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REVIEW

# Accounting for Quadratic and Cubic Invariants in Continuum Mechanics–An Overview

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## ABSTRACT

The differential equations of continuum mechanics are the basis of an uncountable variety of phenomena and technological processes in fluid-dynamics and related fields. These equations contain derivatives of the first order with respect to time. The derivation of the equations of continuum mechanics uses the limit transitions of the tendency of the volume increment and the time increment to zero. Derivatives are used to derive the wave equation. The differential wave equation is second order in time. Therefore, increments of volume and increments of time in continuum mechanics should be considered as small but finite quantities for problems of wave formation. This is important for calculating the generation of sound waves and water hammer waves. Therefore, the Euler continuity equation with finite time increments is of interest. The finiteness of the time increment makes it possible to take into account the quadratic and cubic invariants of the strain rate tensor. This is a new branch in hydrodynamics. Quadratic and cubic invariants will be used in differential wave equations of the second and third order in time.

## KEYWORDS

Quadratic invariant; cubic invariant; continuity equation; generation of periodic waves; N.E. Zhukovsky's hydraulic shock; turbulence

## Nomenclature

| | |
|---|---|
| $I_1$ | Linear invariant |
| $I_2$ | Quadratic invariant |
| $I_3$ | Cubic invariant |
| $J = \dfrac{\partial(u_1, u_2, u_3, \ldots, u_n)}{\partial(x_1, x_2, x_3, \ldots, x_n)}$ | the "n" order Jacobian of the velocity field |
| $u_1$ | u is the fluid velocity component along the $x$-axis |
| $u_2$ | $v$ is the fluid velocity component along the axis $y$ |
| $u_3$ | w is the fluid velocity component along the axis $y$ |
| P | is the fluid pressure |

**e.g.**

| | |
|---|---|
| 0 | Index refers to the start time of the deformation |

## 1 Introduction

The article is devoted to the continuity equation in fluid and gas mechanics. Classical fluid mechanics is based on the continuity hypothesis. This is expressed in the tendency of infinitely small elementary volumes dV and time intervals dt to zero. The idea of using the conservation law in a differential form belongs to d'Alembert. He used it in his work on the cause of the wind.

Euler [1–4] derived a differential continuity equation, which has a general form and is not related to an applied problem. There are materials on the derivation by some researchers of additional terms in the continuity equation. In this regard, it is of interest to consider the effect of the finiteness of processes in time on the Euler equation of conservation.

The terms of the high order of smallness of the continuity equation describe the occurrence of self-oscillations, vibrations, sound and water hammer. The terms of the high order of smallness of the continuity equation describe the possible occurrence of the initial stage of turbulent fluctuations. This can be used to study turbulent stochastic and statistical processes when averaged over time (Taylor) or mass (Favre).

Here, in the approximation of the finiteness of an infinitesimal time interval dt, a number of applied problems are considered. As a result, all three invariants of the strain rate tensor are taken into account. Quadratic and cubic invariants will be used in a second-order differential wave equation in time and in a third-order differential wave equation in time.

The deformation theory of continuum mechanics [5,6] gives a formula for the cubic expansion coefficient, which contains three invariants. These are the linear invariant, the quadratic invariant and the cubic invariant. The formula for the cubic expansion coefficient provides the basis for deriving the equation for the conservation of the amount of matter. However, the solution of hydrodynamic problems contains only a linear invariant. In this, one can see the incompleteness of the solution to the problem. This is an asymmetry between the formulation of the problem and its solution. We can expect the existence of new solutions that reflect the geometric properties of the quadratic and cubic invariants. Restoration of symmetry can give new physical properties of the flow and new regimes.

V.M. Bubnov in 1998 pointed out the presence of terms of a high order of smallness in the continuity equation for an incompressible fluid, which was derived by N.E. Zhukovsky. In 2006, Ovsyannikov [7] found terms of a high order of smallness in the derivation of the continuity equation in Euler's work "Principien motus fluidorum" 1752. Euler calculated the terms of a high order of smallness in terms of the deformation time of the control figure $t - t_0$, and then destroyed them by passing to the limit $t - t_0 \to 0$. We know that the intermediate results have a physical meaning. Therefore, we will use Euler's intermediate results in the derivation of the wave equation. The wave equation can only be derived for a compressible medium.

