Fermatean Hesitant Fuzzy Prioritized Heronian Mean Operator and Its Application in Multi-Attribute Decision Making

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Abstract: In real life, incomplete information, inaccurate data, and the preferences of decision-makers during qualitative judgment would impact the process of decision-making. As a technical instrument that can successfully handle uncertain information, Fermatean fuzzy sets have recently been used to solve the multi-attribute decision-making (MADM) problems. This paper proposes a Fermatean hesitant fuzzy information aggregation method to address the problem of fusion where the membership, non-membership, and priority are considered simultaneously. Combining the Fermatean hesitant fuzzy sets with Heronian Mean operators, this paper proposes the Fermatean hesitant fuzzy Heronian mean (FHFHM) operator and the Fermatean hesitant fuzzy weighted Heronian mean (FHFWHM) operator. Then, considering the priority relationship between attributes is often easier to obtain than the weight of attributes, this paper defines a new Fermatean hesitant fuzzy prioritized Heronian mean operator (FHFPHM), and discusses its elegant properties such as idempotency, boundedness and monotonicity in detail. Later, for problems with unknown weights and the Fermatean hesitant fuzzy information, a MADM approach based on prioritized attributes is proposed, which can effectively depict the correlation between attributes and avoid the influence of subjective factors on the results. Finally, a numerical example of multi-sensor electronic surveillance is applied to verify the feasibility and validity of the method proposed in this paper.

Keywords: Fermatean hesitant fuzzy set; multi-attribute decision-making; Heronian mean operator; prioritized operator

1 Introduction

Due to the limitations of decision makers' cognition and professional differences, various uncertainties often appear in the comprehensive evaluation problems, and the traditional evaluation methods are not applicable or appropriate anymore. To solve the problem of comprehensive evaluation under uncertain circumstances, Zadeh [1] proposed the concept of fuzzy sets, and Atanassov [2]

extended the fuzzy set and developed the concept of intuitionistic fuzzy set (IFS) that includes both the degree of membership and non-membership, which is a good approach for dealing with uncertainty. Recently, the similarity measures of improved intuitionistic hesitant fuzzy sets [3], which are essential techniques to cope with uncertainties and awkward information in practical issues, have arisen the attention of some scholars. Since the sum of the membership degree and non-membership degree of the IFS must be less than or equal to 1, it will be subject to many constraints in solving practical problems. Thus, Yager [4] defined a new form of fuzzy set named Pythagorean fuzzy set (PFS), where the sum of squares of the membership degree and non-membership degree is less than or equal to 1. The PFS has received more and more attention because it can characterize uncertain information more sufficiently and capture stronger fuzziness. Not only have its basic definition, operation rules, and fundamental properties been gradually improved but also there have been extensive studies on the aggregation operators, multi-attribute decision-making (MADM), extensions, information measures and other fields, such as Pythagorean fuzzy soft sets [5], interval-valued picture fuzzy sets [6], Pythagorean fuzzy Einstein aggregation operators [7], distance measures and similarity measures [8].

