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A Meshless Regularized Integral Equation Method (MRIEM) for Laplace Equation in Arbitrary Interior or Exterior Plane Domains

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Summary

A new method is developed to solve the interior and exterior Dirichlet problems for the two-dimensional Laplace equation, namely the *meshless regularized integral equation method* (MRIEM), which consists of three parts: Fourier series expansion, the second kind Fredholm integral equation and an analytically regularized solution of the unknown boundary condition on an *artificial* circle. We find that the new method is powerful even for the problem with very complex boundary shape and with boundary noise.

keywords: Laplace equation, Meshless method, Regularized integral equation, Artificial circle, Arbitrary plane domain.

Introduction

The Dirichlet problem of Laplace equation in the plane domain is a classical one. Although the exact solutions have been found for some simple domains like as circle, ellipse, rectangle, etc., in general, for a given plane domain the finding of closed-form solutions is not an easy task.

Indeed, the explicit solutions are the exception, and if one were to choose an arbitrary shape of the domain, the geometric nonlinearity commences and then typically the numerical solution would be required.

The most widely used numerical methods are finite difference, finite element and boundary element methods. For a complicated shape of the domain they usually require a large number of nodes and elements to match the geometrical shape. In order to overcome these difficulties, the meshless numerical methods are proposed, which are meshes free and only boundary nodes are necessary.

Recently, the meshless local boundary integral equation (LBIE) method (Atluri, Kim and Cho, 1999), and the meshless local Petrov-Galerkin (MLPG) method (Atluri and Shen, 2002) are proposed. Both methods use local weak forms and the integrals can be easily evaluated over regularly shaped domains, like as circles in 2D problems and spheres in 3D problems.

In this paper we are going to propose a new meshless method to treat the Dirichlet problem of Laplace equation in the interior or exterior domain:

$$\Delta u(X) = 0, \;\; X \in \Omega \;\text{ or }\; X \in \mathbb{R}^2/\bar{\Omega}, \tag{1}$$

$$u(P) = h(P), \;\; P \in \Gamma, \tag{2}$$

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where Ω is a simply connected region in $\mathbb{R}^2$ with a contour shape Γ.

It is known that to treat the Dirichlet problem a standard tool is the boundary integral equation (Atkinson, 1997; Kress, 1989). It represents u as a double layer potential:

$$u(X) = \int_\Gamma \rho(Y) \frac{\partial}{\partial n_Y} \log|X-Y| dS_Y, \ \ X \in \Omega, \tag{3}$$

in which n_Y is the unit normal at $Y \in \Gamma$. The density function ρ satisfies

$$\pi\rho(X) + \int_\Gamma \rho(Y) \frac{\partial}{\partial n_Y} \log|X-Y| dS_Y = h(X), \ \ X \in \Gamma. \tag{4}$$

If we can parameterize the contour Γ by $\mathbf{r}(t) = (x(t), y(t))$, $t \in [0, 2\pi]$, we can obtain

$$\pi\rho(t) + \int_0^{2\pi} \rho(s) K(t,s) ds = h(t), \ \ 0 \le t \le 2\pi, \tag{5}$$

where

$$K(t,s) = \begin{cases} \frac{y'(s)[x(s)-x(t)]-x'(s)[y(s)-y(t)]}{[x(s)-x(t)]^2+[y(s)-y(t)]^2} & t \neq s, \\ \frac{y''(t)x'(t)-x''(t)y'(t)}{2[x'(t)^2+y'(t)^2]} & t = s. \end{cases} \tag{6}$$

The various numerical methods for solving the Laplace equation are rapidly developed in the last three decades. Recently, Young, Chen and Lee (2005) have proposed a novel meshless method for solving the Laplace equation in the arbitrary domain through a rather complicated desingularization technique, and Chen, Shen and Chen (2006) utilized the null-field method to calculate the torsion Laplace problem with many holes.

The other parts of the present paper are arranged as follows. In Section 2 we derive the first kind Fredholm intergral equation along a given artificial circle. In Section 3 we consider a direct regularization of the first kind Fredholm intergral equation. Then we derive a two-point boundary value problem, which helps us to derive a semi-analytical solution of the second kind Fredholm intergral equation in Section 4. In Section 5 we use some examples to test the new method, and then, we give some remarkable conclusions in Section 6.

