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Optimal 4-node shell and 3d-shell finite elements for nonlinear analysis

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Summary

First we shortly present several low-order (4-node) shell finite element formulations (based on Reissner-Mindlin kinematics) that allow for accurate and efficient (with coarse and distorted meshes) analysis of shell-like structures subjected to large deformations and rotations. The formulations are based on mixed variational principle, enhanced assumed strain (EAS) method (based on Green-Lagrange strains) and assumed natural strain (ANS) method. The EAS method is used in all formulations in order to improve both membrane and bending behavior of the 4-node element (the formulations differ from one another by the number of assumed EAS parameters), and the ANS method is used to avoid shear locking. An optimal number of membrane/bending EAS parameters is then identified by comparing results of a set of characteristic numerical examples (in this paper we only present results of two illustrative examples). Thus an optimal 4-node EAS/ANS nonlinear shell element is derived. In the second part of the paper we shortly present enhancement of the previously derived optimal shell element leading to an optimal low-order (4-node) nonlinear 3d-shell element; i.e. an element that accounts for through-the-thickness stretching. The enhancement, which introduces incompatible Green-Lagrange strains in the through-the-thickness direction, is based on EAS method. The derived 3d-shell element looks as a surface (with extensible directors) from the outside but it can build fully 3d stress and 3d strain states. Finally, we present a numerical example, which illustrates performance of an optimal 4-node EAS/ANS 3d-shell element.

Introduction

The enhanced assumed strain (EAS) method, introduced by Simo and Rifai [1] and Simo and Armero [2], has been accepted as a relatively simple and efficient tool for performance enhancement of lower order finite elements for linear/nonlinear analysis of solids and structures. Since the initial works many finite elements based on the EAS method have been developed. As concerning shell formulations, the EAS method has been used in two different manners: (i) to obtain 3d-shell and solid-shell formulations that account for through-the-thickness stretching, and (ii) to obtain shell, 3d-shell and solid shell elements with improved membrane performance. Linear/nonlinear EAS shell formulations (elements) have been presented e.g. by Bischoff and Ramm [3], Vu-Quoc and Tan [4], Brank, Korelc and Ibrahimbegovic [5], Sansour and Kollmann [6], among others. In this paper we first study

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two questions: (i) which is an optimal numer of EAS parameters for enhancement of membrane/bending strains for 4-node nonlinear geometrically exact shell element, and (ii) does EAS enhancement of bending strains improve performance of an element. We answer both questions by comparing results of two characteristic numerical examples for several membrane/bending EAS formulations. In such a way an optimal number of EAS parameters is found. In the second part of the paper we extend the optimal shell formulation by including through-the-thickness stretching in order to obtain an optimal 3d-shell nonlinear 4-node finite element.

Shell element formulations

Let the position vector in a shell reference configuration be defined as $\mathbf{X}\left(\xi^1,\xi^2,\xi^3\right) = \mathbf{\Phi}\left(\xi^1,\xi^2\right)+\xi^3\mathbf{T}\left(\xi^1,\xi^2\right)$, where ξ^i are material (convected) coordinates, $\left|\mathbf{T}\left(\xi^1,\xi^2\right)\right| = 1$, $\xi^3\in[-h_0/2,h_0/2]$, and h_0 is initial shell thickness. The position vector in the deformed shell configuration is assumed as $\mathbf{x}\left(\xi^1,\xi^2,\xi^3\right)=\phi\left(\xi^1,\xi^2\right)+\xi^3\mathbf{t}\left(\xi^1,\xi^2\right)$, where $|\mathbf{t}|=1$ and $\mathbf{t}=\mathbf{RT}$, where $\mathbf{R}$ is constrained rotation. Defining $\mathbf{u}=\phi-\mathbf{\Phi}$, $\mathbf{A}_\alpha=\partial\mathbf{\Phi}/\partial\xi^\alpha$ and $\mathbf{A}_\alpha\cdot\mathbf{A}^\beta=\delta_\alpha^\beta$ the Green-Lagrange strains can be given as

$$\mathbf{E}^u=\left(E^{u,m}_{\alpha\beta}+\xi^3E^{u,b}_{\alpha\beta}\right)\mathbf{A}^\alpha\otimes\mathbf{A}^\beta+E^u_{\alpha 3}\mathbf{A}^\alpha\otimes\mathbf{T}+E^u_{3\alpha}\mathbf{T}\otimes\mathbf{A}^\alpha+0\mathbf{T}\otimes\mathbf{T} \qquad (1)$$

