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Transient Response Analysis of Viscoelastic Frames with the Method of Reverberation Ray Matrix

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Summary

In this paper the reverberation ray matrix method, which was developed recently by Professor Pao and his colleagues for analyzing dynamic response of elastic trusses or frames, is extended and used to solve the transient response of frames made of viscoelastic bars. Originally for the solution of elastic structure the matrix (I-R)$^{-1}$ is expanded into Neumann series to circumvent the difficulty of singularity in reversing the matrix in frequency domain. However, it is not necessary to expand this matrix since there is no singularity problem for viscoelastic frame due to viscous damping. The accuracy and effectiveness of applying reverberation ray matrix method is verified by two examples: (1) a fix-free bar subjected to step-like axial load and (2) a viscoelastic frame made of nine members, also subjected to step-like load.

Introduction

The response of a structure subjected to dynamic load, especially impact load is much more complex than that of static load. As the advance of science and technology, polymeric materials and composites have been more and more introduced in, such as civil, aeronautical and astronautical engineering. This type of materials exhibit viscoelastic characteristics. The solution of initial response of trusses or frames made of viscoelastic members is worth attention for engineering application.

There are generally two approaches for dynamic analysis of trusses or frames: One is to treat the structure as a distributive system of multi-connected bars or beams which can be solved by, for instance, transfer matrix method, direct stiffness method, compliance method etc[1]. The other one is to discretize every member of the structure into a number of elements, i.e. finite element method. However, the transcendental function appeared in the matrix of the former approach often results in difficulty of solution, while the latter needs numerous elements to get relatively accurate result. It is likely applicable for low frequency and long term response.

An alternative approach is using wave propagation theory to obtain the response of the structure. Recently, based on the theory of wave propagation Pao [2] proposed the method of reverberation ray matrix (MRRM). The present study extends the reverberation ray matrix method to investigate the transient response of frames made of viscoelastic material. The application of MRRM is verified by examples of a fix-free bar and a frame made of nine viscoelastic beams.

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Basic equations

Without loss of generality, we use the three parameter viscoelastic solid model to represent properties of the bar (Fig. 1). The stress-strain relation of this material can be expressed by the following differential equation:

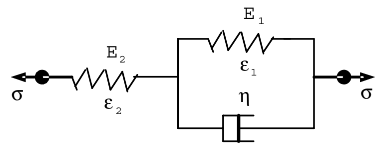

Figure 1: Three parameter viscoelastic solid

$$\sigma + p_1\dot{\sigma} = q_0\varepsilon + q_1\dot{\varepsilon} \tag{1}$$

where $p_1 = \frac{\eta}{(E_1+E_2)}$, $q_0 = \frac{E_1E_2}{E_1+E_2}$, $q_1 = \frac{E_2\eta}{E_1+E_2}$ The complex modulus is

$$\hat{E}(\omega) = \frac{q_0 + i\omega q_1}{1 + i\omega p_1} \tag{2}$$

If Poisson ratio is assumed as constant, then the shear modulus is

$$\hat{G}(\omega) = \frac{\hat{E}(\omega)}{2(1+\mu)} \tag{3}$$

where the top-script '∧' means the variable in frequency domain.

According to the theory of linear viscoelasticity[3], in frequency domain the governing equation of linear viscoelastic material can be obtained by replacing the elastic modulus in the governing equation of corresponding elastic material with the complex modulus. The differential equations for Timoshenko's beam of viscoelastic material can be written as:

$$\frac{\partial^2 \hat{u}(x,\omega)}{\partial x^2} + \frac{\rho}{\hat{E}(\omega)}\omega^2 \hat{u}(x,\omega) = 0 \tag{4}$$