Therefore, in 2006, the continuity equation with terms of a high order of smallness was written for a compressible fluid [7]. The physical meaning of the terms of a high order of smallness in the continuity equation for a compressible fluid was understood. These terms generate density waves and pressure waves against the background of a stationary flow of a compressible fluid. They generate self-oscillations, vibrations and a solitary pressure wave.

Lighthill [8,9] proposed a new method for deriving the wave equation. This is the method of acoustic analogy. It uses the time derivative of the continuity equation for a compressible medium. Such a derivative can be easily obtained from the continuity equation, which was derived by Euler in 1752. The continuity equation with terms of a high order of smallness can be used in Lighthill's acoustic analogy method.

The inhomogeneous wave equation was derived by Ovsyannikov [10] in 2007. The inhomogeneous part of the wave equation contains quadratic and cubic invariants of the strain rate tensor. These invariants describe the generation of density and pressure waves during the flow of a compressible fluid.

Purpose of the review shows that flow regions with different Lagrangian laws of motion of a liquid particle have different additional terms of a high order of smallness in time. It was realized that quadratic invariant terms generate density and pressure waves that are close to harmonic oscillations [11,12]. Reviews [11,12] describe solutions to problems of the formation of sound waves with potential air flowing around a right angle, cylinder. Monograph [10] contains a description of the formation of waves when air flows into narrow gaps. These papers investigate the formation of waves by incorporating a quadratic invariant into the continuity equation. The calculation results are consistent with the experiments.

The main contribution of this work is in the study of the physical meaning of the cubic invariant. There are few works that take into account the cubic invariant [13,14].

Most of the works take into account only the quadratic invariant, which describes harmonic waves [10,11,12,15]. The quadratic invariant of the strain rate tensor is used in the method of regularization of the system of hydrodynamic equations [15] to increase the stability of iterations of T.G. Elizarova's numerical method.

A.V. Dmitrenko noted that these harmonic waves can be used in stochastic methods for calculating turbulent flows.

Thus, calculations taking into account quadratic and cubic invariants may give an extension of the area of application of hydrodynamic equations. The final answer can be given by comparing the calculations with the experimental results.

## 2 Materials and Methods

Section 2 presents simple geometric constructions that show the reason for the appearance of a high-order term of smallness in the continuity equation for a plane two-dimensional flow with a linear law of deformation of the control figure in time. Here is the reason why mathematicians are divided into two camps. Newton's followers considered the differential to be an infinitesimal quantity. They had to replace it with zero. Leibniz's followers considered the differential to be a small but finite value. They were required to perform arithmetic operations with differentials according to the rules for quantities of finite size. This is discussed in Carnot's book.

There are two main derivations of the differential continuity equation for an incompressible fluid. This is Euler's derivation and Ostrogradsky's derivation. Both conclusions are based on geometric constructions. If Euler's geometrical derivation is used in exact form, then it gives terms of a high order of smallness. This article studies the physical meaning of terms of a high order of smallness.

The reason for the appearance of terms of a high order of smallness in the equation of continuity lies in the use by Euler, Zhukovsky, Ostrogradsky of the Lagrangian law of motion of a liquid particle linear in time. We will demonstrate this by deforming the control square. The side of this square is equal to one. We will take into account only compression and tension deformations.

The side of the square, which is parallel to the $x$ axis, changes its length according to such a linear law with time $(t - t_0)$

$$1 + \frac{\partial u}{\partial x}(t - t_0) \tag{1a}$$

Here $u$ is the fluid velocity component along the $x$-axis, $t$ is the time, $t_0$ is the start time of the deformation. The side of the square, which is parallel to the $y$ axis. Changes its length according to such a linear law with time $(t - t_0)$

$$1 + \frac{\partial v}{\partial y}(t - t_0) \tag{1b}$$

Here $v$ is the fluid velocity component along the axis $y$. Change in the area of the control figure with time

$$S = \left[ 1 + \frac{\partial u}{\partial x}(t - t_0) \right] \left[ 1 + \frac{\partial v}{\partial y}(t - t_0) \right] \tag{1c}$$

Let us equate the area of the deformed control figure and the area of the initial square

$$\left[ 1 + \frac{\partial u}{\partial x}(t - t_0) \right] \left[ 1 + \frac{\partial v}{\partial y}(t - t_0) \right] = 1 \tag{1d}$$

A simplification of this formula gives the continuity equation with a term of high order of smallness in time

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + (t - t_0) \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} = 0 \tag{1e}$$

Euler made similar calculations for a three-dimensional flow with allowance for the shear deformation in 1752.

