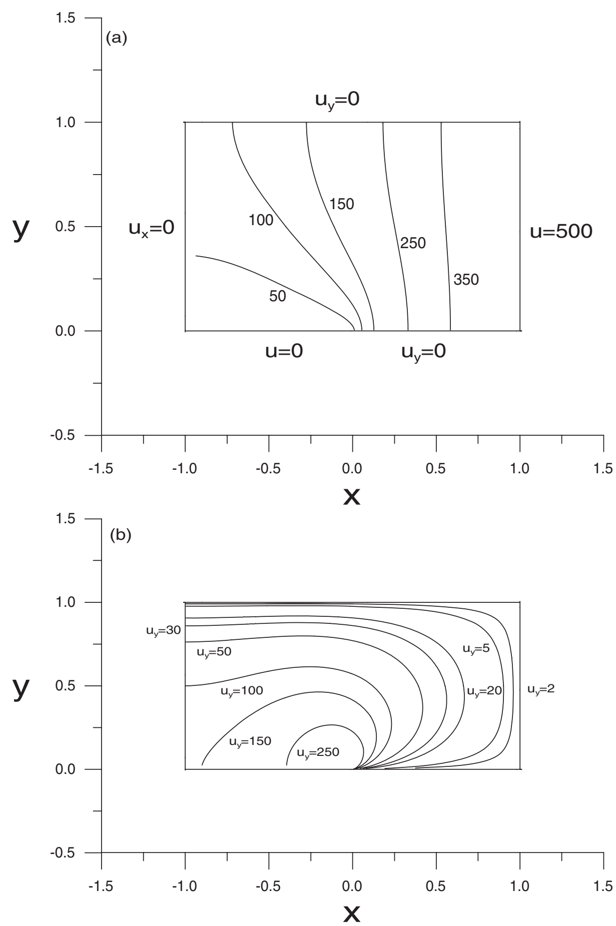

Figure 2: For the Motz problem (a) plotting the contour levels of u, (b) plotting the contour levels of u_y.

According to the above method we have calculated the Motz problem by using $m = 400$. In Fig. 1(g) we have plotted the Fourier coefficients. The amplitude of a_k converges very rapidly as k increases. When we plotted the different contour levels of $u = 50, 100, 150, 250, 350$ in Fig. 2(a), the different contour levels of $u_y = 2, 5, 20, 30, 50, 100, 150, 250$ are also plotted in Fig. 2(b). It can be seen that there has a narrow region near to the singular point that the solution has a high gradient with u_y very large. In a practical calculation we found that $u_y \approx 3976$ at the point $(x, y) = (-0.003846155, 0)$. Therefore, along the x-axis, u_y undergoes a large variation from a very large value to zero, when x passes the singular point $x = 0$. This fact makes the Motz problem not easy to handle by the conventional numerical methods.

Because in our numerical method it does not require any domain or surface meshing, the new meshless method would be very convenient for the engineering application in the computation of singular Motz's problem.

Conclusions

We have employed a new idea to treat the Motz problem by a collocation method, which is supplemented with a linear interpolation technique to enhance the accuracy. Even we were only considered the singular problem in a rectangle, the idea used here can be extended to other singular problem in a more complex region. The new method developed here provids us a semi-analytical solution in terms of the Fourier series, which renders a rather compendious numerical implementation to solve the singular problems. The new method was found to be accurate and effective.

References

1. **Arad, M.; Yosibash, Z.; Ben-Dor, G.; Yakhot, A.** (1998): Computing flux intensity factors by a boundary method for elliptic equations with singularities. *Comm. Num. Meth. Eng.*, vol. 14, pp. 657-670.

2. **Dosiyev, A. A.** (2004): The high accurate block-grid method for solving Laplace's boundary value problem with singularity. *SIAM J. Num. Anal.*, vol. 42, pp. 153-178.

3. **Georgiou, G.; Olson, L.; Smyrlis, Y. S.** (1996): A singular function boundary integral method for the Laplace equation. *Comm. Num. Meth. Eng.*, vol. 12, pp. 127-134.

4. **Georgiou, G.; Boudouvis, A.; Poullikkas, A.** (1997): Comparison of two methods for the computation of singular solutions in elliptic problems. *J. Comp. Appl. Math.*, vol. 79, pp. 277-287.

5. **Huang, H. T.; Li, Z. C.** (2003): Global superconvergence of Adini's elements coupled with the Trefftz method for singular problems. *Eng. Anal.*

Boun. Elem., vol. 27, pp. 227-240.

6. **Huang, H. T.; Li, Z. C.** (2006): Effective condition number and superconvergence of the Trefftz method coupled with high order FEM for singularity problems. *Eng. Anal. Boun. Elem.*, vol. 30, pp. 270-283.
7. **Li, Z. C.** (1996): Combinations of the Ritz-Galerkin and finite difference methods. *Int. J. Num. Meth. Eng.*, vol. 39, pp. 1839-1857.
8. **Li, Z. C.** (1998): Combined Methods for Elliptic Equations with Singularities, Interfaces and Infinities. Kluwer Academic Publishers, Netherlands.
9. **Lu, T. T.; Hu, H. Y.; Li, Z. C.** (2004): Highly accurate solutions of Motz's and the cracked beam problems. *Eng. Anal. Boun. Elem.*, vol. 28, pp. 1387-1403.
10. **Motz, H.** (1947): The treatment of singularities in relaxation methods. *Quart. Appl. Math.*, vol. 4, pp. 371-377.
11. **Yosibash, Z.; Arad, M.; Yakhot, A.; Ben-Dor, G.** (1998): An accurate semi-analytic finite difference scheme for two-dimensional elliptic problems with singularities. *Num. Meth. Par. Diff. Eq.*, vol. 14, pp. 281-296.