for axial movement and

$$\begin{cases} \kappa\hat{G}(\omega)\frac{\partial^2 \hat{v}_s(x,\omega)}{\partial x^2} + \rho\omega^2\hat{v}(x,w) = 0 \\ \hat{E}(\omega)I\frac{\partial^3 \hat{v}_b(x,\omega)}{\partial x^3} + \kappa A\hat{G}(\omega)\frac{\partial \hat{v}_s(x,\omega)}{\partial x} + \rho I\omega^2\frac{\partial \hat{v}_b(x,\omega)}{\partial x} = 0 \end{cases} \tag{5}$$

for flexural movement. In the above equations $\hat{u}$,$\hat{v}_b$, $\hat{v}_s$ are axial, flexural bending and flexural shear displacement ($\hat{v} = \hat{v}_b + \hat{v}_s$), I, A, κ, ρ are second moment, area, shear coefficient of the cross section and density respectively. The amplitudes of axial wave a_1, d_1 and that of flexural waves a_2 , d_2, a_3, d_3 are assumed as basic unknowns, where 'a' means incident wave while 'd' means departing wave. The axial and flexural displacements can be expressed as[2]

$$\hat{u}(x,\omega) = a_1(\omega)e^{ik_1x} + d_1(\omega)e^{-ik_1x} \tag{6}$$

$$\begin{cases} \hat{v}_b(x,\omega) = a_2(\omega)e^{ik_2x} + d_2(\omega)e^{-ik_2x} + a_3(\omega)e^{ik_3x} + d_3(\omega)e^{-ik_3x} \\ \hat{v}_s(x,\omega) = \chi_2 a_2(\omega)e^{ik_2x} + \chi_2 d_2(\omega)e^{-ik_2x} + \chi_3 a_3(\omega)e^{ik_3x} + \chi_3 d_3(\omega)e^{-ik_3x} \end{cases} \tag{7}$$

where $k_1 \sim k_3$are wave numbers, χ_2, χ_3 are coefficients of deflection ratio. The axial force $\hat{F}$, shear force $\hat{V}$, bending moment $\hat{M}$ and rotation $\hat{\phi}$ are

$$\hat{F} = \hat{E}A\frac{\partial \hat{u}}{\partial x}, \quad \hat{V} = \kappa A\hat{G}\frac{\partial \hat{v}_s}{\partial x}, \quad \hat{M} = -\hat{E}I\frac{\partial^2 \hat{v}_b}{\partial x^2}, \quad \hat{\phi} = \frac{\partial \hat{v}_b}{\partial x} \tag{8}$$

The conditions of momentum equilibrium as well as compatibility of node $J$of the frame can be written in matrix form as[4]

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}^{\mathrm{J}} \begin{Bmatrix} d^{\mathrm{J}} \\ H^{\mathrm{J}} \end{Bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^{\mathrm{J}} \left\{ a^{\mathrm{J}} \right\} - \begin{Bmatrix} 0 \\ f^{\mathrm{J}} \end{Bmatrix} \tag{9}$$

where $\boldsymbol{D}$ and $\boldsymbol{A}$ are coefficient matrixes with material constants and geometric parameters, $\boldsymbol{H}$ is displacement vector and $\boldsymbol{f}$ is load vector of the node.$\left\{d^{\mathrm{J}}\right\}$ and $\left\{H^{\mathrm{J}}\right\}$can be solved from Eq.(9), and $\left\{d^{\mathrm{J}}\right\}$ can be reduced as:

$$d^{\mathrm{J}} = S^{\mathrm{J}} a^{\mathrm{J}} + s^{\mathrm{J}} \tag{10}$$

where S^{J} is the scattering matrix, s^{J} is the source vector of the node. Because the incident wave of a node is just the departing wave of a neighboring node, taking the whole structure of all nodes into consideration, $\boldsymbol{a}$ can be expressed with $\boldsymbol{d}$ as $\boldsymbol{a} = \boldsymbol{PUd}$, where $\boldsymbol{P}$ is named as phase shift matrix and $\boldsymbol{U}$ as permutation matrix. Equation (10) can be further reduced as[2]