IFS and PFS have been widely applied to numerous fields in real life. However, they also have some limitations related to the membership and non-membership grades, so Riaz et al. [9] introduced the concept of linear Diophantine fuzzy set (LDFS) with the addition of reference parameters. This was gradually expanded into a variety of forms, including the spherical LDFS [10], the LDFS rough sets [11], LDFS Einstein aggregation operators [12], and so on. This type of set can not only relax the strict constraints of satisfaction and dissatisfaction grades but also classify a physical system with the help of reference parameters. Therefore, it is an effective tool for expressing the evaluation values of experts during the decision-making process. Later, Senapati et al. [13] put forward the Fermatean fuzzy set (FFS), which extends the evaluation information, weakens the data requirements for the membership degree and non-membership degree, and broadens the conditions such that the cubic sum is less than or equal to 1. Many scholars have started work in this direction and confirmed its scientificity and effectiveness [14–20], mainly focusing on its applications in MADM [14,15] or multicriteria decision-making (MCDM) [16–18] and the extensions such as Fermatean fuzzy linguistic sets [19] and interval-valued Fermatean fuzzy sets [20]. Due to the different knowledge and experience of decision-makers, the decision-making group tends to be hesitant when confronted with challenging problems, and the membership function of fuzzy sets allows assessment information to contain multiple possible values [21,22]. Torra [21] first defined the hesitant fuzzy set (HFS) in terms of a function that returns a set of membership values for each element and then expanded the membership from a single real-valued set to a multi-real-valued set, which is more helpful for dealing with ambiguity and uncertainty. So far, HFS has been extensively studied in the domain of aggregation operators and decision-making. A Fermatean hesitant fuzzy set (FHFS) is stemmed from the concept of FFS and HFS. Liu et al. [23] effectively combined FFS and HFS to achieve comprehensive evaluations of green restoration levels in five provinces and cities along the Yangtze River Economic Belt. Mishra et al. [24] proposed a generalized distance measure by merging FHFS with a modified VIKOR method. In addition, various models have been investigated to review the information under uncertainty. For instance, the soft multisets (SMS) and SMS topology have been extensively applied to soft computing, fuzzy modeling and decision-making. Riaz et al. [25] established several MCDM algorithms with aggregation operators based on the SMS topology. The rough set is a fundamental tool for dealing with incomplete and inaccurate information because of its inherent characteristics. Sahu et al. [26] proposed a rough set-based approach to help students choose an appropriate subject. Sun et al. [27] developed a new method based on the multilabel fuzzy neighborhood rough sets and maximum relevance and minimum redundancy to deal with data with missing labels.

In the comprehensive evaluation (decision) problems, the information aggregation operators have been applied to a variety of theoretical and practical studies related to FFSs. Among them, Senapati et al. [28] proposed the Fermatean fuzzy weighted average (FFWA) operator and Fermatean fuzzy weighted geometric (FFWG) operator and gave the application of them in MADM. Aydemir et al. [29] combined the Dombi aggregation operators with FFS and presented a Fermatean fuzzy TOPSIS decision-making method. By extending the Hamacher aggregation operators and the prioritized aggregation (PA) operator [30]. Jan et al. [31] proposed the Fermatean fuzzy Hamacher prioritized weighted average (FFHPWA) operator and Fermatean fuzzy Hamacher prioritized weighted geometric (FFHPWG) operator. Rani et al. [32,33] proposed several Fermatean fuzzy Heronian mean operators and Fermatean fuzzy Einstein information aggregation operators based on Heronian mean (HM) operators and Einstein aggregation operators, and then applied them to the choice of food waste treatment technology. Induced by the Hamacher operations and FFS, Hadi et al. [34] proposed several Fermatean fuzzy Hamacher arithmetic and geometric aggregation operators, such as Fermatean fuzzy Hamacher ordered weighted average (FFHOWA) operator, Fermatean fuzzy Hamacher hybrid weighted (FFHHW) operator and Fermatean fuzzy Hamacher ordered weighted geometric (FFHOWG) operator. In addition, Shit et al. [35] developed various Fermatean fuzzy Dombi aggregation operators and applied them to MADM. Based on FFSs and probabilistic hesitant fuzzy sets, Liu et al. [23] proposed the probabilistic hesitant Fermatean fuzzy Dombi Choquet integral geometric (PHFFDCIG) operator and the probabilistic hesitant Fermatean fuzzy Dombi Choquet integral average (PHFFDCIA) operator, which were applied to the evaluation of regional green restoration level. Since the Fermatean fuzzy linguistic (FFL) set theory provides an efficient tool for modeling a higher level of uncertain information, Verma [36] defined several new aggregation operators, including the FFL-weighted average (FFLWA) operator, FFL-weighted geometric (FFLWG) operator, FFL-ordered weighted average (FFLOWA) operator, FFL-hybrid average (FFLHA) operator, and so on. Akram et al. [37] proposed a series of Hamy-inspired operators based on 2-tuple linguistic Fermatean fuzzy information. With the popularization of FFSs, more and more scholars pay attention to the application in solving MCDM problems [38–40]. At present, FFSs have been extended to various fields since they are better able to describe fuzziness, such as blockchain platform evaluation [41], treatment of COVID-19 [42], green construction supplier evaluation [43] and pattern recognition [44]. To further extend FFSs, Gul et al. [45] proposed an improved TOPSIS methodology with FFSs and applied it to the real-life manufacturing risk evaluation problem. Further, Yang et al. [46] investigated the properties of continuous Fermatean fuzzy information and examined their continuity, derivatives and differentials. Akram et al. [47] defined a triangular interval-valued Fermatean fuzzy number and its arithmetic operations. Undoubtedly, these operators have been successfully applied in MCDM or MADM under a Fermatean fuzzy environment, but they also have several drawbacks in their applications. That is, they do not take the interrelationships between the attributes into account during the process of aggregation.

