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Transient Heat Conduction in Materials with Linear Power-Law Temperature-Dependent Thermal Conductivity: Integral-Balance Approach

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Abstract Closed form approximate solutions to nonlinear transient heat conduction with linear power-law $k = k_0\left(1 \pm \beta T^m\right)$ temperature-dependent thermal diffusivity have been developed by the integral-balance integral method under transient conditions. The solutions use improved direct approaches of the integral method and avoid the commonly used linearization by the Kirchhoff transformation. The main steps in the new solutions are improvements in the integration technique of the double-integration technique and the optimization of the exponent of the approximate parabolic profile with unspecified exponent. Solutions to Dirichlet boundary condition problem have been developed as examples by the classical Heat-balance Integral method (HBIM) and the Double-integration method (DIM).

Keywords: Non-linear heat conduction, integral-balance solutions, temperature-dependent thermal diffusivity.

a	Thermal diffusivity, $m^2 s^{-1}$
a_0	Thermal diffusivity at the reference temperature $m^2 s^{-1}$
C_p	Heat capacity, $Jkg^{-1}K^{-1}$
k	Heat conductivity, $W.m^{-1}K^{-1}$
k_0	Heat conductivity, at the reference temperature $W.m^{-1}K^{-1}$
m	Dimensionless parameter of non-linearity
n	Dimensionless exponents of the assumed profile
T	Temperature, K
T_0	Reference temperature at $t = 0$, K
T_s	Surface temperature at $x = 0$, K

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T_{reff} Reference temperature, K

t Time, s

Greek Symbols

β Dimensionless factor

δ Penetration depth, m

ρ_s Mass density, $kg.m^{-3}$

Subscripts

HBIM Heat Balance Integral Method

DIM Double Integration Method

1 Introduction

Most diffusion models concerning transport of heat (or mass) occur nonlinearly. Except some limited number of problems, there are no exact analytical solutions and, in general, numerical approaches have to be applied. However, in some cases approximate analytical solutions are possible. The present work reports new solutions of a non-linear transient heat conduction by Heat-balance Integral method (HBIM) [Goodman (1964)] and the Double-integration method (DIM) [Volkov and Li-Orlov (1970); Myers (2009)].

The communication considers a transient heat condition problem in a semi-infinite medium with temperature-dependent thermal diffusivity modelled by the equation

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k(T) \frac{\partial T}{\partial x} \right) \tag{1a}$$

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(a(T) \frac{\partial T}{\partial x} \right) \tag{1b}$$

$$a = a_0 \left(1 + \beta T^m\right) , \ m > 0 \tag{1c}$$

The relationship (1c) is related mainly to the temperature-dependent thermal conductivity $k = k_0 \left(1 + \beta T^m\right)$ assuming the product ρC_p temperature-independent [Cobble (1967); Sucec and Hedge (1978); Lin (1978); Noda (1993)].

The difficulties inherent in obtaining solutions for this class of equations have motivated a variety of solution methods, both exact and approximate ones. There exist several approaches to solve Eq. (1a) , among them: orthogonal collocation method [Lin(1978)], Green function method [Liu et al. (2015)], perturbation method [Khaleghi et al. (2007); Aziz and Benzies (1976); Aziz (1977)] variational iteration method [Khaleghi et al. (2007)], homotopy-perturbation method [Khaleghi et al. (2007)], direct variational method [Krajewski (1975)], the least squares method [Aziz and Bouaziz (2011)], networks models [Alhama and Zueco (2007)], iterative solutions with the solution of the linear problem as initial approximation [Mehta (1979)], finite difference solutions [Sucec and Hedge (1978)], Lattice Boltzmann method [Das et al. (2009)], numerical solutions [Parveen and Alim (2013)], etc. The Kirchhoff transformation [Tomatis (2013)] is the common approach to transform eq. (1c) into a linear diffusion equation by applying (2a), namely

$$w = \int_0^T a_0 \left(1 + \beta T\right) dT = a_0 T + \frac{\beta T^2}{2} \Rightarrow \frac{\partial w}{\partial t} = a_0 \frac{\partial^2 w}{\partial x^2} \qquad \text{(2a,b,c)}$$

The heat-balance integral method is among the approximate analytical methods allowing to develop closed-form solutions of the problem (1), but its simple form known as *Heat-balance integral method* [Goodman (1964)] has been used accidentally [Goodman (1964); Yen (1989)] for solutions of the problem (1). The main approach in these solutions is the initial non-linear transform $\mathrm{v} = \int_0^T \rho C_p dT$ to linearize eq. (1b) and then applying HBIM with assumed cubic polynomial profiles. It is worthnoting, that in these solutions [Goodman (1964); Yen (1989)] the non-linear transform is mainly applied to the surface temperature (at $x = 0$). These solutions are not popular due to the inherent property of HBIM to predetermine the accuracy of approximation when a fixed order of the assumed profiles is used [Goodman (1964); Myers (2009)] as well as due to the difficulties emerging in the reversion of the solution in the terms of T, when the order of the polynomial approximation is high [Cobble (1967)].