$$d = \frac{s}{[I-R]} \tag{11}$$

in which $\boldsymbol{R} = \boldsymbol{SPU}$ is named as the reverberation ray matrix of the structure. As far as elastic structure is concerned, the matrix $[I-R]^{-1}$ contains infinite poles in frequency complex plane. It is impractical to directly reverse the matrix. In order to circumvent this difficulty, Ref.[2] proposes expanding $[I-R]^{-1}$ into a Neumann Series:

$$[I-R]^{-1} = I + R + R^2 + \ldots + R^N + \ldots \tag{12}$$

Once $\boldsymbol{d}$ is solved, the stress, strain and displacement of all members as well as displacements or constraint forces of every node can be calculated. These variables can be transformed back to time domain by the inverse fast Fourier procedure. However, for a structure made of viscoelastic materials, there is no singularity problem in the denominator of Eq. (11). The matrix $[I-R]^{-1}$ can be reversed directly so that calculation is reduced dramatically.

Verification of MRRM in viscoelastic problem

Shown in Fig.2 is a fix-free bar with square cross section ($0.1\times0.1\text{m}^2$) subjected to suddenly applied step-like stress at the right end. The boundary condition is: $\sigma(l,t)=\sigma_0 h(t)$, $h(t)$ is the Heaviside function, and $\sigma_0=100$MPa. The length of the bar is l=0.75m. The material properties are E_1 = 16GPa, E_2 = 20GPa, $\eta = 8.062\times10^7\text{Pa}\cdot\text{s}$, κ=0.833, ρ =6500Kg/m^3.

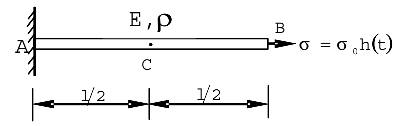

Figure 2: A fix-free bar subjected to suddenly applied axial load

As shown by Christenson[3] the stress in Laplace domain can be written as

$$\bar{\sigma}(x,s)=\frac{\sigma_0}{s}+\sum_{n=1}^{\infty}\frac{8\rho\sigma_0 l^3(-1)^n\cos\left[(2n-1)\pi x/2l\right]}{(2n-1)\pi\left[(2n-1)^2\pi^2\bar{E}(s)+4l^2\rho s\right]} \tag{13}$$

where $\bar{E}(s)=\bar{A}(s)/\bar{B}(s)$, $\bar{A}(s)=q_0+sq_1$, $\bar{B}(s)=s+s^2p_1$

Denote the polynomial of the denominator of Eq.(13) multiplied by $\bar{B}(s)$ as

$$\bar{P}_n(s)=(2n-1)^2\pi^2\bar{A}(s)+4\rho l^2 s\bar{B}(s)=D_n\prod_{j=1}^{N+2}(s-a_{jn}) \tag{14}$$

For the three parameter viscoelastic solid, P_n is a cubic polynomial of $s.D_n$ is the coefficient of $\bar{P}_n$. After inverse Laplace transformation, we have

$$\sigma(x,t)=\sigma_0 h(t)\left\{1+\frac{8\rho l^3}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\cos[(2n-1)\pi x/2l]}{D_n(2n-1)}\right.$$
$$\left.\times\sum_{k=1}^{N+2}\frac{\left(a_{k,n}e^{a_{k,n}t}\right)}{\lim\limits_{s\to a_{k,n}}\prod_{j=1}^{N+2}\left(s-a_{j,n}\right)\Big/\left(s-a_{k,n}\right)}\right\} \tag{15}$$

Shown in Fig.3 is the axial stress of the mid-point C obtained from MRRM compared with solution of Eq.(15). The unit time is $t_0=0.403\times10^{-3}$s. The two curves coincide well. The amplitude of the axial stress reduces with time due to viscous damping of the material and approaches quasi static value σ_0. It is verified that MRRM can be applied to solve the transient response of frames made of viscoelastic materials (such as polymers, composites).