The Fredholm integral equation

In this paper we consider a new meshless method to solve the Dirichlet problem which consists of the Laplace equation and the Dirichlet boundary condition given at a non-circular boundary:

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0,$$
$$r < \rho \ \text{ or } \ r > \rho, \ \ 0 \le \theta \le 2\pi, \tag{7}$$

$$u(\rho, \theta) = h(\theta), \ \ 0 \le \theta \le 2\pi, \tag{8}$$

where $h(\theta)$ is a given function, and $\rho = \rho(\theta)$ is a given contour describing the boundary shape of the interior or exterior domain. The contour Γ in the polar coordinates is given by $\Gamma = \{(r,\theta)|r = \rho(\theta),\ 0 \le \theta \le 2\pi\}$.

We replace Eq. (8) by the following boundary condition:

$$u(R_0,\theta) = f(\theta),\ \ 0 \le \theta \le 2\pi, \tag{9}$$

where $f(\theta)$ is an unknown function to be determined, and R_0 is a given positive constant, such that the disk $D = \{(r,\theta)|r \le R_0,\ 0 \le \theta \le 2\pi\}$ can cover Ω for the interior problem, or is inside in the complement of Ω, that is, $D \in \mathbb{R}^2/\bar{\Omega}$ for the exterior problem. The advantage of this replacement is that we have a closed-form solution in terms of the Poisson integral:

$$u(r,\theta) = \pm\frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - R_0^2}{R_0^2 - 2R_0 r\cos(\theta-\xi) + r^2} f(\xi)d\xi. \tag{10}$$

Here, R_0 can be viewed as the radius of an *artificial circle*, and $f(\theta)$ is an unknown function to be determined on this artificial circle. In the above, the positive sign is used for the exterior problem, and conversely the minus sign is used for the interior problem.

By utilizing the technique of separation of variables we can write a Fourier series expansion of $u(r,\theta)$ satisfying Eqs. (7) and (9):

$$\begin{aligned} u(r,\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\Bigg[& a_k\left(\frac{R_0}{r}\right)^{\pm k}\cos k\theta \\ & + b_k\left(\frac{R_0}{r}\right)^{\pm k}\sin k\theta\Bigg], \end{aligned} \tag{11}$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\xi)d\xi, \tag{12}$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(\xi)\cos k\xi d\xi, \tag{13}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(\xi)\sin k\xi d\xi. \tag{14}$$

Similarly, in Eq. (11) the positive sign before k is used for the exterior problem, and conversely the minus sign before k is used for the interior problem.

By imposing the condition (8) on Eq. (11) we obtain

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}\left[a_k\left(\frac{R_0}{\rho}\right)^{\pm k}\cos k\theta \right.$$
$$\left. +b_k\left(\frac{R_0}{\rho}\right)^{\pm k}\sin k\theta\right]=h(\theta). \tag{15}$$

Substituting Eqs. (12)-(14) into Eq. (15) leads to the first kind Fredholm integral equation:

$$\int_0^{2\pi} K(\theta,\xi)f(\xi)d\xi = h(\theta), \tag{16}$$

where

$$K(\theta,\xi)=\frac{1}{2\pi}+\sum_{k=1}^{\infty}\left\{B_k\left[\cos k\theta\cos k\xi+\sin k\theta\sin k\xi\right]\right\} \tag{17}$$

is a kernel function, and

$$B_k(\theta):=\frac{1}{\pi}\left(\frac{R_0}{\rho(\theta)}\right)^{\pm k}. \tag{18}$$

Our starting point in Eq. (11) is similar to the Trefftz method. However, the Trefftz method expands the solution by using the following T-complete basis functions:

$$\{1, r^{\mp k}\cos k\theta, r^{\mp k}\sin k\theta,\ k=1,2,\ldots\},$$

which just merely satisfy the governing equation and the unknown coefficients are determined by satisfying the boundary conditions in some manners as by means of the collocation, the least square or the Galerkin method, etc. (Kita and Kamiya, 1995). Huang and Shaw (1995) have derived an integral representation of the Trefftz method on the so-called embedding surface. However, as remarked by Huang and Shaw (1995) their method is simply an alternative derivation of the Trefftz method, and their approach is still in a conceptual level. On the other hand, the method of fundamental solution (MFS), also called the F-Trefftz method, utilizes the fundamental solutions as basis functions to expand the solution. In spite of the minor and major differences between the Trefftz and MFS methods, Chen, Wu, Lee and Chen (2006) have proved the equivalence of these two methods for Laplace and biharmonic equations.