The terms of the high order of smallness reflect the simultaneity of deformations that occur in perpendicular directions. The terms of the high order of smallness reflect the volume that is obtained due to the deformation of the deformations. Hydrodynamicists must calculate the basic problems of wave formation, the results of which can be verified experimentally.

When using the exponential Lagrangian law of motion of a liquid particle, terms of a high order of smallness do not arise. In this case, the continuity equation contains three terms of the velocity vector divergence. This conclusion is contained in Section 5.

Thus, we propose to use the completely geometric characteristics of the flow, which are contained in the quadratic and cubic invariants.

Similar terms of a high order of smallness can be seen in the geometric constructions of Ostrogradsky before making the transition from geometric objects of finite size to infinitely small objects. It is also necessary to point out the historical reasons for the elimination by Euler of the terms of the high order of smallness of the continuity equation. Modern integral and differential calculus appeared as a result of the merging of Leibniz's theory of infinitesimals and Newton's theory of vanishingly small quantities. They were created independently at the same time. There are slight differences in these theories.

The analysis of the differences was made by Lazar Carnot. Leibniz considers the differential to be a small quantity with which mathematical operations can be carried out according to the rules of arithmetic of finite quantities.

Newton in his theory considers the analogue of the differential as a vanishingly small value, which is equal to zero. A mathematician must consider the result of multiplying a number by such a differential equal to zero.

L. Carnot divided the mathematicians and mechanics of his era into two lists. Euler is written on the list of mathematicians who supported Newton's mathematical theory. This position of Euler can explain the reason for his exclusion of terms of the high order of smallness of the continuity equation by passing to the limit $(t - t_0) \to 0$.

In engineering calculations, it is necessary to choose the most dangerous situation. For example, a seismologist must predict the possibility of an earthquake using the continuity equation, which contains terms of a high order of smallness.

## 3 Results

### *3.1 Euler's Continuity Equation with Terms of High Order of Smallness*

Euler's 1752 derivation of the continuity equation for the three-dimensional flow of an incompressible fluid, taking into account tensile and shear deformations, gave the following result:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + (t - t_0)\left\{ \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} + \begin{vmatrix} \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial y & \partial w/\partial z \end{vmatrix} + \begin{vmatrix} \partial w/\partial z & \partial w/\partial x \\ \partial u/\partial z & \partial u/\partial x \end{vmatrix} \right\} + (t - t_0)^2 \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix} = 0$$

Let us analyze a similar continuity equation for a compressible fluid, published in 2006. Methods for deriving the continuity equation in hydrodynamics and elasticity theory are similar in geometric constructions.

The theory of deformations studies the general approach to the construction of equations for the theory of elasticity and for hydrodynamics. The textbook by Sedov [5] considers these disciplines at the same time. In this textbook, the theory of deformations for a solid elastic body and for an incompressible fluid is given from the standpoint of the general properties of the strain tensor and the strain rate tensor.

These properties are the same since the components of the strain tensor and the strain rate tensor are related to each other. When deriving the expression for the coefficient of cubic expansion $\theta$ in the theory of elasticity in paragraph 5 of chapter 2 of the first volume of the textbook [5], the formula (5.37) was obtained

$$\theta = (1 + 2I_1 + 4I_2 + 8I_3)^{0,5} - 1 \tag{1f}$$

In this formula, $I_1,\ I_2, I_3$ are the linear, quadratic and cubic invariants of the strain tensor $\| \varepsilon_{ij} \|$. The coefficient of cubic expansion for finite deformations of the control figure is equal to the relative change in its volume. After deriving Eq. (1f), Sedov [5] considers only infinitesimal deformations. For infinitesimal deformations, the quadratic and cubic invariants are eliminated from the formula for the cubic expansion coefficient by using the passage to the limit when the deformation time tends to zero. Sedov obtains an approximate formula for the coefficient of cubic expansion in the theory of elasticity $\theta \approx I_1$.

Let us pay attention to the inexact equal sign in this formula. In the sections that study Hydrodynamics, Sedov considers the strain rate tensor $\| e_{ij} \|$. Sedov does not take into account the quadratic and cubic invariants of the strain rate tensor for ease of calculation with low accuracy. The components of the strain rate tensor invariants are multiplied by the deformation time increment in various powers. A high invariant number corresponds to a high degree of time increment.