The recent applications of the simple heat-balance integral method (HBIM) [Hristov (2005a) and the double-integral balance method (DIM) [Hristov (2016)] to eq. (1b), when the thermal diffusivity is a of a power-law functional dependence of the temperature $a = a_0 T^m$, demonstrate a new solution strategy were the non-linear Kirchhoff transformation can be avoided. The present article reports new solutions to equation (1b) with the additive functional relationship (1c) about the temperature-dependent diffusivity, using the technique of HBIM and DIM and the solution strategies developed in [Hristov (2016)].

The general task of the present study is the development of approximate integral-balance solutions of the model (1a,b) in avoiding the Kirchhoff linearization transformation of the thermal diffusivity (1c). The solutions developed refer to the cases with Dirichlet and Neumann boundary conditions. The approximate solution are developed by HBIM and DIM applying an assumed parabolic profile with unspecified exponent [[Hristov (2005a); Hristov (2009, 2015b); Mitchell and Myers (2008, 2010); Sadoun et al. (2006)]

1.1 Background of the integral-balance solution

The integral-balance method is based on the concept that the diffusant (heat or mass) penetrates the undisturbed medium at a final depth δ. Therefore, the common boundary conditions at infinity

$$T(\infty) = 0 \text{ and } \frac{\partial T}{\partial x}(\infty) = 0 \qquad (3)$$

can be replaced by

$$T(\delta)=0 \text{ and } \frac{\partial T}{\partial x}(\delta)=0 \tag{4a,b}$$

The conditions (4a,b) define a sharp-front movement $\delta(t)$ of the boundary between disturbed and undisturbed medium when an appropriate boundary condition at $x=0$ is applied. The position $\delta(t)$ is unknown and should be determined trough the solution. When the thermal diffusivity is temperature-independent, the integration of eq.(1b) over a finite penetration depth δ yields (5a)

$$\int_0^{\delta}\frac{\partial T(x,t)}{\partial t}dx=\int_0^{\delta}a_0\frac{\partial^2 T}{\partial x^2}dx \tag{5a}$$

or

$$\frac{d}{dt}\int_0^{\delta}T(x,t)dx=-a_0\frac{\partial T}{\partial x}(0,t) \tag{5b}$$

Applying the Leibniz rule to the left-side of (5a) we get (5b) .Equation (5b) is the principle relationship of the simplest version of the integral-balance method known as *Heat-balance Integral Method* (**HBIM**) [1]. After this first step, replacing T by an assumed profile T_a (expressed as a function of the relative space co-ordinate x/δ) the integration in (5b) results in an ordinary differential equation about $\delta(t)$ [1,3,20]. The principle problem emerging in application of (5a,b) is that its right-side depends on the type of the assumed profile.

An improvement, avoiding the principle problem of HBIM is the double integration approach (**DIM**) [Volkov and Li-Orlov (1970)] recently renewed by Myers as Refined integral Method (**RIM**) [Mitchell and Myers (2008, 2010)]. The first step of DIM is integration of (1b) from 0 to x and then the resulting equation is integrated again from 0 to δ . The principle equation of **DIM**, following Myers [Mitchell and Myers (2008, 2010)]is

$$\frac{d}{dt}\int_0^{\delta}xT(x,t)=a_0T(0,t) \tag{6}$$

An alternative expression of the DIM principle relationship can be easily derived, too. Representing the integral in the left-side of (5a) as $\int_0^{\delta}f(\bullet)dx=\int_0^{x}f(\bullet)dx+\int_x^{\delta}f(\bullet)dx$ [Hristov (2015b, 2016)] we get

$$\int_0^{x}\frac{\partial T}{\partial t}dx+\int_x^{\delta}\frac{\partial T}{\partial t}dx=-a_0\frac{\partial T(0,t)}{\partial x} \tag{7}$$

Subtracting (5b) from (7) and integrating the resulting equation from 0 to δ one obtain

$$\int_0^{\delta}\left(\int_x^{\delta}\frac{\partial T}{\partial t}dx\right)dx = a_0 T(0,t) \tag{8}$$

If the thermal diffusivity is non-linear, as a power-law $a = a_0 T^m$ ($m > 0$) then (6) and (8) take the forms [Hristov (2016)]

$$\frac{d}{dt}\int_0^{\delta} xT(x,t) = \frac{a_0}{m+1}\left[T(0,t)\right]^{m+1} \tag{9a}$$

$$\int_0^{\delta}\left(\int_x^{\delta}\frac{\partial T}{\partial t}dx\right)dx = \frac{a_0}{m+1}\left[T(0,t)\right]^{m+1} \tag{9b}$$

The integral relations presented by (9a) and (9b) will be used further in this work in the development of the problems at issue.