Basically, these methods are of the too-early discretized method, of which the governing equations are discretized into a linear equation system in a rather earlier stage, and not to be continued into the integral equation as we will do in this paper. Therefore, many inherent drawbacks of these methods as explained by Liu

(2005) can be avoided here by the new method, of which we would provide a semi-analytical solution of the unknown data on the artifical circle in the next two sections.

The Trefftz method is a special case of our method with $R_0 = 1$. In a forthcoming paper we will develop a modified Trefftz method, which improves the stability problem of the Trefftz method.

Two-point boundary value problem

In order to obtain $f(\theta)$ we have to solve the first kind Fredholm integral equation (16). However this integral equation is known to be ill-posed. We assume that there exists a regularized parameter α, such that Eq. (16) can be regularized by

$$\alpha f(\theta) + \int_0^{2\pi} K(\theta,\xi) f(\xi) d\xi = h(\theta), \tag{19}$$

which is known as one of the second type Fredholm integral equation. The above regularization method to obtain a regularized solution by solving the singularly perturbed operator equation is usually called the Lavrentiev regularization method (Lavrentiev, 1967).

Up to this point we can remark the differences between Eqs. (19) and (5). In Eq. (5) the kernel function requires the contour curve to be twicely differentiable, which is however a rather stringent constraint. But in Eq. (19) the kernel function is well-defined for all contour curves. The kernel function in Eq. (6) is not separable, but the kernel function in Eq. (17) is separable, which makes an easier solution of the integral equation (19) than Eq. (5).

Our method is different from the the other boundary-type solution procedure, which is used as a general terminology to include the boundary element method, the Trefftz method, the method of fundamental solution, as well as different type meshless methods. The new method is more easy to handle because it is an integral equation on a given artificial circle, instead of on the contour Γ. As we know that in the open literature there has no report to connect the Laplace problem to this type integral equation.

We assume that the kernel function can be approximated by m terms with

$$K(\theta,\xi) = \frac{1}{2\pi} + \sum_{k=1}^{m} \left\{ B_k \left[\cos k\theta \cos k\xi + \sin k\theta \sin k\xi\right] \right\}. \tag{20}$$

This assumption is for the convenience of our derivation but is not an essential one. Moreover, the numerical solutions are usually dominated by the first few leading terms.

By inspection we have

$$K(\theta,\xi) = \mathbf{P}(\theta) \cdot \mathbf{Q}(\xi), \tag{21}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are $2m+1$-vectors given by

$$\mathbf{P} := \begin{bmatrix} \frac{1}{2\pi} \\ B_1 \cos\theta \\ B_1 \sin\theta \\ B_2 \cos 2\theta \\ B_2 \sin 2\theta \\ \vdots \\ B_m \cos m\theta \\ B_m \sin m\theta \end{bmatrix}, \quad \mathbf{Q} := \begin{bmatrix} 1 \\ \cos\xi \\ \sin\xi \\ \cos 2\xi \\ \sin 2\xi \\ \vdots \\ \cos m\xi \\ \sin m\xi \end{bmatrix}, \tag{22}$$

and the dot between $\mathbf{P}$ and $\mathbf{Q}$ denotes the inner product, which is sometime written as $\mathbf{P}^{\mathrm{T}}\mathbf{Q}$, where the superscript T signifies the transpose.

With the aid of Eq. (21), Eq. (19) can be decomposed as

$$\alpha f(\theta) + \int_0^\theta \mathbf{P}^{\mathrm{T}}(\theta)\mathbf{Q}(\xi) f(\xi) d\xi + \int_\theta^{2\pi} \mathbf{P}^{\mathrm{T}}(\theta)\mathbf{Q}(\xi) f(\xi) d\xi = h(\theta). \tag{23}$$

Let us define

$$\mathbf{u}_1(\theta) := \int_0^\theta f(\xi)\mathbf{Q}(\xi) d\xi, \tag{24}$$

$$\mathbf{u}_2(\theta) := \int_{2\pi}^\theta f(\xi)\mathbf{Q}(\xi) d\xi, \tag{25}$$

and then Eq. (23) can be expressed as

$$\alpha f(\theta) + \mathbf{P}^{\mathrm{T}}(\theta)[\mathbf{u}_1(\theta) - \mathbf{u}_2(\theta)] = h(\theta). \tag{26}$$