To solve problems of non-stationary hydrodynamics, taking into account quadratic and cubic invariants, it is necessary to use differential equations of the second and third order in time. Before excluding terms with higher invariants, it is necessary to understand their physical meaning. This became possible, when Lighthill [8,9] proposed the method of acoustic analogy for deriving the wave equation from the system of equations of motion and continuity.

In the same years, Truesdell [2] translated from Latin into English the first version of Euler's classic work Principia motus fluidorum [1]. There is now a translation of the first version of Euler's work Principia motus fluidorum 1752 [1] into various languages [3]. A detailed translation of Euler's work into English [4] was made in 2008.

C. Truesdell drew attention to Euler's intermediate result in the form of a complete continuity equation, which contains terms of a high order of smallness in time

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+(t-t_0)\left[\frac{\partial(u,v)}{\partial(x,y)}+\frac{\partial(v,w)}{\partial(y,z)}+\frac{\partial(w,u)}{\partial(z,x)}\right]+(t-t_0)^2\partial(u,v,w)/\partial(x,y,z)=0 \tag{2}$$

Here $u$, $v$, $w$ are the velocity components along the $x$, $y$, z axes; $t-t_0$ is the time interval of deformation of the control figure; $\frac{\partial(u,v)}{\partial(x,y)}$ and $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ are the second and third-order Jacobians of the velocity field, respectively. C. Truesdell combined the terms of this equation into the Jacobians of the second $\frac{\partial(u,v)}{\partial(x,y)}$ and the third $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ orders.

Lectures on hydroaeromechanics by Wallander [6] contain such formulas for the invariants of the strain rate tensor $\| e_{ij} \|$.

Linear invariant is $I_1=e_{11}+e_{22}+e_{33}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$. Quadratic invariant is

$$I_2=\begin{vmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{vmatrix}+\begin{vmatrix} e_{22} & e_{23} \\ e_{32} & e_{33} \end{vmatrix}+\begin{vmatrix} e_{33} & e_{21} \\ e_{13} & e_{11} \end{vmatrix}=\frac{\partial(u,v)}{\partial(x,y)}+\frac{\partial(v,w)}{\partial(y,z)}+\frac{\partial(w,u)}{\partial(z,x)}\ . \tag{3}$$

Cubic invariant

$$I_3=\det \| e_{ij} \| = \partial(u,v,w)/\partial(x,y,z) \tag{4}$$

If we use the expressions for the invariants of the strain rate tensor (3), (4), then the Euler continuity equation for an incompressible fluid (2) takes the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+(t-t_0)\ I_2\ +(t-t_0)^2\ \ I_3\ \ =0. \tag{5}$$

After deriving the continuity Eq. (2) or (5), Euler made the following passage to the limit $\mathrm{t}-\mathrm{t}_0\to 0$. Eq. (5) eliminates the quadratic and cubic invariants. Eq. (5) takes such an approximate form for an incompressible fluid $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\ \approx 0$. This approximate equation of hydrodynamics is consistent with the approximate equation of the theory of elasticity, which Sedov wrote in the first volume of the textbook [5] $\theta\approx I_1$.

The change in density can be taken into account by replacing the velocity components $u,\ v,\ w$ in the continuity equation with the products of the velocity components and the density $\rho u,\ \rho v, \rho w$. The continuity equation will look like this

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}+\ (t-t_0)\ \rho^{-1}\left[\frac{\partial(\rho u,\ \rho v)}{\partial(x,y)}+\frac{\partial(\rho v,\rho w)}{\partial(y,z)}+\frac{\partial(\rho w,\rho u)}{\partial(z,x)}\right] \\ +(t-t_0)^2\ \rho^{-2}\partial(\rho u,\rho v,\rho w)/\partial(x,y,z)=0 \tag{6}$$

Factors $\rho^{-1}$ and $\rho^{-2}$ are introduced in front of the Jacobians to preserve the dimension of various terms. We can simplify the Eq. (6) to this form [7]

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}+(t-t_0)\rho\left[\frac{\partial(u,v)}{\partial(x,y)}+\frac{\partial(v,w)}{\partial(y,z)}+\frac{\partial(w,u)}{\partial(z,x)}\right]+(t-t_0)^2\rho\partial(u,v,w)/\partial(x,y,z)=0$$

or

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}+(t-t_0)\rho I_2+(t-t_0)^2\rho\, I_3=0 \tag{7}$$

Here ρ is the density. The deformation time $(t-t_0)$ of the control figure does not exceed the time differential in the derivative $\frac{\partial \rho}{\partial t}$. In a compressible medium, density waves and pressure waves can occur. Euler obtained such a formula for the equation of continuity of a compressible fluid

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \tag{8}$$

The continuity Eqs. (6) and (7) for a compressible fluid are more accurate.