We further make the enhancement of membrane and bending strains: $\mathbf{E}=\mathbf{E}^u+\tilde{\mathbf{E}}$ where $\tilde{\mathbf{E}}=\left(\tilde{E}^m_{\alpha\beta}+\xi^3\tilde{E}^u_{\alpha\beta}\right)\mathbf{A}^\alpha\otimes\mathbf{A}^\beta$, so that total strains are $E^m_{\alpha\beta}=E^{u,m}_{\alpha\beta}+\tilde{E}^m_{\alpha\beta}$ and $E^b_{\alpha\beta}=E^{u,b}_{\alpha\beta}+\tilde{E}^b_{\alpha\beta}$. By introducing the above strain enhancement into the Hu-Washizu functional for shells one can obtain the following mixed functional

$$\begin{aligned}&\Pi\left(\mathbf{u},\mathbf{t},\tilde{E}^m_{\alpha\beta},\tilde{E}^b_{\alpha\beta},n^{\alpha\beta},m^{\alpha\beta}\right)=\int_A W_s\left(\mathbf{E}^u(\mathbf{t},\mathbf{u})+\tilde{\mathbf{E}}\left(\tilde{E}^m_{\alpha\beta},\tilde{E}^b_{\alpha\beta}\right)\right)dA\\&-\int_A n^{\alpha\beta}\left(E^{u,m}_{\alpha\beta}(\mathbf{u})-E^m_{\alpha\beta}\left(\mathbf{u},\tilde{E}^m_{\alpha\beta}\right)\right)dA-\int_A m^{\alpha\beta}\left(E^{u,b}_{\alpha\beta}(\mathbf{u},\mathbf{t})-E^b_{\alpha\beta}\left(\mathbf{u},\mathbf{t},\tilde{E}^b_{\alpha\beta}\right)\right)dA\\&+\Pi_{ext}(\mathbf{u},\mathbf{t})\end{aligned} \qquad (2)$$

where $n^{\alpha\beta}$ and $m^{\alpha\beta}$ are the 2^{nd} Piola-Kirchhoff membrane forces and bending moments, $\mathbf{n}=n^{\alpha\beta}\mathbf{A}_\alpha\otimes\mathbf{A}_\beta$, $\mathbf{m}=m^{\alpha\beta}\mathbf{A}_\alpha\otimes\mathbf{A}_\beta$. The strain energy of a hyperelastic shell is $W_s\left(E^m_{\alpha\beta},2E_{\alpha 3},E^b_{\alpha\beta}\right)=W^m\left(E^m_{\alpha\beta}\right)+W^s\left(2E_{\alpha 3}\right)+W^b\left(E^b_{\alpha\beta}\right)$. The finite element approximation of (2) is carried out by 4-node isoparametric finite elements. The enhanced strains and the stress resultants are approximated such that the following orthogonality conditions hold

$$\int_A n^{\alpha\beta}\tilde{E}^m_{\alpha\beta}dA=0 \qquad \int_A m^{\alpha\beta}\tilde{E}^b_{\alpha\beta}dA=0 \qquad (3)$$

The enhancement of membrane and bending strains in the element isoparametric space $(\xi,\eta)\in[-1,1]\times[-1,1]$ is performed as $\tilde{\Xi}^m=\left(\begin{array}{ccc}\tilde{\Xi}^m_{11} & \tilde{\Xi}^m_{22} & 2\tilde{\Xi}^m_{12}\end{array}\right)^T=$

$\Gamma^m \alpha_e^m$ and $\tilde{\Xi}^b = \begin{pmatrix} \tilde{\Xi}_{11}^b & \tilde{\Xi}_{22}^b & 2\tilde{\Xi}_{12}^b \end{pmatrix}^T = \Gamma^b \alpha_e^b$, where the above matrices are having one of the following forms ($A = |\mathbf{A}_1 \times \mathbf{A}_2|$)

$$\Gamma^m = \Gamma^b = \frac{1}{\sqrt{A}} \begin{bmatrix} \xi & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \xi & \eta \end{bmatrix}, \qquad \begin{array}{l} \alpha_e^{m,T} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ \alpha_e^{b,T} = (\beta_1, \beta_2, \beta_3, \beta_4) \end{array}$$

$$\Gamma^m = \quad \Gamma^b = \frac{1}{\sqrt{A}} \begin{bmatrix} \xi & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & \xi & \eta & \xi\eta \end{bmatrix}, \qquad \begin{array}{l} \alpha_e^{m,T} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \\ \alpha_e^{b,T} = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) \end{array} \tag{4}$$

$$\Gamma^m = \Gamma^b = \frac{1}{\sqrt{A}} \begin{bmatrix} \xi & 0 & 0 & 0 & \xi\eta & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 & \xi\eta & 0 \\ 0 & 0 & \xi & \eta & 0 & 0 & \xi\eta \end{bmatrix}, \quad \begin{array}{l} \alpha_e^{m,T} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) \\ \alpha_e^{b,T} = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7) \end{array}$$

