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Normal deflection of a notched plate under uniform tensile stress

J.W. Choi[1], S.H. Yoo[1], J.B. Kim[1] and E.J. Park[1]

Summary

A semi-analytical solution by beam theory and finite element solutions are obtained for normal deflection of a plate with a surface crack under tensile stress. The semi-analytical solution consists of the deflection of a beam and the half-space or full-space solution. Also finite element solutions are obtained and compared with these analytical solutions. These solutions can be used as reference data for non-destructive evaluation of a surface crack.

Introduction

The free-edge effects have been extensively studied in the linear elastic fracture mechanics. Most of the studies are on the fracture parameter, K, J and COD[1]. The numerical collocation formulation and weight function technique are the primary methods to obtain stress intensity factors of the single edge crack problem [2]. The deformation of crack edge or crack-like notch edge is necessary for crack detection in MEMS, semiconductor, biomedical applications, etc [3,4]. The theoretical normal displacements of a free surface were first examined by Steele and Yoo [5]. The vertical displacement combined with those of cracked layer and remaining half-space solution is the total deflection of the plate.

Governing Equations

A notched plate under remote uniform tension can be modeled as the addition of plates with notch face traction and remote uniform tension with no notch by the principle of superposition. The notch face traction is separated into two parts. One is a beam on rigid foundation having a horizontal stress and the other is half-space having interacting stresses on surface as shown in Figure 1. The plain strain condition is assumed and Timoshenko beam theory is adopted for the beam under horizontal stress as in Figure 1(b). The equilibrium equations are

$$\frac{dQ}{dx} = \sigma_o \tag{1}$$

$$\frac{dM}{dx} = Q - \frac{t}{2}\tau_o \tag{2}$$

$$\frac{dN_x}{dx} = \tau_o \tag{3}$$

where Q is a shear stress resultant and σ_0, τ_0 are interacting normal stress and shear stress at $z = -t/2$ respectively. M is a bending moment resultant. N_x is an axial stress resultant. From the kinematics and constitutive equations, the following

[1]Department of Mechanical Engineering, Ajou University, Wonchon , San 5, Suwon, Korea.

relations are obtained.

$$N_x = \frac{Et}{(1-\nu^2)} \frac{du_0}{dx} \tag{4}$$

$$M = \frac{Et^3}{12(1-\nu^2)} \frac{d\chi}{dx} \tag{5}$$

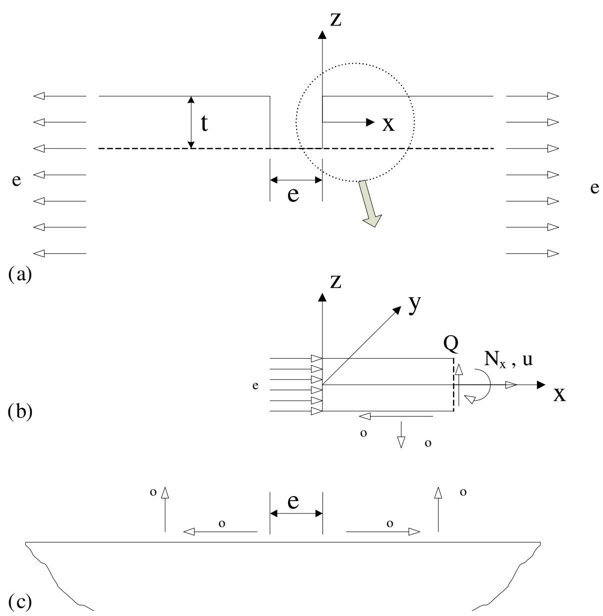

Figure 1: (a) Notched plate subject to uniform tensile stress. (b) Beam model with shear deformation and stretching of normal for notched layer. (c) Interface stresses acting on the remaining half-space.

χ is rotation of the plate and u_0 is displacements of the mid-surface. By considering constitutive relations, we obtain $\sigma_x = N_x/t + z \cdot 12M/t^3$. The two dimensional equation of equilibrium for the case in which there are no internal body forces is considered and substituting into Eq.1, we obtain $\sigma_z = \sigma_o(1/2 - z/t)$. After integrating the strain in z direction for plain strain, we obtain the equation of σ_0 in terms of the displacement of the centerlinew_0 as

$$\sigma_0 = \frac{8E}{3t(1-\nu^2)}[w_0 + \frac{\nu(1+\nu)}{2E}(N_x - \frac{3M}{t})] \tag{6}$$