Must have a finite time increment value $(t-t_0)>t_*$ to take into account the quadratic $I_2$ and cubic $I_3$ invariants in Eq. (7). But the deformation time of the control figure $(t-t_0)$ should not exceed the time differential $t_{**}$ in the derivative $\frac{\partial \rho}{\partial t}$. Therefore, the wave processes under consideration must arise and develop within a time limited by the values of $t_*$ and $t_{**}$

$$t_*<(t-t_0)<t_{**}$$

Large values of additional terms in the continuity Eq. (7) can arise with large values of the invariants $I_2$ and $I_3$, which arise when there is strong heterogeneity in the $x$, $y$, $z$ coordinates of the stationary velocity field.

### *3.2 Derivation of the Wave Equation by Lighthill's Acoustic Analogy Method*

Lighthill's method consists in creating a d'Alembert operator on the left side of the wave equation. To preserve the square and cubic invariants in the system of fluid dynamics equations, it is necessary to take the time derivative of the right and left sides of the continuity Eq. (7).

When taking the time derivative, the time increment $t-t_0$ in front of the quadratic invariant $I_2$ disappears and the order of smallness of the time increment $t-t_0$ in front of the cubic invariant $I_3$ changes from second to first.

$$\frac{\partial^2 \rho}{\partial t^2}+\rho I_2+2(t-t_0)\rho\, I_3=0 \tag{9}$$

The time derivative is taken from the continuity equation for an unsteady flow of a compressible fluid. Taking the derivative creates a second-order derivative of the density with respect to time $\frac{\partial^2 \rho}{\partial t^2}$. It will be converted to the second derivative of pressure with respect to time using the speed of sound. This is the first step in Lighthill's acoustic analogy method for deriving the wave equation. The derivative with respect to coordinates is taken from the equation of motion at the second step of deriving the wave equation. The equation of motion for an inviscid fluid has the form $\rho\frac{d\boldsymbol{V}}{dt}=-\,grad\,p$. Here $\frac{d\boldsymbol{V}}{dt}$ is the total derivative of the velocity vector $\boldsymbol{V}$, $p$ is the pressure. As a result, the second-order derivatives of the pressure along the $x$, $y$, $z$ coordinates are obtained.

Lighthill's acoustic analogy method yields an inhomogeneous second-order differential wave equation in time

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}-\left(\frac{1}{c_0^2}\right)\frac{\partial^2 p}{\partial t^2}=\rho_0\left[\frac{\partial(u,v)}{\partial(x,y)}+\frac{\partial(v,w)}{\partial(y,z)}+\frac{\partial(w,u)}{\partial(z,x)}\right]+(t-t_0)\rho_0 2\frac{\partial(u,v,w)}{\partial(x,y,z)}$$

or

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}-\left(\frac{1}{c_0^2}\right)\frac{\partial^2 p}{\partial t^2}=\rho_0\, I_2+(t-t_0)\rho_0 2 I_3 \tag{10}$$

Here $c_0$ is the speed of sound, $\rho_0$ is thermodynamic density.

### *3.3 Physical Meaning of the Quadratic Invariant*

We will use the method of successive approximations to calculate the intensity of the generation of density waves and pressure waves for a stationary velocity field. We will use in the inhomogeneous part of the wave equation the invariants $I_2$ and $I_3$, which are calculated for a stationary velocity field. Solution of the wave equation without a cubic invariant $I_3$

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}-\left(\frac{1}{c_0^2}\right)\frac{\partial^2 p}{\partial t^2}=\rho_0\, I_2 \tag{11}$$

described in articles [11,12]. The right side of the wave equation with the d'Alembert operator describes the process of transmission through the elementary volume of the waves existing in the fluid. The quadratic invariant $I_2$ describes the emergence of new harmonic density and pressure waves. The wave Eq. (11) has a solution of the lagging potential type for the sound pressure $p(\boldsymbol{r},\boldsymbol{t})=\frac{\rho_0}{4\pi}\int I_2\,|_{t-R/c}R^{-1}dW$. Here $\boldsymbol{R}=|\boldsymbol{r}-\boldsymbol{r_1}|$, $\boldsymbol{r}$ is the radius vector of the observation point, $\boldsymbol{r_1}$ is the radius vector of the point in the region of integration.