Taking the differential of Eqs. (24) and (25) with respect to θ we obtain

$$\mathbf{u}_1'(\theta) = \mathbf{Q}(\theta) f(\theta), \tag{27}$$

$$\mathbf{u}_2'(\theta) = \mathbf{Q}(\theta) f(\theta). \tag{28}$$

Inserting Eq. (26) for $f(\theta)$ into the above two equations we obtain

$$\alpha \mathbf{u}_1'(\theta) = \mathbf{Q}(\theta)\mathbf{P}^{\mathrm{T}}(\theta)[\mathbf{u}_2(\theta) - \mathbf{u}_1(\theta)] + h(\theta)\mathbf{Q}(\theta), \tag{29}$$

$$\alpha \mathbf{u}_2'(\theta) = \mathbf{Q}(\theta)\mathbf{P}^{\mathrm{T}}(\theta)[\mathbf{u}_2(\theta) - \mathbf{u}_1(\theta)] + h(\theta)\mathbf{Q}(\theta), \tag{30}$$

$$\mathbf{u}_1(0) = \mathbf{0}, \ \ \mathbf{u}_2(2\pi) = \mathbf{0}, \tag{31}$$

where the last two conditions follow from Eqs. (24) and (25) immediately. The above equations constitute a two-point boundary value problem.

Semi-analytical solution

In this section we will find a semi-analytical solution of $f(\theta)$. From Eqs. (27) and (28) it can be seen that $\mathbf{u}_1' = \mathbf{u}_2'$, which means that

$$\mathbf{u}_1 = \mathbf{u}_2 + \mathbf{c}, \tag{32}$$

where $\mathbf{c}$ is a constant vector to be determined. By using the final condition in Eq. (31) we find that

$$\mathbf{u}_1(2\pi) = \mathbf{u}_2(2\pi) + \mathbf{c} = \mathbf{c}. \tag{33}$$

From Eqs. (24) and (33) it follows that

$$\mathbf{c} = \int_0^{2\pi} f(\xi)\mathbf{Q}(\xi)d\xi. \tag{34}$$

The mathematical meaning of $\mathbf{c}$ is that it is a vector of the Fourier coefficients of the unknown function $f(\theta)$.

Substituting Eq. (32) into (29) we have

$$\alpha\mathbf{u}_1'(\theta) = -\mathbf{Q}(\theta)\mathbf{P}^{\mathrm{T}}(\theta)\mathbf{c} + h(\theta)\mathbf{Q}(\theta). \tag{35}$$

Integrating and using the initial condition in Eq. (31) it follows that

$$\mathbf{u}_1(\theta) = \frac{-1}{\alpha}\int_0^{\theta} \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi\,\mathbf{c} + \frac{1}{\alpha}\int_0^{\theta} h(\xi)\mathbf{Q}(\xi)d\xi. \tag{36}$$

Taking $\theta = 2\pi$ in the above equation and imposing the condition (33) one obtains a governing equation for $\mathbf{c}$:

$$\mathbf{R}\mathbf{c} = \int_0^{2\pi} h(\xi)\mathbf{Q}(\xi)d\xi, \tag{37}$$

where

$$\mathbf{R} := \alpha\mathbf{I}_{2m+1} + \int_0^{2\pi} \mathbf{Q}(\xi)\mathbf{P}^{\mathrm{T}}(\xi)d\xi. \tag{38}$$

It is straightforward to write

$$\mathbf{c} = \mathbf{R}^{-1}\int_0^{2\pi} h(\xi)\mathbf{Q}(\xi)d\xi. \tag{39}$$

By Eq. (34) we have

$$\int_0^{2\pi} f(\xi)\mathbf{Q}(\xi)d\xi = \mathbf{R}^{-1}\int_0^{2\pi} h(\xi)\mathbf{Q}(\xi)d\xi, \tag{40}$$

which describes the relation between the Fouier coefficients vectors of two boundary functions $f(\theta)$ and $h(\theta)$.