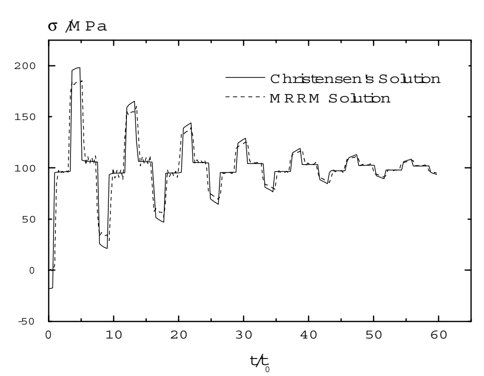

Figure 3: Christenson's solution compared with MRRM for the fix-free bar

Transient response of a nine member plane frame

In order to verify that the response of a viscoelastic structure can be obtained by direct inverse of Eq.(11) , the strain response of a plane frame made of nine viscoelastic members (Fig.4) is analyzed with Neumann series expansion of Eq.(12) as well as direct inverse of Eq.(11). The nine member frame has the same material properties and cross-section as the fix-free bar, except $\eta = 4.031 \times 10^8 \mathrm{Pa} \cdot \mathrm{s}$, subjected to a step-like load (800N) at the node 4. The transient response of axial strain and bending strain (the normal strain at the top or bottom surface due to flexural deformation of Timoshenko's beam) at the mid-point of member 34 are plotted and compared in Figures 5∼6.

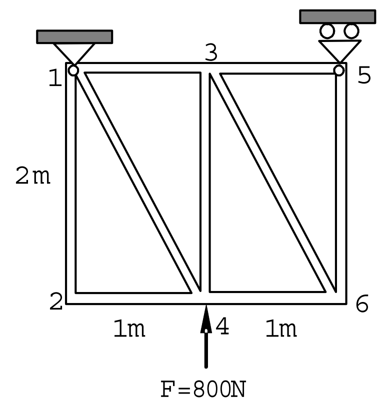

Figure 4: Nine member plan frame

As shown in Eq.(12) direct inverse of matrix $[I-R]^{-1}$is equivalent to infinite number of terms of Neumann series expansion. It is time saving and accurate. The results from the two methods coincide very well in initial stage, however, they depart from each other after a period of time, especially for bending strain as shown in Fig.6.

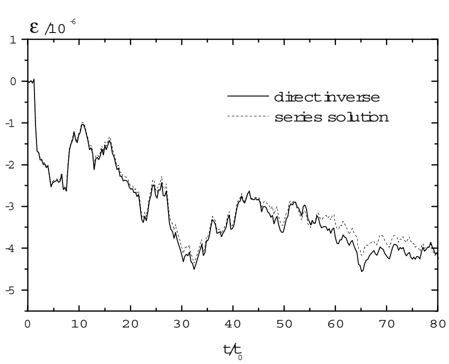

Figure 5: Axial strain at the mid-point of member 34

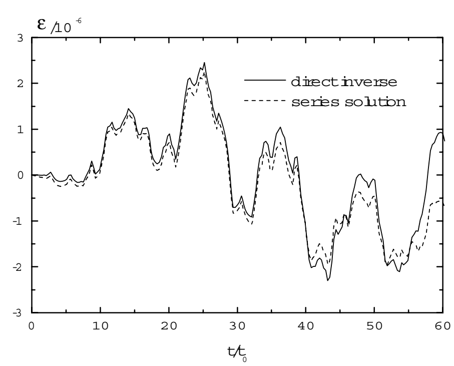

Figure 6: Bending strain at the mid-point of member 34

Conclusion

The MRRM which is originally developed for analysis of dynamic response of elastic trusses or frames is extended to viscoelastic frames. The feasibility of this extension is verified first with an example of a fix-free viscoelastic bar subjected to a suddenly applied tensile load, by comparing with the result obtained with Christensen's method. Then a plane frame of nine viscoelastic members is solved to demonstrate that the MRRM is efficient in analyzing dynamic problem of viscoelastic structures.

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