1.2 Aim

In this study, the focus will be on a temperature dependent diffusivity namely:

$$a(T) = a_0\left[1+\beta(T/T_s)^m\right] \tag{10}$$

where T_s is the temperature at $x = 0$ and β could be either positive or negative.

In order to simplify the problem development, the dimensionless variable $u = T/T_s$ is used and therefore, the governing equation can be presented as

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\left[\left(1+\beta u^m\right)\frac{\partial u}{\partial x}\right] \tag{11a}$$

with boundary conditions

$$u(0,1) = 1,\ u(\delta,t) = 0 \text{ and } k\left(\frac{\partial u}{\partial x}\right)_{x=\delta} = 0 \tag{11b,c,d}$$

2 Solution strategies

2.1 HBIM solution

If we apply the classical HBIM to our equation we obtain:

$$\int_0^{\delta}\frac{\partial u}{\partial t}dx = \int_0^{\delta}\frac{\partial}{\partial x}\left(a_0(1+\beta u^m)\frac{\partial u}{\partial x}\right)dx \tag{12}$$

Now applying the solution strategy of [21] represented by (9) and (10) we get

$$\frac{\partial}{\partial t}\int_0^{\delta} u dx = -a_0\left(\frac{\partial u}{\partial x} + \frac{\beta}{m+1}\frac{\partial u^{m+1}}{\partial x}\right)_{x=0} \tag{13}$$

Equation (13) is the principle equation of the DIM solution relevant to the problem at issue. Indeed if the transformation is not applied we do not find the same result as it is in the literature when a is temperature independent. Hence it is possible to solve non-linear heat conduction equations. Moreover this method will be tested next for a fixed temperature boundary condition.

2.2 DIM solution

Applying the DIM we get

$$\int_0^{\delta}\left(\int_0^{x}\frac{\partial u}{\partial t}dx\right)dx=\int_0^{\delta}a_0\left((1+\beta u^m)\frac{\partial u}{\partial x}\right)-a_0\left((1+\beta u^m)\frac{\partial u}{\partial x}\right)_{x=0} \tag{14}$$

If the double integral of (14) is integrated by parts, we obtain:

$$\frac{\partial}{\partial t}\int_0^{\delta}xudx=-a_0(u+\frac{\beta}{m+1}u^{m+1}) \tag{15}$$

Equation (15) is the principle equation of the DIM solution relevant to the problem at issue.

2.3 Assumed profile

The solutions use an assumed parabolic profile with unspecified exponent [Hristov (2009, 2015a, 2005b, 2016); Mitchell and Myers (2008, 2010); Sadoun et al. (2006)]

$$T_a=T_s\left(1-\frac{x}{\delta}\right)^n \tag{16}$$

The profile (16) satisfies the Goodman boundary conditions (4a, b), namely

$$T(0,t)=T_s \quad \text{or} \quad k\left(\frac{\partial T}{\partial x}\right)_{x=0}=q_0 \tag{17a,b}$$

$$T(\delta,t)=T_0=0 \quad \text{or} \quad k\left(\frac{\partial T}{\partial x}\right)_{x=\delta}=0 \tag{18a, b}$$

2.4 Scaling and governing equations

The scaled thermal diffusivity is commonly expressed as $a=f\left[\left(T/T_{ref}\right)^m\right]$ where T_{ref} is a reference temperature which differs from the initial medium temperature T_0 . The functional relationship can be expressed as a simple power-law $a=a_0T^m$ [Alhama and Zueco (2007); Hristov (2015a, 2016] or as a linear relationship $a=a_0\left[1+\beta\left(T/T_{ref}\right)^m\right]$, where m and β are dimensionless constants. In this context, when $T_{ref}\neq T_0\neq 0$ the power-law can be rescaled as $a_{eff}u^m=a_0k_T\left(T/T_0\right)^m$ where $u=\left(T/T_0\right)$,

$k_T = (T_0/T_{ref})^m = const.$ and $a_{eff} = a_0 k_T$. Therefore the relationship for $m > 1$ can be presented as $a = a_0\left[1 + \beta k_T (T/T_{ref})^m\right]$.When $T_{ref} = T_0 \neq 0$, we get $k_T = 1$.