Substituting Eqs.1, 4 and 5 into Eq.6 and considering boundary condition at $z = -t/2$, following non-dimensional equation for rotation is obtained.

$$\frac{d^3\chi}{d\xi^3} - \frac{\nu}{(1-\nu)} \frac{d\chi}{d\xi} = \frac{8w_0}{t} \tag{7}$$

The non-dimensional parameter ξ is $\xi = x/t$. By considering shear deformation and boundary condition at $z = -t/2$, following non-dimensional equation is obtained.

$$\frac{\mu}{3(1-\nu^2)}\frac{d^2\chi}{d\xi^2} = \frac{d}{d\xi}\frac{w_0}{t} + \chi \tag{8}$$

where μ is effective transverse shear modulus. If Poisson's ratio $\nu = 0.3$ and $\mu = 2.6$, the solution of the rotational equation at the centerline is

$$\chi = \frac{2\sigma_e(1-\nu^2)}{E}\frac{\beta_1\beta_2}{(\beta_2-\beta_1)}[\frac{1}{\beta_1^2}e^{-\beta_1\xi} - \frac{1}{\beta_2^2}e^{-\beta_2\xi}] \tag{9}$$

where $\beta_1 = 1.077,\ \beta_2 = 2.624$. By integrating the strain in z direction for plain, the normal displacement, w_b at the free surface is

$$\frac{Ew_b}{\sigma_e t} = \frac{(1-\nu^2)\beta_1\beta_2}{\beta_2-\beta_1}[(-\frac{\beta_1}{6} + \frac{\nu}{2(1-\nu)}\frac{1}{\beta_1})e^{-\beta_1\xi} + (\frac{\beta_2}{6} - \frac{\nu}{2(1-\nu)}\frac{1}{\beta_2})e^{-\beta_2\xi}] \tag{10}$$

From Eqs.1, 3, and 9, the interfacing stresses are

$$\sigma_o = \frac{2\sigma_e\beta_1\beta_2}{3\beta_2-\beta_1}[-\beta_1 e^{-\beta_1\xi} + \beta_2 e^{-\beta_2\xi}] \tag{11}$$

$$\tau_o = \frac{\sigma_e\beta_1\beta_2}{\beta_2-\beta_1}[e^{-\beta_1\xi} - e^{-\beta_2\xi}] \tag{12}$$

Next, we consider the half-space to the remaining interfacing part of the plate. We use bi-harmonic equation of a stress function $\varphi(\xi,\zeta)$ as

$$\varphi(\xi,\zeta) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{\lambda\zeta}(A + B\lambda\zeta)\cos\lambda\xi\, d\lambda \tag{13}$$

where $\xi = x/t$, $\zeta = z/t$ and A,B are constants. By considering σ_0, τ_0 respectively, we can determine constants A, B by taking Fourier transform. Normal displacement w_n loaded only for the σ_0 is

$$\frac{Ew_n(\xi,0)}{\sigma_e t} = -\frac{2\beta_1\beta_2}{3\beta_2-\beta_1}\frac{4}{\pi}(1-\nu^2)\int_0^\infty \left[\begin{array}{c} \frac{\beta_1(\beta_1\cos\lambda\delta - \lambda\sin\lambda\delta)}{\beta_1^2+\lambda^2} \\ -\frac{\beta_2(\beta_2\cos\lambda\delta-\lambda\sin\lambda\delta)}{\beta_2^2+\lambda^2}\end{array}\right]\frac{\cos\lambda\xi}{\lambda}d\lambda \tag{14}$$

where $\delta = \frac{e}{2t}$. Normal displacement w_s loaded only for the τ_0 is

$$\frac{Ew_s(\xi,0)}{\sigma_e t} = \frac{\beta_1\beta_2}{\beta_2-\beta_1}(1+\nu)(1-2\nu)\frac{2}{\pi}\int_0^\infty \left[\begin{array}{c} \frac{\beta_1\sin\lambda\delta+\lambda\cos\lambda\delta}{\beta_1^2+\lambda^2} \\ -\frac{\beta_2\sin\lambda\delta+\lambda\cos\lambda\delta}{\beta_2^2+\lambda^2}\end{array}\right]\frac{\cos\lambda\xi}{\lambda}d\lambda \tag{15}$$

From Eqs.10, 14 and 15, the total deflection of the plate is the sum of the displacements from beam solution w_b and half-space solutions w_n ,w_s. When we consider full-space,w_s due to shear stress τ_0 will be zero by symmetry. w_n from full-space is reduced to 0.398 of the half-space solution. Fig. 2 shows the results of the cases considered. We consider the half-space solution as the upper limit and the full-space solution as the lower limit for the real normal deflection as shown in Table 1.