The strains $\tilde{\Xi}^m = \tilde{\Xi}_{\alpha\beta}^m \mathbf{A}_0^\alpha \otimes \mathbf{A}_0^\beta$ and $\tilde{\Xi}^b = \tilde{\Xi}_{\alpha\beta}^b \mathbf{A}_0^\alpha \otimes \mathbf{A}_0^\beta$ are referred to the contravariant basis of the centre of the element $\mathbf{A}_0^\alpha = \mathbf{A}^\alpha (\xi = 0, \eta = 0)$. The strains $\tilde{E}_{e,\alpha\beta}^m$ and $\tilde{E}_{e,\alpha\beta}^b$, which are referred to the basis $\mathbf{A}^\alpha (\xi, \eta)$, are obtained by transformation $\tilde{E}_{e,\alpha\beta}^m = \left(\mathbf{A}_\alpha \cdot \mathbf{A}_0^\gamma\right) \tilde{\Xi}_{\gamma\delta}^m \left(\mathbf{A}_\beta \cdot \mathbf{A}_0^\delta\right)$, $\tilde{E}_{e,\alpha\beta}^b = \left(\mathbf{A}_\alpha \cdot \mathbf{A}_0^\gamma\right) \tilde{\Xi}_{\gamma\delta}^b \left(\mathbf{A}_\beta \cdot \mathbf{A}_0^\delta\right)$. For constant membrane forces and constant bending moments conditions (3) are

$$\int_{-1}^{1}\int_{-1}^{1} \Gamma^m \sqrt{A} d\xi d\eta = \mathbf{0} \qquad \int_{-1}^{1}\int_{-1}^{1} \Gamma^b \sqrt{A} d\xi d\eta = \mathbf{0} \tag{5}$$

The orthogonality conditions (3) thus hold at least for constant membrane forces and bending moments, and the elements with enhanced strains pass the patch test.

Table 1: Derived EAS/ANS shell elements.

Element	No. of EAS parameters for membrane strains	No. of EAS parameters for bending strains	ANS concept
M4	4	0	Yes
M4B4	4	4	Yes
M5	5	0	Yes
M5B5	5	5	Yes
M7	7	0	Yes
M7B7	7	7	Yes
ANS	0	0	Yes

Assesement of optimal number of EAS parameters through numerical examples

The derived EAS/ANS shell elements are summarized in Table 1. The problem data of two examples is presented in Figure 1. Other data for example (a) is *r*=10,

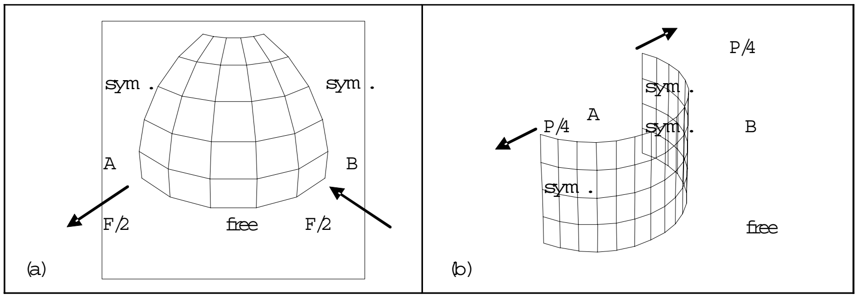
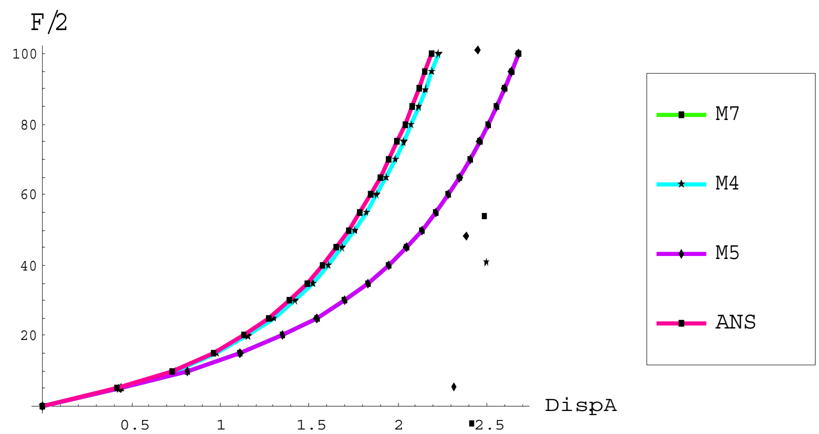