In real life, the lack of complete information, the inaccurate data, and the preferences of decision-makers during qualitative judgment would have a significant influence on the results. Most of the Fermatean fuzzy information aggregation methods mentioned above are proposed under the condition that the attributes are independent. However, in practical problems, there may be different degrees of correlation between the attributes, such as complementarity, redundancy and preference. HM operators can effectively capture the correlations between input variables and aggregate multiple variables into one variable. In the decision-making, decision-makers will inevitably encounter situations where the attributes are interrelated and influence each other. If conventional operators are still used for information aggregation, the results may be inaccurate. Considering the interconnection of numerous

criteria or input attributes, the HM operator has attracted wide attention, which can effectively capture the correlation between input variables and aggregate multiple variables.

In recent years, many scholars have paid attention to extending the HM operator with different fuzzy environments and proposed numerous aggregation operators. Li et al. [48] studied the generalized HM operators under the Pythagorean fuzzy environment and put forward some Pythagorean fuzzy HM operators. Fan et al. [49] proposed two HM operators to aggregate the linguistic neutrosophic multisets and then discussed their properties. As for interval-valued fuzzy sets, Hu et al. [50] presented a study of MADM based on the correlations between attributes under the interval-valued intuitionistic fuzzy environment. Narang et al. [51] merged an improved generalized weighted HM operator and generalized geometric weighted HM operators with the traditional combined compromise solution (CoCoSo) method to present a new decision-making model. Since the neutrosophic cubic numbers (NCNs) can easily express incomplete, indeterminate and inconsistent information, Gulistan et al. [52] combined HM operators with NCNs. Additionally, HM operators have been extended to a variety of new forms in the process of MADM, such as cubic fuzzy HM Dombi aggregation operators [53], linguistic HM operators [54], and power generalized HM operators [55]. Most existing Fermatean fuzzy aggregation methods assume that the attribute weights are known. Considering that the priority relationship between attributes is often easier to obtain than the attribute weights, the prioritized aggregation (PA) operators [31] are effectively used to determine the attribute weights, which can make the decision-making results objective and fair. Chen et al. [56] proposed a prioritized measure-guided aggregation operator based on the ordered weighted average (OWA) operator and the Choquet integral. As an extension of PA operators, He et al. [57] used the priority labels to express the prioritized relationships between different criteria and then presented some scaled PA operators. The PA operators have always been one of the hot spots in MADM and MCDM problems and have also been applied to the intuitionistic fuzzy environment [58] and Pythagorean fuzzy environment [59,60]. To comprehensively take advantage of HM operators and PA operators, this paper developed a Fermatean hesitant fuzzy aggregation method based on these two operators.

From the literature review, this paper can draw the following conclusions. First, from the literature in the WOS database, FFSs have been preliminarily studied by scholars, mainly focusing on aggregation operators and their applications in MADM and the extensions, while few studies have aggregated the Fermatean fuzzy numbers based on HM operators. Second, as far as Fermatean hesitant fuzzy sets are concerned, the research is still at the initial stage. Thus, only a few papers have shown relevant investigations, and even fewer scholars have studied the combination of FHFS and HM operators. Finally, as a special generalization of the Bonferroni mean operator, HM operators have attracted significant attention from scholars and have been expanded into a variety of forms, such as linguistic Pythagorean HM operators, cubic HM operators, power generalized HM operators, and so on [48,52,55]. In addition, many authors have proposed MCDM approaches with these operators above. Li et al. [48] extended the generalized HM operators to the Pythagorean fuzzy context and developed several Pythagorean fuzzy HM operators. Later, Fan et al. [49] introduced four generalized HM operators for bipolar neutrosophic numbers and discussed their properties. However, most of the existing Fermatean fuzzy information aggregation methods assume that the attribute weights are known. Although the priority relationship between attributes is often easier to obtain than the attribute weights, there is almost no research concerning the prioritized HM operators.