Table 1: Upper and lower limit of the deflection when notch width$\delta = 0$.

	$\frac{Ew_b}{t\sigma_e}$	$\frac{Ew_n}{t\sigma_e}$	$\frac{Ew_s}{t\sigma_e}$	$\frac{Ew_T}{t\sigma_e}$
Half-space	0.623	1.256	0.520	2.399
Full-space	0.623	0.499	0	1.122

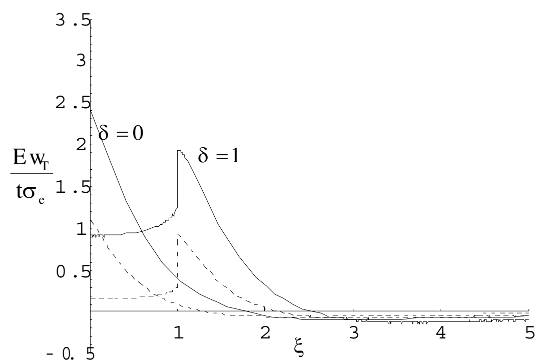

Figure 2: Plain strain normal deflection under uniform tensile stress according to crack width $\delta = 0, 1$

Numerical Model

Figure 3 shows the plate model. The 4-node plane stain CPE4 of the commercial FEM program ABAQUS are used[6]. The model consists of 13,200 elements and 13,431 nodes. A half model created by taking advantage of symmetry. The width and depth of the notch are increased $\delta = \frac{e}{2t} = 0 \sim 1$ and $\frac{t}{H} = 0.1 \sim 0.5$.

Results and Discussion

The Figure 4 shows the result of the normal displacements of the notch edge of the plate by the semi-analytical model and FE analysis. As the notch width is increased, the normal deflection of the plate is decreased. As the depth is increased, the normal deflection is increased and the edge effect is shown in the Figure 4(b). About at two times distance from notch edge, the effect is diminished of which the normal displacement of the plate is zero. The upper and lower limits are well compared with the finite element results and can provide good reference for non-destructive examination.

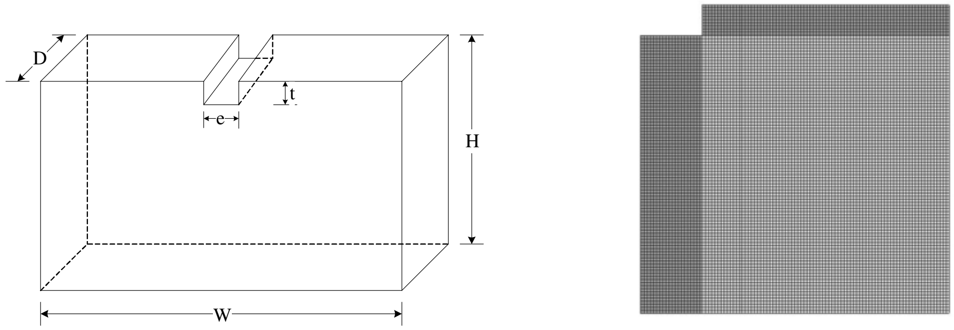

Figure 3: (a) Geometry of the notched plate. (b) FEM Model

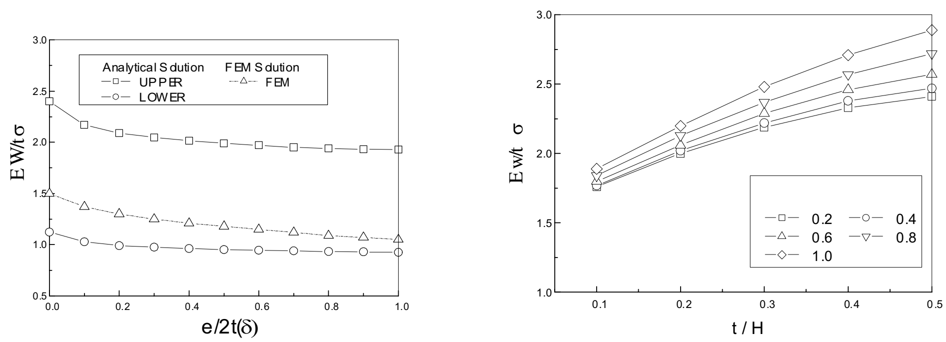

Figure 4: (a) Normal displacements according to the notch width. (b) Normal displacements according to the notch depth

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