In order to be correct in the solutions performed next and for the sake of clarity of the expressions, we have to mention that the common literature data about the heat conductivity of the materials are presented in dimensional form as $k = k_0\left[1 + \beta T^m\right]$. It is easy, to transform this relationship into $a = a_0\left[1 + \beta k_T (T/T_{ref})^m\right]$ by a simple rescaling procedure which affects only the pre-factor βk_T.

3 Solution example

3.1 Dirichlet problem

In the case of the Dirichlet problems using the dimensionless variable $u = (T - T_0)/(T_s - T_0)$ or $u = T/T_s$ the dimensionless assumed profile is

$$u_a = (1 - x/\delta)^n, \ u_a = T_a/T_s \tag{19a}$$

satisfying the boundary conditions (11), namely

$$u_a(0,1) = 1, \quad u(\delta, t) = 0, \quad k\left(\frac{\partial u}{\partial x}\right)_{x=\delta} = 0 \tag{19b}$$

Now, the governing equation is eq. 10.

3.1.1 HBIM solution

The application of eq. (15) of HBIM with a dimensionless profile $u_a = (1 - x/\delta)^n$ and fixed temperature boundary condition yields

$$-a_0\left(\frac{\partial u}{\partial x} + \frac{\beta}{m+1}\frac{\partial u^{m+1}}{\partial x}\right)_{x=0} = -a_0\frac{n}{\delta}(1+\beta) \tag{20a}$$

$$\frac{\partial}{\partial t}\left(\int_0^\delta u dx\right) = -\frac{\partial}{\partial t}\left(\frac{\delta}{n+1}\right) \tag{20b}$$

$$\frac{\partial \delta^2}{\partial t} = a_0 n(n+1)(1+\beta) \tag{20c}$$

$$\delta_{HBIM} = \sqrt{a_0 t}\sqrt{2n(n+1)(1+\beta)} \tag{20d}$$

If the thermal diffusivity is a constant with respect to temperature (i.e. $\beta = 0$), then (20d) reduces to the solution developed by Myers [23], i.e. $\delta_{HBIM(\beta=0)} = \sqrt{a_0 t}\sqrt{2n(n+1)}$.

It is worthnoting that the non-linearity represented by the exponent m disappears and therefore the approach of HBIM cannot be applied when $m \neq 1$. Hence, for the Dirichlet problem at issue an advanced solution should be developed, as it is demonstrated next. Moreover, commonly $|\beta| < 1$ and according to (20d) we have $\delta_{HBIM} > \delta_{HBIM(\beta=0)}$ when $\beta > 0$. Oppositely, there is a front retardation when $\beta < 0$.

3.1.2 DIM solution

The application of the DIM relationship (15) yields

$$-a_0\left(u + \frac{\beta}{m+1}u^{m+1}\right)_{x=0} = -a_0\left(1 + \frac{\beta}{m+1}\right) \tag{21a}$$

$$\frac{\partial}{\partial t}\int_0^{\delta} xudx = \frac{\partial \delta^2}{\partial t}\frac{1}{(2+n)(1+n)} \tag{21b}$$

$$\frac{\partial \delta^2}{\partial t}\frac{1}{(n+1)(n+2)} = a_0\left(1 + \frac{\beta}{m+1}\right) \tag{21c}$$

$$\delta_{DIM} = \sqrt{a_0 t}\sqrt{\frac{(n+1)(n+2)}{m+1}(m+1+\beta)} \tag{21d}$$

For $\beta = 0$ we have the linear case and we find the result developed by Myers [Mitchell and Myers (2010)], i.e. $\delta_{DIM(\beta=0)} = \sqrt{a_0 t}\sqrt{(n+1)(n+2)}$. Hence if $m > 0$ and $\beta > 0$ it is obvious that $\delta_{DIM\,\beta=0} < \delta_{DIM}$ which is related to the fact that if m or β increases the depth penetration decreases due to the increased thermal diffusivity of the medium.

3.2 Approximate profiles

Therefore the approximate solutions are

HBIM

$$u_{a(HBIM)} = \left(1 - \frac{x}{\sqrt{a_0 t}\sqrt{2n(n+1)(1+\beta)}}\right)^n \tag{22}$$

DIM

$$u_{a(DIM)} = \left(1 - \frac{x}{\sqrt{a_0 t}\sqrt{(n+1)(n+2)(1+\frac{\beta}{m+1})}}\right)^n \tag{23}$$

4 Optimization of the approximate profile

In the general moment method [Ames (1965); Prasad and Salomon (2005)] a desired accuracy in approximation can be attained be increasing the number of terms in the series $u_a = \sum_{j=1}^{N} a_j \left(1 - x/\delta\right)^{n_j}$ (see ref. 22 for comments), i.e. by increase in the order of the moments involved in the solution. Since both HBIM and DIM are restricted to the zeroth moment they use only the first term of the series. Therefore, the accuracy of approximation depends on the values of the exponent n because the coefficient $a_1 = u_s$ depends on the boundary condition $x = 0$. The classical applications [Goodman (1964); Hristov (2009, 2015b)] are with $n = 2$ and $n = 3$. However, when the exponent n is stipulated the approximation error is predetermined.