The intensity $I$ of generation of periodic oscillations in a two-dimensional flow can be estimated from the following formula $I=\frac{p^2}{c_0\rho_0}=\frac{\rho_0 I_2^2\, W^2}{16\pi^2 c_0 \boldsymbol{r^2}}$ if the difference between the quadratic invariant $I_2$ from zero is observed only in the volume $W$. In this case, the intensity of the emerging waves is proportional to the square of the quadratic invariant $I_2$. Wave Eq. (11) and its solution do not depend on the deformation time of the control figure $t-t_0$.

The situation will be different when the cubic invariant $I_3$ is taken into account.

## 4 Physical Meaning of the Cubic Invariant

In recent years, reference [13] have been carried out to study the physical meaning of a term with a cubic invariant $I_3$. It is obtained that it describes the occurrence of hydraulic shock of N.E. Zhukovsky by a differential equation of the third order in time.

Let us assume that the solution to the wave Eq. (10) is the sum of the solution to the inhomogeneous Eq. (11) and the solution to Eq. (12)

$$-\left(\frac{1}{c_0^2}\right)\frac{\partial^2 p}{\partial t^2}=(t-t_0)\rho_0 2\frac{\partial(u,v,w)}{\partial(x,y,z)}$$

or

$$-\left(\frac{1}{c_0^2}\right)\frac{\partial^2 p}{\partial t^2}=(t-t_0)\rho_0 2\, I_3$$

or

$$-\left(\frac{1}{c_0^2}\right)\frac{\partial^3 p}{\partial t^3}=\rho_0 2\, I_3 \tag{12}$$

The solution of Eq. (11) has a complex form and is obtained using retarded potential method. But this solution does not tend to strongly increase in time. The solution of Eq. (12) will contain the deformation time of the control figure $t-t_0$ to a high degree. Therefore, let us estimate the pressure increase regime by solving a simple Eq. (12).

The wave Eq. (12) describes the possibility of increasing the pressure as a cubic function of time

$$p=-\frac{\rho_0}{3}\,{c_0}^2\, I_3\,(t-t_0)^3 \tag{13}$$

The solution of the differential Eq. (12) depends on the value of the deformation time of the control figure $t-t_0$. The deformation time is chosen differently for different tasks. Zhukovsky's classical algebraic formula $\Delta p=\rho_0 u c_0$ gives an increase in hydraulic shock pressure during one-dimensional fluid motion along the pipeline. Here $u$ is the fluid flow velocity along the pipeline before the valve closes. The new differential Eq. (12) describes the occurrence of hydraulic shock in a three-dimensional flow. The cubic invariant $I_3$ controls the rate of pressure rise.

It is necessary to solve the system of four differential Eqs. (8), (9), (11), (12) in order to use the three invariants of the strain rate tensor. In this case, a complete solution to the problem is obtained. We use the method of successive approximations. We will find the velocity and pressure field from the system of the classical continuity Eq. (8) $\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$ and the equation of motion $\rho\frac{d\boldsymbol{V}}{dt}=-\,grad\,p$. Then we find the wave pressure from the differential wave Eq. (11) of the second order in time

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}-\left(\frac{1}{c_0^2}\right)\frac{\partial^2 p}{\partial t^2}=\rho_0\, I_2 \tag{14}$$

and the differential Eq. (12) of the third order in time. The solution of Eq. (12) has the form of a power function of time.

$$p=-\,\frac{\rho_0}{3}\,c_0^2\, I_3\,(t-t_0)^3 \tag{15}$$

This function increases as the third power of time. This deformation time $t-t_0$ is determined by the content of the problem. The power function describes Zhukovsky's hydraulic shock. The approximate similarity of the solution to the wave equation of the third order in time with the Zhukovsky water hammer wave is shown by d'Alembert's method of selecting a solution. A more accurate solution of the wave equation can be made by developing the retarded potential method and practicing solving numerical methods of finite-difference equations corresponding to the complete Euler continuity equation.