On the other hand, from Eqs. (26) and (32) we have

$$\alpha f(\theta) = h(\theta) - \mathbf{P}(\theta)\cdot\mathbf{c}. \tag{41}$$

Inserting Eq. (39) into the above equation we obtain

$$\alpha f(\theta) = h(\theta) - \mathbf{P}(\theta)\cdot\mathbf{R}^{-1}\int_0^{2\pi} h(\xi)\mathbf{Q}(\xi)d\xi. \tag{42}$$

Then, the conjugate gradient method is used to solve the following normal equation:

$$\mathbf{A}\mathbf{c} = \mathbf{b}, \tag{43}$$

where

$$\mathbf{A} := \mathbf{R}^{\mathrm{T}}\mathbf{R}, \;\; \mathbf{b} := \mathbf{R}^{\mathrm{T}}\int_0^{2\pi} h(\xi)\mathbf{Q}(\xi)d\xi. \tag{44}$$

Inserting the calculated $\mathbf{c}$ into Eq. (41) we thus have a semi-analytical solution of $f(\theta)$, upon substituting which into Eqs. (12)-(14) we can easily evaluate all the integrals over the circular domain to obtain the semi-analytical solution of u. Because in these processes we do not require any domain or surface meshing, the new meshless method would be very convenient for the engineering application in the computation of complex boundary shape problems.

If we do not want to know the boundary data $f(\theta)$ on the artificial circle, we can directly skip to the solution of u given by

$$\begin{aligned} u(r,\theta) \;\; = \;\; & \frac{c_1}{2\pi} + \sum_{k=1}^{m}\left[\frac{c_{2k}}{\pi}\left(\frac{R_0}{r}\right)^{\pm k}\cos k\theta\right. \\ & \left. + \frac{c_{2k+1}}{\pi}\left(\frac{R_0}{r}\right)^{\pm k}\sin k\theta\right], \end{aligned} \tag{45}$$

after solving $\mathbf{c}$ from Eq. (43), where $(c_1,\ldots,c_{2m+1})$ are the components of $\mathbf{c}$. The above equation can be derived immediately by comparing Eq. (34) with Eqs. (12)-(14) and using Eq. (22) and Eq. (11) with ∞ replaced by m.

Numerical examples

Before embarking the numerical study of the new method, we are concerned with the stability of MRIEM, in the case when the boundary data are contaminated by random noise, which is investigated by adding the different levels of random noise on the boundary data. We use the function RANDOM_NUMBER given in

Fortran to generate the noisy data $R(i)$, where $R(i)$ are random numbers in $[-1,1]$. Hence we use the simulated noisy data given by

$$\hat{h}(\theta_i) = h(\theta_i) + \varepsilon R(i), \tag{46}$$

where $\theta_i = 2i\pi/n_b,\ i = 0,1,\ldots,n_b$, and ε is defined as

$$\varepsilon = \max|h(\theta)| \times \frac{s}{100}, \tag{47}$$

where s is the percentage of additive noise on the data.

Example 1 (exterior problem)

In this example we investigate a discontinuous boundary condition on the unit circle:

$$h(\theta) = \begin{cases} 1 & 0 \le \theta < \pi, \\ -1 & \pi \le \theta < 2\pi. \end{cases} \tag{48}$$

For this example an analytical solution is given by

$$u(x,y) = \frac{2}{\pi} \arctan\left(\frac{2y}{x^2+y^2-1}\right). \tag{49}$$

We have applied the new method to this example by fixing $R_0 = 1$, $m = 20$ and $\alpha = 10^{-10}$. In Fig. 1 we compared the exact solution with numerical solutions with $s = 0,5$ along a circle with radius 2.5. It can be seen that the numerical solutions are very close to the exact solution.

Example 2 (exterior problem)

In this example we consider a complex epitrochoid boundary shape

$$\rho(\theta) = \sqrt{(a+b)^2 + 1 - 2(a+b)\cos(a\theta/b)}, \tag{50}$$

$$x(\theta) = \rho\cos\theta, \quad y(\theta) = \rho\sin\theta \tag{51}$$

with $a = 3$ and $b = 1$. The analytical solution is given by

$$u(x,y) = \exp\left(\frac{x}{x^2+y^2}\right)\cos\left(\frac{y}{x^2+y^2}\right). \tag{52}$$

The exact boundary data can be easily derived by inserting Eqs. (50) and (51) into the above equation.