To sum up, the main contributions and theoretical innovations of this paper are as follows. On the one hand, this paper introduces hesitance on the basis of Fermatean fuzziness and fully considers the impact of hesitant fuzzy data on decision-making, which can effectively address the uncertainty of associated attributes under the Fermatean fuzzy environment. On the other hand, in order to effectively

use prioritized operators to determine the attribute weights and ensure more objective and accurate results, this paper proposes the HM operator with the priority relationship, which further improves the theoretical system of the aggregation operators of FFSs. Additionally, the attributes tend to be related to each other rather than independent of each other in MADM problems, and the interactions will directly affect the results. The method proposed in this paper can effectively describe the correlation between attributes and avoid the influence of subjective factors on the results, so it has a critical reference significance for decision-making in reality. Therefore, this paper presents some aggregation operators under the Fermatean fuzzy environment for MADM and provides a scientific approach for information evaluation when attributes are associated with each other.

To further expand the application of HM operators and PA operators in a new fuzzy environment, the remainder of this paper is organized as follows. Some basic concepts and operational rules of FFS are defined in Section 2. Section 3 extends HM operators to the Fermatean fuzzy set, and introduces the Fermatean hesitant fuzzy Heronian mean (FHFHM) operator and the Fermatean hesitant fuzzy weighted Heronian mean (FHFWHM) operator, followed by related properties. In Section 4, the paper proposes the Fermatean hesitant fuzzy prioritized Heronian mean (FHFPHM) operator based on the FHFWHM operator and PA operators. In Section 5, the paper proposes a MADM method based on the FHFPHM operator and utilizes a numerical example to verify the method proposed in this paper. Section 6 concludes the paper with some remarks.

2 Preliminaries and Basic Theory

In this section, some basic concepts and operational rules about the HFS, IFS, PFS, FFS, and the HM operator are recalled briefly.

Definition 1 [21]: Assume $E = (\mu_1, \mu_2, \ldots, \mu_n)$ is a set of membership functions, then the HFS related to E is an expression of the form

$$h_E(x) = \cup_{\mu \in E} \{\mu_E(x)\} \tag{1}$$

Definition 2 [2]: Let X be a non-empty set, the IFS defined on X is an object having the form

$$A = \{\langle x, \alpha_A(x), \beta_A(x)\rangle : x \in X\} \tag{2}$$

where $\alpha_A: X \to [0, 1]$ and $\beta_A: X \to [0, 1]$ are the membership and non-membership of each element $x \in X$. And for all $x \in X$, $0 \leq \alpha_A(x) + \beta_A(x) \leq 1$.

Definition 3 [4]: The PFS defined on a non-empty set X is an object having the form

$$P = \{\langle x, \alpha_P(x), \beta_P(x)\rangle : x \in X\} \tag{3}$$

where $\alpha_P: X \to [0, 1]$ and $\beta_P: X \to [0, 1]$ denote, respectively, the degree of membership and non-membership for each element $x \in X$ to the set P, and $0 \leq \alpha_P^2(x) + \beta_P^2(x) \leq 1$ for all x. To give a specific example, assume that one expresses his preference for an alternative A_i $(i = 1, 2, \ldots, n)$ on a criterion $C_j(j = 1, 2, \ldots, n)$. Now he gives 0.7 as the degree to which the alternative A_i $(i = 1, 2, \ldots, n)$ satisfies the criterion $C_j(j = 1, 2, \ldots, n)$ and gives 0.6 as the degree to which it dissatisfies the criterion. The values 0.7 and 0.6 are termed as the membership degree and non-membership degree, and $(0.7 + 0.6) = 1.3 > 1$ but $[(0.7)^2 + (0.6)^2] = 0.85 < 1$. That is, PFS can describe uncertainty better than IFS.