Figure 1: (a) Mesh of one quarter of a pinched half-sphere with an 18^0 hole. (b) Mesh of one quarter of a pinched cylinder with free ends.



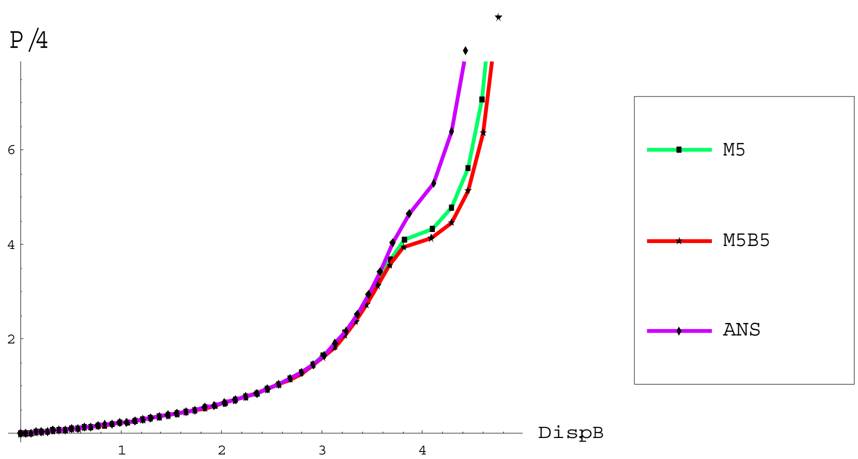


Figure 2: Top: Load-displacement curves for example (a). Bottom: Load-displacement curves for example (b).

h=0.04, *E*=6.825 10^7, $\nu = 0.3$, and for example (b) is *r*=4.953, *h*=0.094, *L*=10.35, *E*=1.05 104, $\nu = 0.3125$. The results are presented in Figure 2. It can be seen for example (a) that M4 gives only slight improvement with respect to ANS and that M5 and M7 give almost identical results. It can be seen for example (b) that membrane and bending enhancement M5B5 gives only slightly different results than membrane enhancement M5. The optimal 4-node element is therefore M5.

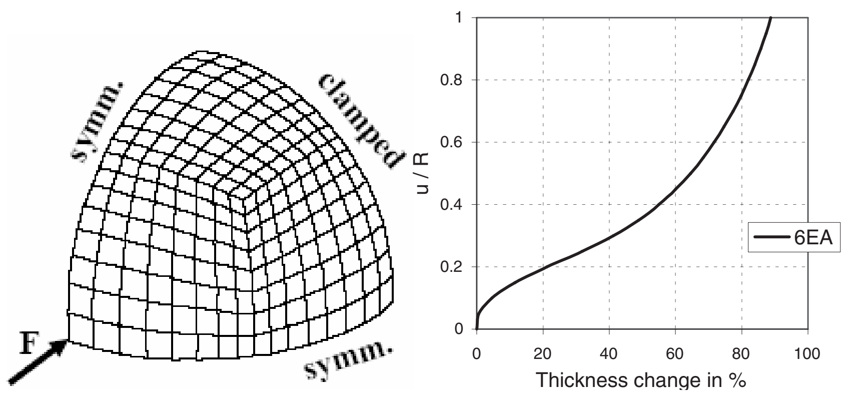


Figure 3: Left: Mesh of one quarter of a pinched clamped half-sphere. Right: Thickness change under the force with respect to normalized displacement under the force.

3d-shell element formulation

The above described optimal shell element M5 can be extended to take into account the through-the-thickness stretching; we refer to Brank et al. [5] for details on 3d-shell formulations. In the present case refinement of kinematics in the through-the-thickness direction is made by using EAS method. The resulting 3d-shell 4-node finite element is capable to reproduce behavior of M5 shell element for thin shells and can provide 3d stress and strain state for thicker shells (as a good approximation of complete 3d solution). Here we present an example computed by M5 element with through-the-thickness stretching. The data and results are given in Figure 3.

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