We have applied the new method to this example by fixing $R_0 = 3$, $m = 20$ and $\alpha = 10^{-5}$. In Fig. 2 we compared the exact solution with numerical solution along a circle with radius 10. It can be seen that the numerical solution is almost

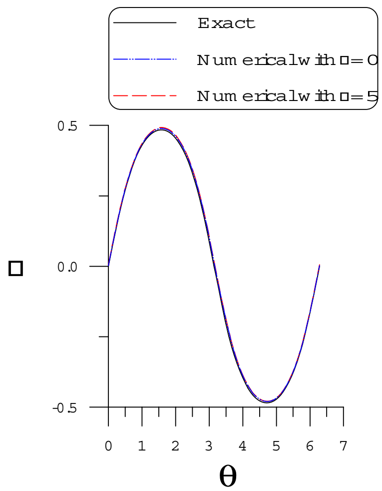

Fig. 1. Comparing the numerical and exact solutions for example 1.

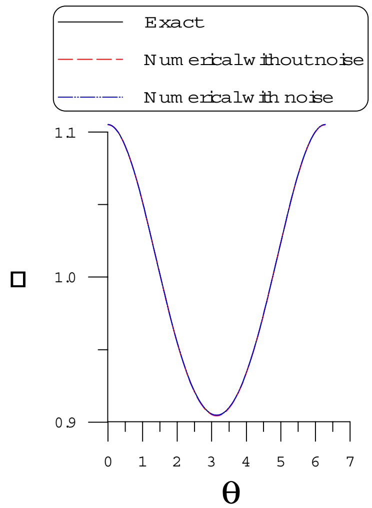

Fig. 2. Comparing the numerical and exact solutions for example 2.

coincident with the exact solution, of which the L^2 error is about 10^{-4}. Also we are imposed a random noise with intensity $\sigma = 0.003$ by

$$\hat{h}(\theta_i) = h(\theta_i)[1 + \sigma R(i)] \tag{53}$$

on the exact boundary data, of which the numerical solution was shown in the same figure by the dashed-dotted line. The new method is robust to against the disturbance on the boundary data.

Example 3 (interior problem)

In this example we consider another epitrochoid boundary shape with $a = 4$ and $b = 1$; see Fig. 3. The analytical solution is given by

$$u(x,y) = x^2 - y^2. \tag{54}$$

The exact boundary data can be easily derived by inserting Eqs. (50) and (51) into the above equation.

In the numerical computation we have fixed $R_0 = 6$, $m = 5$ and $\alpha = 10^{-15}$. In Fig. 3 we compared the contour levels of potential $u = -4$ and $u = 2$ for exact solutions and numerical solutions. It can be seen that the numerical results are almost coincident with the exact ones. The accurcay of the numerical solutions are found to be good with the L^2 error about 1.36×10^{-13}.

Conclusions

In this paper we have proposed a new meshless method to calculate the solutions of Laplace equation in the arbitrary plane domains. It was demonstrated

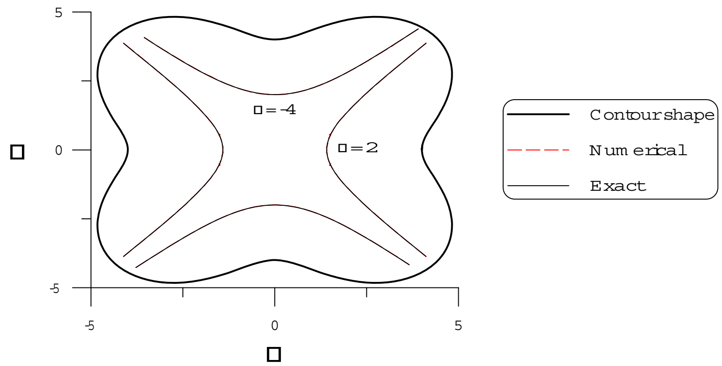

Fig. 3. Comparing the numerical and exact contour lines for example 3.

that in the regularized sense we can find a semi-analytical solution of the boundary condition on an artificial circle, and thus by the Poisson integral we can calculate the solution at any point inside the domain. The numerical examples show that the effectiveness of the new method and the accuracy is very good. The new method possesses several advantages than the conventional boundary-type solution methods, including mesh-free, singularity-free, non-illposedness, semi-analyticity of solution, efficiency, accuracy and stability.

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