Definition 4 [13]: Let X be a universe of discourse, then the FFS defined on X is an object holding the following structure

$$F = \{\langle x, \alpha_F(x), \beta_F(x)\rangle : x \in X\} \tag{4}$$

where $\alpha_F: X \to [0, 1]$ and $\beta_F: X \to [0, 1]$ respectively indicate the function of membership and non-membership, which satisfy the following condition $0 \le \alpha_F{}^3(x) + \beta_F{}^3(x) \le 1$. Here the paper gives a concrete example to help readers further understand the differences between FFS, IFS and PFS. Now an expert needs to make decisions based on his preference for an alternative $A_i\,(i = 1, 2, \ldots, n)$ on a criterion $C_j(j = 1, 2, \ldots, n)$. He may take 0.9 as the degree to which the alternative meets the criterion $C_j(j = 1, 2, \ldots, n)$ and 0.6 as the degree to which it does not meet. Obviously, $0.9 + 0.6 = 1.5 > 1$ and $(0.9)^2 + (0.6)^2 = 1.17 > 1$ can be obtained, which do not follow the conditions of IFS and PFS. However, $(0.9)^3 + (0.6)^3 = 0.729 + 0.216 = 0.945 < 1$. Therefore, among the three different types of fuzzy sets, FFS have the broadest application space and can deal with the strongest fuzziness.

Definition 5 [31]: Let $d = (\mu, v)$ be a Fermatean fuzzy set, then the score function and accuracy function of F can be described as follows:

$$S(d) = \frac{1 + \mu^3 - v^3}{2} \tag{5}$$

$$p(d) = \mu^3 + v^3 \tag{6}$$

The larger the score function, the better the FFS. In particular, researchers can further compare the accuracy functions if the score functions are the same. The FFS is superior when the accuracy function is more prominent.

According to the properties of FFS, Senapati et al. proposed several basic algorithms.

Definition 6 [28]: Let $d = (h, g)$, $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$ be three FFSs, then

1) $d_1 \oplus d_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{\left\{[\gamma_1{}^3 + \gamma_2{}^3 - \gamma_1{}^3\gamma_2{}^3]^{1/3}\right\}, \{\eta_1\eta_2\}\right\}$

2) $d_1 \otimes d_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{\{\gamma_1\gamma_2\}, \left\{[\eta_1{}^3 + \eta_2{}^3 - \eta_1{}^3\eta_2{}^3]^{1/3}\right\}\right\}$

3) $nd = \cup_{\gamma \in h, \eta \in g} \left\{\left\{[1 - (1 - \gamma^3)^n]^{1/3}\right\}, \{\eta^n\}\right\}$

4) $d^n = \cup_{\gamma \in h, \eta \in g} \left\{\{\gamma^n\}, \left\{[1 - (1 - \eta^3)^n]^{1/3}\right\}\right\}$

Definition 7 [47]: Assume that $a_i(i = 1, 2, \cdots, n)$ is a set of non-negative real numbers, $p, q > 0$, then the HM operator is defined as

$$HM^{p,q}(a_1, a_2, \ldots, a_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} a_i^p a_j^q\right)^{\frac{1}{p+q}} \tag{7}$$

Considering the influence of priority, Yager defined the PA operator as follows [31].

Definition 8 [31]: Let $C = \{C_1, C_2, \ldots, C_n\}$ $(n \in N)$ be an attribute set with the priority relationship, which satisfies $C_1 \succ C_2 \succ \ldots \succ C_n$. That is, when C_j has a higher priority than C_k for $j < k$, the PA operator is an object hosting the structure:

$$PA(C_1, C_2, \cdots, C_n) = \frac{T_1}{\sum_{j=1}^{n} T_j} C_1 \oplus \frac{T_2}{\sum_{j=1}^{n} T_j} C_2 \oplus \cdots \oplus \frac{T_n}{\sum_{j=1}^{n} T_j} C_n = \overset{n}{\underset{j=1}{\oplus}} \frac{T_j C_j}{\sum_{j=1}^{n} T_j} \tag{8}$$

where $w = \left(\dfrac{T_1}{\sum\limits_{j=1}^{n} T_j}, \dfrac{T_2}{\sum\limits_{j=1}^{n} T_j}, \ldots, \dfrac{T_n}{\sum\limits_{j=1}^{n} T_j} \right)^T$ designates the corresponding priority weight of attributes and

$C_k (h_k)$ represents the comprehensive score of C_k when $T_j = \prod_{k=1}^{j-1} C_k (h_k)$ $(j = 2, 3, \cdots, n)$, $T_1 = 1$ for all $c \in C$.

3 Fermatean Hesitant Fuzzy Aggregation Operators Based on the Heronian Mean Operator

Since the membership and non-membership degrees of FFS often show fuzziness in many cases, this section not only proposes the concept of Fermatean hesitant fuzzy set (FHFS) but also puts forward the Fermatean hesitant fuzzy Heronian mean (FHFHM) operator combined with the HM operators.