Mathematicians must give the equation of continuity with terms of high order of smallness to seismologists and acousticians. For rigorous quantitative calculations of the intensity of generated waves, engineers need to obtain from mathematicians various solutions to the wave equation using the retarded potential method. We pose this problem to mathematicians in this review.

## 5 A Flow for Which the Continuity Equation Does Not Contain Terms of a High Order of Smallness

Let us show that the continuity equation can have different forms in flow regions with different Lagrangian laws of motion of a liquid particle and with different Euler velocity fields. Demonstration of the differences in the continuity equation for different places in the velocity field is the main goal of this review.

There are two ways to describe the motion of a fluid [13]. This is Lagrange's law of motion of a liquid particle. This is the Euler velocity field. We know the Lagrange variables

$$x = f_1(a,\ b,\ c,\ t)$$

$$y = f_2(a,\ b,\ c,\ t)$$

$$Z = f_3(a,\ b,\ c,\ t)$$

We know the Euler variables

$$u = F_1(x,\ y,\ z,\ t)$$

$$v = F_2(x,\ y,\ z,\ t)$$

$$w = F_3(x,\ y,\ z,\ t)$$

Euler used the linear-in-time Lagrange law of motion of a fluid particle in deriving the 1752 continuity equation. These are the Cauchy-Helmholtz formulas with a linear dependence of the coordinates $x$, $y$ of points during deformation on time $t$. For a plane two-dimensional flow, they have the form

$$\begin{aligned} x &= (1 + \alpha t)x_b + bty_b \\ y &= \gamma t x_b + (1 + \kappa t)y_b \end{aligned} \tag{16}$$

Here $x_b,\ y_b$ are the initial values of the coordinates of the point at $t = 0$.

$$\alpha = \frac{\partial u}{\partial x}\ ,\ \beta = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{y}}\ ,\ \gamma = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\ ,\ \kappa = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{y}}$$

Let us calculate the velocity components along the coordinate axes

$$\begin{aligned} u &= \frac{dx}{dt} = \alpha x_b + \beta y_b \\ v &= \frac{dy}{dt} = \gamma x_b + \kappa y_b \end{aligned} \tag{17}$$

The linearity in time of the Lagrange law of motion of a liquid particle [1–3] gives terms of a high order of smallness in the continuity Eq. (2). The exponential law of motion of a liquid particle gives the exact fulfillment of the continuity equation without additional terms.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

The exponential law of motion of a liquid particle can be obtained by replacing the initial values $x_b$ and $y_b$ of the coordinates by $x$ and $y$ on the right side of the system of Eq. (17). The system of equations will have the following form:

$$\begin{aligned} \frac{dx}{dt} &= \alpha x + \beta y \\ \frac{dy}{dt} &= \gamma x + \kappa y \end{aligned} \tag{18}$$

Initial conditions

$$x = x_b, \quad y = y_b \ \text{at} \ t = 0 \tag{19}$$

The solution of the system of Eq. (18) with initial conditions (19) is contained in the book [10] and gives the exponential Lagrange law of motion of a liquid particle with time.

$$\begin{aligned} x &= \left(\frac{s_2 - \alpha}{s_2 - s_1} e^{ts_1} - \frac{s_1 - \alpha}{s_2 - s_1} e^{ts_2}\right) x_b + \frac{\beta}{s_2 - s_1}(e^{ts_2} - e^{ts_1})y_b \\ y &= \frac{(s_1 - \alpha)(s_2 - \alpha)}{\beta(s_2 - s_1)}(e^{ts_1} - e^{ts_2})x_b + \left(\frac{s_2 - \alpha}{s_2 - s_1} e^{ts_2} - \frac{s_1 - \alpha}{s_2 - s_1} e^{ts_1}\right) y_b \end{aligned} \tag{20}$$

Here

$$\begin{aligned} s_1 &= (1/2)\left\{\alpha + \kappa + \left[(\alpha - \kappa)^2 + 4\beta\gamma\right]^{0.5}\right\} \\ s_2 &= (1/2)\left\{\alpha + \kappa - \left[(\alpha - \kappa)^2 + 4\beta\gamma\right]^{0.5}\right\} \end{aligned} \tag{21}$$

Let us derive the continuity equation for Lagrange's exponential law of motion (20), (21). Let the control figure have the shape of a unit square with the following coordinates of the corner points (0, 0), (0, 1), (1, 1), (1, 0). The control figure will have the shape of a parallelogram after being deformed for time t. Parallelogram corner points 2 and 4 will have these coordinates.