4.1 Definition of n by matching HBIM and DIM penetration depths

Since the penetration depth is physically well defined, it should be one and the same irrespective of the method of integration applied. From this point of view and the results developed, we have $\delta_{HBIM} = \delta_{DIM}$, that is

$$\sqrt{2(n+1)n(1+\beta)} = \sqrt{(n+1)(n+2)(1+\frac{\beta}{m+1})} \tag{24}$$

solution of (24) with respect to n is

$$n_{m,\beta} = \frac{2(m+1+\beta)}{m+1+\beta+2m\beta} \tag{25a}$$

If $m >> |\beta|$ then we may simplify (25a) as

$$n_{m,\beta} = 2\frac{1+\dfrac{\beta}{m+1}}{1+2m\dfrac{\beta}{m+1}} \Rightarrow n_{m,\beta} \approx 2 \tag{25b}$$

For $\beta = 0$ we get the Goodman exponent $n_{\beta=0} = 2$ and this is the limit when $m >> |\beta|$. Therefore, the increase in the nonlinearity through the values of m does affect significantly the values of $n_{m,\beta}$ because in general $m >> |\beta|$ because the classical values of beta are around 10^{-4} [Shelton, 1934; Y et al. (2011)] (see more details in Fabre and Hristov, 2016)]. The data summarized in Tab.1 and the plots shown in Fig 1 are related to the most frequent case of $m = 1$.

Table 1: Exponent determined by matching the penetration depths determined of HBIM and DIM

β	**0.01**	**0.1**	**1**	**0.01**	**0.1**	**1**
$n_{m,\beta}$	1.98	1.826	1.2	1.97	1.74	0.91
Approximation Error, e_n	0.0222	0.0557	0.025	0.0204	0.0415	0.0594

The approximation error e_n presented in Tab.1 is the mean squared error over the penetration depth. Details about calculations of e_n are presented in the next point.

4.2 Global minimization approach

4.2.1 Approximation error and restrictions on the exponent

Now, we focus the attention on the optimization of the exponent n in order to obtain solutions with minimal approximation errors bearing in mind that the approximate profile satisfies the integral-balance relations (13) and (15b) but not the original heat conduction equation (10). Hence, the residual function $\varphi\left(u_a\left(x,t\right)\right)$ can be defined from the requirement that the approximate solution should satisfy the governing equation (10)

$$\varphi\left(u_a\left(x,t\right)\right)=\frac{\partial u_a}{\partial t}-\frac{\partial}{\partial x}\left(a_0\left(1+\beta u_a^m\right)\frac{\partial u_a}{\partial x}\right) \tag{26a}$$

If u_a matches the exact solution then $\varphi\left(u_a\left(x,t\right)\right)=0$, otherwise it should attain a minimum for a certain value of the exponent n, which is the only unspecified parameter of the approximate profile.

With $u_a=\left(1-x/\delta\right)^n$, at the boundary $x=0$, we have

$$\varphi(0,t)=-\frac{n(n-1)+\beta n(n(m+1)+1)}{\delta^2} \tag{26b}$$

and looking for positive n we get

$$n=\frac{\beta+1}{1+\beta(m+1)} \tag{27a}$$

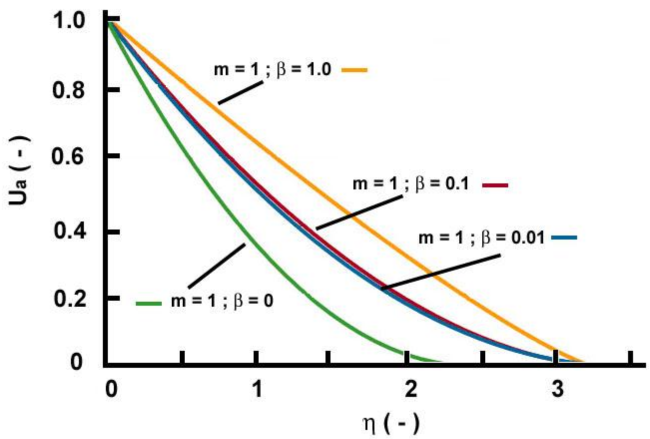

Figure1: Dimensionless temperature profiles with exponents established by matching the penetration depths of HBIM and DIM for fixed temperature boundary condition and $m = 1$. $\eta = x\big/\sqrt{a_0 t}$ is the Boltzmann -similarity variable.