Definition 9: The Fermatean hesitant fuzzy set interpreted on a non-empty set X is an object of the form

$$D_E (x) = \{ \langle x, \mu_h (x_i), v_h (x_i) \rangle : x \in X \} \tag{9}$$

where $\mu_h (x_i)$ and $v_h (x_j)$ meet the condition of $0 \leq \mu_h^3 (x_i) + v_h^3 (x_j) \leq 1$. Specifically, $\mu_h (x_i)$ and $v_h (x_j)$ are known as the membership function and non-membership function containing multiple hesitant fuzzy numbers respectively, and satisfy the following conditions: $0 \leq \mu_h (x_i) \leq 1$ and $0 \leq v_h (x_j) \leq 1$.

Definition 10: Let $d_j = (\mu_j, v_j)$ $(j = 1, 2, \ldots, n)$ be a Fermatean hesitant fuzzy set, then the Fermatean hesitant fuzzy Heronian mean (FHFHM) operator is an expression of the form

$$FHFHM (d_1, d_2, \ldots, d_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} d_i^p \otimes d_j^q \right)^{\frac{1}{p+q}} \tag{10}$$

Theorem 1: Assume that $p, q > 0$, and $d_j = (\mu_j, v_j)$ $(j = 1, 2, \ldots, n)$ is a set of Fermatean hesitant fuzzy numbers, then

$FHFHM (d_1, d_2, \ldots, d_n)$

$$= \cup_{\xi_i \in \mu_i, \xi_j \in \mu_j, \eta_i \in v_i, \eta_j \in v_j} \left\{ \left\{ \left[\left(1 - \prod_{i=1, j=i}^{n} \left(1 - \xi_i^{3p} \xi_j^{3q} \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right]^{1/3} \right\}, \right.$$

$$\left. \left\{ \left[1 - \left(1 - \prod_{i=1, j=i}^{n} \left(1 - \left(1 - \eta_i^3 \right)^p \left(1 - \eta_j^3 \right)^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right]^{1/3} \right\} \right\} \tag{11}$$

Proof: According to Definition 4, the following conclusions can be drawn.

$$d_i^p = \cup_{\xi_i \in \mu_i, \eta_i \in v_i} \left\{ \{ \xi_i^p \}, \left\{ \left[1 - \left(1 - \eta_i^3 \right)^p \right]^{1/3} \right\} \right\} \tag{12}$$

Then

$$d_i^p \otimes d_j^q = \cup_{\xi_i \in \mu_i, \xi_j \in \mu_j, \eta_i \in v_i, \eta_j \in v_j} \left\{ \{ \xi_i^p \xi_j^p \}, \left\{ \left[1 - \left(1 - \eta_i^3 \right)^p \left(1 - \eta_j^3 \right)^q \right]^{1/3} \right\} \right\} \tag{13}$$

So,

$$\sum_{i=1}^{n} \sum_{j=i}^{n} d_i^p \otimes d_j^q$$

$$= \cup_{\xi_i \in \mu_i, \xi_j \in \mu_j, \eta_i \in \nu_i, \eta_j \in \nu_j} \left\{ \left\{ \left[1 - \prod_{i=1,j=i}^{n} \left(1 - \xi_i^{3p} \xi_j^{3q} \right) \right]^{1/3} \right\}, \left\{ \left[\prod_{i=1,j=i}^{n} \left(1 - \left(1 - \eta_i^3 \right)^p \left(1 - \eta_j^3 \right)^q \right)^{\frac{2}{n(n+1)}} \right]^{1/3} \right\} \right\} \tag{14}$$

$$\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} h_i^p \otimes h_j^q$$

$$= \cup_{\xi_i \in \mu_i, \xi_j \in \mu_j, \eta_i \in \nu_i, \eta_j \in \nu_j} \left\{ \left\{ \left[1 - \prod_{i=1,j=i}^{n} \left(1 - \xi_i^{3p} \xi_j^{3q} \right)^{\frac{2}{n(n+1)}} \right]^{1/3} \right\}, \left\{ \left[\prod_{i=1,j=i}^{n} \left(1 - \left(1 - \eta_i^3 \right)^p \left(1 - \eta_j^3 \right)^q \right)^{\frac{2}{n(n+1)}} \right]^{1/3} \right\} \right\} \tag{15}$$