$$x_2 = \frac{\beta}{s_2 - s_1}(e^{ts_2} - e^{ts_1})$$

$$y_2 = \frac{s_2 - \alpha}{s_2 - s_1} e^{ts_2} - \frac{s_1 - \alpha}{s_2 - s_1} e^{ts_1}$$

$$x_4 = \left(\frac{s_2 - \alpha}{s_2 - s_1} e^{ts_1} - \frac{s_1 - \alpha}{s_2 - s_1} e^{ts_2}\right)$$

$$y_4 = \frac{(s_1 - \alpha)(s_2 - \alpha)}{\beta(s_2 - s_1)}(e^{ts_1} - e^{ts_2})$$

The area of a parallelogram can be calculated using this formula.

$$S = y_2 x_4 - y_4 x_2$$

Equation of conservation of area in time

$$\left\{\frac{s_2 - \alpha}{s_2 - s_1} e^{ts_2} - \frac{s_1 - \alpha}{s_2 - s_1} e^{ts_1}\right\}\left\{\frac{s_2 - \alpha}{s_2 - s_1} e^{ts_1} - \frac{s_1 - \alpha}{s_2 - s_1} e^{ts_2}\right\} + \left\{\frac{(s_1 - \alpha)(s_2 - \alpha)}{(s_2 - s_1)^2}(e^{ts_2} - e^{ts_1})^2\right\} = 1$$

The area conservation equation after simplifying the notation has the following form:

$$e^{t(s_1 + s_2)} = 1$$

or

$t(s_1 + s_2) = 0$

or

$$t\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

We get the continuity equation after dividing by time

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

This continuity equation is realized in the case of a linear field of velocity components (18) along the coordinates $x,\ y$

$u = \alpha x + \beta y$

$v = \gamma x + \ \kappa y$

Liquid particles have an accelerated Lagrange law of motion. There will be no turbulent pulsations in the place of the accelerated Lagrange law of motion. In place of the accelerated law of motion of Lagrange, there will be no generation of sound vibrations.

The real velocity field in problems of hydrodynamics cannot be linear in the entire flow region. Terms of a high order of smallness will appear in the equation of continuity at the point of violation of the linear field of the velocity components. The terms of a high order of smallness will generate sound waves and disturbances at the point of violation of the linear field of the velocity components.

Koppel et al.'s experiments [16] on fluid flow with acceleration demonstrate an increase in the critical Reynolds number for the transition from laminar to turbulent flow by a factor of 30–100. This is an experimental confirmation of the disappearance of terms of a high order of smallness in the continuity equation for a flow with acceleration. In the place of the linear field of the velocity components along the coordinates, there will be no generation of sound vibrations, self-oscillations.

Here we also emphasize that for stochastic processes in hydrodynamics, the question of finiteness in space and time of infinitesimal perturbations plays a key role [17–21] in the statistical theory and [22–26] in nonline theory. This leads to the appearance of additional fluctuation terms in the right part of the equations of conservation laws [27–31] in the chaos theory and in the theory of attractors [32–34]. As a result, on the basis of the law of interaction between deterministic and random motion [35–39], the transition from deterministic motion to chaotic, turbulent is realized [40–44]. The solutions considered in the article, albeit in the first approximation, but raise questions for numerical methods such as RANS [45–50], LES [51–53] and DNS methods for bifurcation of periodic solutions [54–58] and methods for simulation of a complete transition to turbulence [59–61]. Note that at the same time, the numerical TDNS method is emerging [36,37,62]. In it, differential equations with a random term on the right side have the ability to take into account the influence of various invariants.

## 6 Conclusions

1. A review of articles that take into account the quadratic and cubic invariants of the strain rate tensor in the continuity equation for a compressible fluid is presented.

2. The quadratic invariant describes self-oscillations, which are similar to harmonic oscillations. These are vibrations, sound generation, and the initial stage of turbulence.

3. The cubic invariant describes the occurrence of pressure waves that rapidly increase in time. They are similar to a soliton or Zhukovsky hydraulic shock.

4. These invariants can be taken into account in solving the wave equations, which are derived by the method of acoustic analogy, taking into account the terms of the high order of smallness of the continuity equation.

5. Taking into account quadratic and cubic invariants is important when analyzing emergency situations.

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