Hence, n must be positive and therefore the heat conduction equation will be satisfied for $n = (\beta + 1)/[1 + \beta(m+1)]$.

Meanwhile, we have to verify the Goodman boundary condition at $x = \delta$ where $n > n_{\min} = (\beta + 1)/[1 + \beta(m+1)]$. Thus, the estimation of n through matching the penetration depth of HBIM and DIM solutions, established in the previous paragraph (see eqs. 25a, b) is satisfied too because comparing (28a) and (25a) we have

$$\frac{2(m+1+\beta)}{m+1+\beta+2m\beta} > (\beta+1)/[1+\beta(m+1)] \tag{27b}$$

when $m >> |\beta|$

Further, for $x \to \delta$ we have:

$$\varphi(\delta, t) = \frac{n(n-1)}{\delta^2} \lim_{x \to \delta} (1 - x/\delta)^{n-2} + \frac{\beta n (n(m+1)-1)}{\delta^2} \lim_{x \to \delta} (1 - x/\delta)^{n(m+1)-2} = 0 \tag{28}$$

Thus, the heat diffusion equation is satisfied at $x = \delta$ when $n > 2/m + 1$. This restriction is obeyed when $m = 0$ where $n > 2$ is the principle condition of the HBIM solution [Hristov (2009, 2016); Mitchell and Myers (2010)].

4.2.2 Optimal exponents

The optimization will be done thanks to the minimization method of Langford [Langford (1973)]. The method consists in minimization the square of the function φ over the entire penetration depth δ

$$E_L(n,\beta,t)=\int_0^{\delta}\left[\frac{\partial u_a}{\partial t}-\frac{\partial}{\partial x}\left(a_0(1+\beta u_a^m)\frac{\partial u_a}{\partial x}\right)\right]^2 dx \quad (29)$$

As example demonstrating the approach we will use the case with $m=1$.

4.2.2.1 The HBIM solution

From the solution developed we have

$$\frac{\partial u_a}{\partial t}=\frac{n^2 x(n+1)(\beta+1)}{\delta_{HBIM}^3}\left(1+\frac{x}{\delta_{HBIM}}\right)^{n-1} \quad (30a)$$

$$\frac{\partial}{\partial x}\left((1+\beta u_a)\frac{\partial u_a}{\partial x}\right)=\frac{n(n-1)}{\delta_{HBIM}^2}\left(1-\frac{x}{\delta_{HBIM}}\right)^{n-2}+\frac{\beta n(2n+1)}{\delta_{HBIM}^2}\left(1-\frac{x}{\delta_{HBIM}}\right)^{2n-2} \quad (30b)$$

Then, from (30) we get

$$E_L(n,\beta)=AF_1(n,\beta)+BF_2(n,\beta) \quad (31a)$$

$$F_1(n,\beta)=2n^4-7n^3+6n^2++2n-1+\beta(2n^3+11n^2-3n+ +\beta(2n^4+n^3+2n^2-3n)) \quad (31b)$$

$$F_2(n,\beta)=96n^5-448n^4+614n^3-401n^2+115n-12+ +\beta(60n^5-264n^4+339n^3-231n^2+66n-6) \quad (31c)$$

$$A=\frac{\sqrt{2n(n+1)(\beta+1)}\ t^{-3/2}}{4(n+1)^2(\beta+1)^2(2n-3)(4n^2-1)} \quad (32a)$$

$$B=\frac{\beta\sqrt{2n(n+1)(\beta+1)}t^{-3/2}}{4(n+1)^2(\beta+1)^2(3n-1)(3n-2)(3n-3)(4n-3)} \quad (32b)$$

Hence $E_L(n,\beta)=e_n t^{-3/2}$, where e_n depends only on n and β. Therefore, the function e_n has to be minimized with respect to n for given β in order to find the optimal exponent. The optimal exponents assuring minima of the squared errors function of approximations e_n are summarized in Tab. 2.

Table 2: Exponents defined by the global minimization approach for $m=1$

β	**0**	**0.0001**	**0.001**	**0.01**	**0.1**	**1**
e_n	0.0169	0.0169	0.017	0,0178	0.022	0.007
$n_{\min}$	1	1	1	1	0.0257	0.094
n_{opt}	2.335	2.234	2.235	2.249	2.339	2.404

In the literature the coefficient β is between 0.001 and 0.0001 [Shelton (1934); Y et al. (2011); Fabre and Hristov (2016)], and for these classical values, β has no significant effect on the determination of the optimal exponent.