Therefore,

$FHFHM(d_1, d_2, \ldots, d_n)$

$$= \cup_{\xi_i \in \mu_i, \xi_j \in \mu_j, \eta_i \in \nu_i, \eta_j \in \nu_j} \left\{ \left\{ \left[\left(1 - \prod_{i=1,j=i}^{n} \left(1 - \xi_i^{3p} \xi_j^{3q} \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right]^{1/3} \right\}, \right.$$

$$\left. \left\{ \left[1 - \left(1 - \prod_{i=1,j=i}^{n} \left(1 - \left(1 - \eta_i^3 \right)^p \left(1 - \eta_j^3 \right)^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right]^{1/3} \right\} \right\} \tag{16}$$

The FHFHM operator satisfies the following excellent properties: idempotency, monotonicity, invariance, boundedness, and so on.

Property 1 (Idempotency): Let $d_j = (\mu_j, \nu_j)$ $(j = 1, 2, \ldots, n)$ be a Fermatean hesitant fuzzy set, if $d_j = d = \cup_{\xi \in \mu, \eta \in \nu} \{\{\xi\}, \{\eta\}\}$ holds permanently, $p > 0$, $q > 0$, then

$$FHFHM(d_1, d_2, \ldots, d_n) = FHFHM(d, d, \ldots, d) = d \tag{17}$$

Property 2 (Monotonicity): Let $d_i = \cup_{\xi_i \in \mu_i, \eta_i \in \nu_i} \{\{\xi_i\}, \{\eta_i\}\}$ $(i = 1, 2, \ldots, n)$ and $k_i = \cup_{\alpha_i \in s_i, \beta_i \in t_i} \{\{\alpha_i\}, \{\beta_i\}\}$ $(i = 1, 2, \ldots, n)$ be two Fermatean hesitant fuzzy sets. Then

$$FHFHM(h_1, h_2, \ldots, h_n) \leq FHFHM(k_1, k_2, \ldots, k_n) \tag{18}$$

when $\xi_i^3 \leq \alpha_i^3$ and $\eta_i^3 \geq \beta_i^3$ are established.

Proof: Since $\xi_i^3 \leq \alpha_i^3$ is a true statement, it can be inferred that

$$\xi_i^{3p}\xi_j^{3q} \leq \alpha_i^{3p}\alpha_j^{3q}$$

$$\Rightarrow \left[\prod_{i=1,j=i}^n \left(1 - \xi_i^{3p}\xi_j^{3q}\right)^{\frac{2}{n(n+1)}}\right]^{1/3} \geq \left[\prod_{i=1,j=i}^n \left(1 - \alpha_i^{3p}\alpha_j^{3q}\right)^{\frac{2}{n(n+1)}}\right]^{1/3}$$

$$\Rightarrow \left[1 - \prod_{i=1,j=i}^n \left(1 - \xi_i^{3p}\xi_j^{3q}\right)^{\frac{2}{n(n+1)}}\right]^{1/3} \leq \left[1 - \prod_{i=1,j=i}^n \left(1 - \alpha_i^{3p}\alpha_j^{3q}\right)^{\frac{2}{n(n+1)}}\right]^{1/3}$$

$$\Rightarrow \left[\left(1 - \prod_{i=1,j=i}^n \left(1 - \xi_i^{3p}\xi_j^{3q}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right]^{1/3} \leq \left[\left(1 - \prod_{i=1,j=i}^n \left(1 - \alpha_i^{3p}\alpha_j^{3q}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right]^{1/3}$$

$$\Rightarrow \cup_{\xi_i \in \mu_i, \xi_j \in \mu_j} \left\{\left[\left(1 - \prod_{i=1,j=i}^n \left(1 - \xi_i^{3p}\xi_j^{3q}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right]^{1/3}\right\}$$

$$\leq \cup_{\alpha_i \in s_i, \alpha_j \in s_j} \left\{\left[\left(1 - \prod_{i=1,j=i}^n \left(1 - \alpha_i^{3p}\alpha_j^{3q}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right]^{1/3}\right\} \tag{19}$$