4.2.2.2 The DIM solution

With the DIM solution and for $m=1$ we have

$$\frac{\partial u_a}{\partial t}=\frac{nx(n+1)(n+2)(\beta+2)}{4\delta_{DIM}^3}\left(1+\frac{x}{\delta_{DIM}}\right)^{n-1} \tag{33a}$$

$$\frac{\partial}{\partial x}\left((1+\beta u_a)\frac{\partial u_a}{\partial x}\right)=\frac{n(n-1)}{\delta_{DIM}^2}\left(1-\frac{x}{\delta_{DIM}}\right)^{n-2}+\frac{\beta n(2n+1)}{\delta_{DIM}^2}\left(1-\frac{x}{\delta_{DIM}}\right)^{2n-2} \tag{33b}$$

The values of optimal exponents are summarized in Tab. 3.

Table 3: Optimal exponents of the DIM solution for $m=1$

β	**0**	**0.0001**	**0.001**	**0.01**	**0.1**	**1**
$n_{\min}$	0.0167	0.0167	0.0169	0,0191	0.0419	0.319
n_{opt}	2.218	2.2186	2.2182	2.214	2.173	2

For the case $m\neq 1$ the effect on the optimal exponents is shown in Fig. 2.

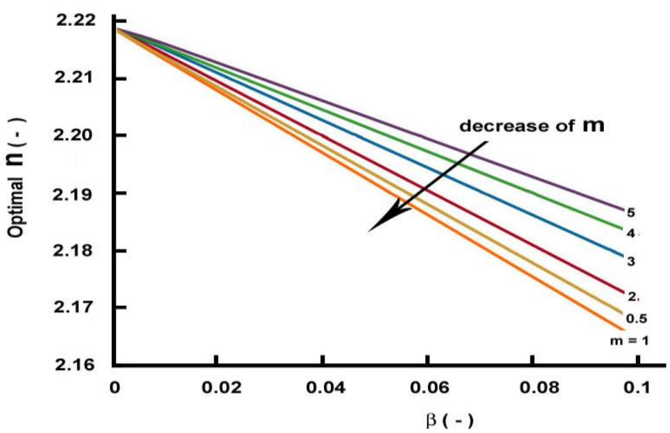

Figure 2: Effect of the parameter $m\neq 1$ on the optimal exponents (DIM solution)

The comparisons with the approximate solutions to the numerical ones demonstrate acceptable accuracy of approximation sufficient for the thermal engineering practice (details are availed elsewhere [Fabre and Hristov (2016)]).

5 Comparison to numerical solution with real values of β

The numerical simulations in the preceding sections were performed for values of β

larger than those corresponding to some real materials (see Table 4) because the goal as to demonstrate the features of the approximate solutions and thus presenting disguisable approximate profiles.

Now, in order to demonstrate the effect of the factor β related to real materials plots of pointwise (absolute) errors are shown in Fig.3. Since, the problem at issue has no exact solutions, the accuracy of the approximate one were compared to numerical calculation carried out by the Runge-Kutta solutions of 4th order as it is explained next.

With the Boltzmann transform $\eta = x \big/ \sqrt{a_0 t}$ and $X = x/\delta = \eta / f(n)$ the governing equation (1a) (with the relationship 1c) can be presented in the form (case of $m=1$)

$$X \frac{f(n)^2}{2}\frac{\partial u(X)}{\partial X} + \frac{\beta}{1+\beta u(X)}\left(\frac{\partial u(X)}{\partial X}\right)^2 + \frac{\partial^2 u(X)}{\partial X^2} = 0 \ , \quad f\left(n\right) \neq 1 \tag{34}$$

$$\frac{1}{2}\frac{\partial u(\eta)}{\partial \eta} + \frac{\beta}{1+\beta u(\eta)}\left(\frac{\partial u(\eta)}{\partial \eta}\right)^2 + \frac{\partial^2 u(\eta)}{\partial \eta^2} = 0 \ , \quad f\left(n\right) = 1 \ \text{ and } \eta = X \tag{35}$$

Here, the normalizing function $f(n)$ is introduced for consistency with the concept of the finite penetration depth δ which is missing in the classical solution of the linear equation expressed by the Gaussian error function. Actually, with $f\left(n\right) \neq 1$ the initial problem is transformed to a boundary value problem with $u(X=0)=1$ and $u(X=1)=0$ allowing to compare the integral-balance solutions with the numerical ones in the domain $0 \leq X \leq 1$.

The solutions were developed by Maple 13 where Runge-Kutta solutions of 4th order are possible with absolute error less than 10^{-6}. The normalizing function $f(n)$ for each β is expressed through the optimal n developed by minimization of the residual function (see Table 2 and Table 3) and it is equal either to the numerical factors of the penetration depth $F_{HBIM}\left(n,\beta\right) = \delta_{HBIM} / \eta$ or $F_{DIM}\left(n,\beta\right) = \delta_{DIM} / \eta$ which are dependent on the integration method applied.

As a general outcome of these numerical experiments it may be stated that when $\beta < 0$, that is the case of most native materials as pure metals [Shelton (1934); Y et al. (2011); Fabre and Hristov (2016)] the accuracy of the integral-balance solutions is better than when $\beta > 0$ (the case of alloys, composites, wood, etc. –see Table 4). Generally, the conductivity-temperature relationship can be written as $a = a_0\left(1 \pm \beta T\right)$ where the positive sign means that the thermal diffusivity increases with the temperature, while the negative sign corresponds to the opposite tendency.

As an expected result, the accuracy of DIM is better that that exhibited by HBIM (see Fig. 3b). The range of variations of the pointwise errors is typical for the integral-balance solutions [Goodman (2009); Mitchell and Myers (2010)], i.e. less than 0.003. In general, the increase in the absolute value of β leads to increased errors of approximations.

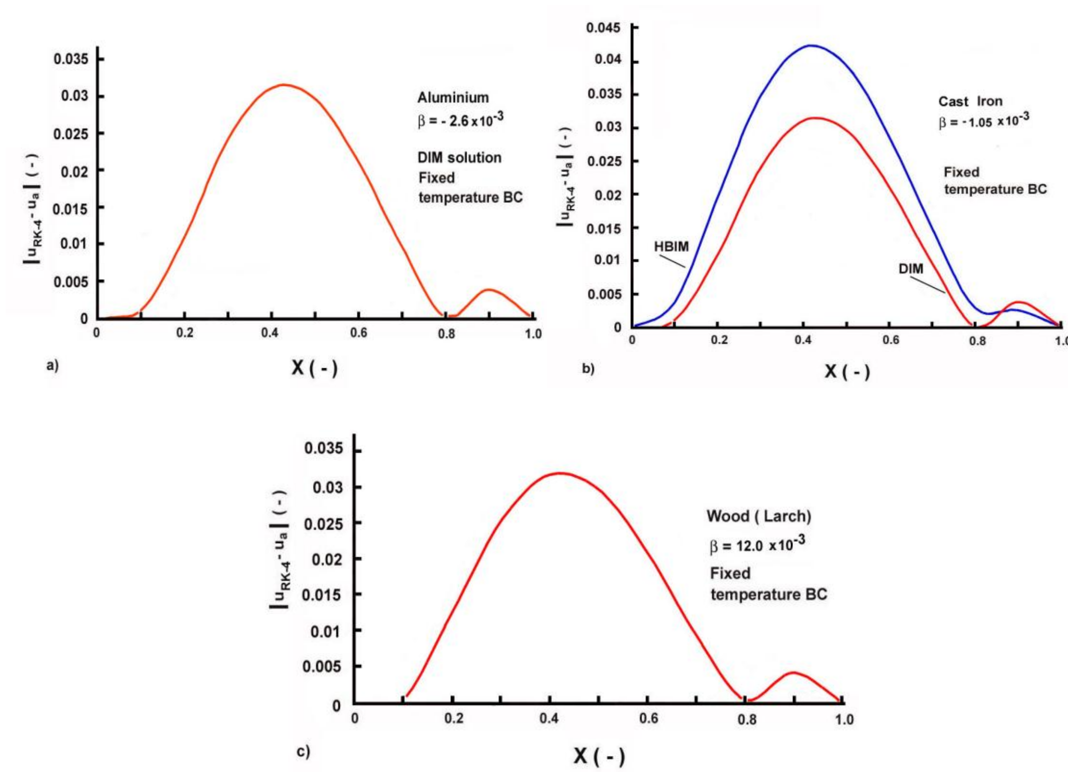

Figure 3: Pointwise errors of DIM solution calculated with respect to numerical (Runge-Kutta – 4th order) solutions for real values of β in cases of . a) Pure Aluminum ($\beta < 0$), b) Pure Iron ($\beta < 0$) and c) for wood with $\beta > 0$. $X = x/\delta = \eta / f(n)$.

6 Conclusions

The article presents approximate integral-balance solutions to one-dimensional non-linear transient heat conduction problem.

The technique developed in [Fabre and Hristov (2016)] allows the integral-balance methods (mainly DIM) to handle non-linear problems and provide approximate closed form solution.

The DIM solution handles successfully the solution of the non-linear heat-conduction problems, under Dirichlet boundary condition, with well distinguished effect in the final closed form solution. In contrast, the simple integration technique (HBIM) cannot be applied successfully because the non-linear effect vanishes trough the solution.

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