Similarly, the non-membership degree can be concluded as follows:

$$\cup_{\eta_i \in v_i, \eta_j \in v_j} \left\{\left[1 - \left(1 - \prod_{i=1,j=i}^n \left(1 - \left(1 - \eta_i^3\right)^p \left(1 - \eta_j^3\right)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right]^{1/3}\right\}$$

$$\geq \cup_{\beta_i \in t_i, \beta_j \in t_j} \left\{\left[1 - \left(1 - \prod_{i=1,j=i}^n \left(1 - \left(1 - \beta_i\right)^p \left(1 - \beta_j\right)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right]^{1/3}\right\} \tag{20}$$

According to the comparison rules of Definition 3, it can be inferred that

$$FHFHM\left(d_1, d_2, \ldots, d_n\right) \leq FHFHM\left(k_1, k_2, \ldots, k_n\right) \tag{21}$$

Property 3 (Invariance): Assume $d_i = \cup_{\xi_i \in \mu_i, \eta_i \in v_i}\{\{\xi_i\}, \{\eta_i\}\}$ $(i = 1, 2, \ldots, n)$ is a Fermatean hesitant fuzzy set, then

$$FHFHM\left(d_1, d_2, \ldots, d_n\right) = FHFHM\left(\dot{d}_1, \dot{d}_2, \ldots, \dot{d}_n\right) \tag{22}$$

where $\left(\dot{d}_1, \dot{d}_2, \ldots, \dot{d}_n\right)$ is any permutation of $(d_1, d_2, \ldots, d_n)$.

Property 4 (Boundedness): Let $d_i = \cup_{\xi_i \in \mu_i, \eta_i \in v_i}\{\{\xi_i\}, \{\eta_i\}\}$ $(i = 1, 2, \ldots, n)$ be a Fermatean hesitant fuzzy set, and

$$d^- = \cup_{\xi_i \in \mu_i, \eta_i \in v_i}\{\{\min(\xi_i)\}, \{\max(\eta_i)\}\}$$

$$d^+ = \cup_{\xi_i \in \mu_i, \eta_i \in v_i}\{\{\max(\xi_i)\}, \{\min(\eta_i)\}\} \tag{23}$$

Then,

$$d^- \leq FHFHM\left(h_1, h_2, \ldots, h_n\right) \leq d^+ \tag{24}$$

Proof: Refer to the proof process of Properties 1–3, this is easily proven.

4 The Fermatean Hesitant Fuzzy Prioritized Heronian Mean Operator

Although the attribute weights need to be taken into account during the MADM process, the FHFHM operator mentioned above does not actually consider this effect. Therefore, the Fermatean hesitant fuzzy weighted Heronian mean (FHFWHM) operator is defined. Compared with the weight of attributes, the priority relationship between attributes is easier to obtain. This paper further combines the FHFWHM operator with PA operators and proposes a Fermatean hesitant fuzzy prioritized Heronian mean (FHFPHM) operator.

Definition 11: Let $d_j = (\mu_j, v_j)$ $(j = 1, 2, \ldots, n)$ be a Fermatean hesitant fuzzy set. The corresponding attribute weight is $w = (w_1, w_2, \ldots, w_n)^T$, and it satisfies $w_i > 0$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$.

Then

$$FHFWHM(h_1, h_2, \cdots, h_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} (w_i d_i)^p \otimes (w_j d_j)^q \right)^{\frac{1}{p+q}} \tag{25}$$

is termed as a Fermatean hesitant fuzzy weighted Heronian mean (FHFWHM) operator.

Definition 12: Let $d_j = (\mu_j, v_j)$ $(j = 1, 2, \ldots, n)$ be a Fermatean hesitant fuzzy set, $C = \{C_1, C_2, \ldots, C_n\}$ $(n \in N)$ represents the set of prioritized attributes, which meets $C_1 \succ C_2 \succ \ldots \succ C_n$. If C_j has a higher priority level than C_k for $j < k$, the FHFPHM operator is introduced as

$$FHFPHM(h_1, h_2, \cdots, h_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} \left(\frac{T_i}{\sum_{r=1}^{n} T_r} d_i \right)^p \otimes \left(\frac{T_j}{\sum_{r=1}^{n} T_r} d_j \right)^q \right)^{\frac{1}{p+q}} \tag{26}$$

where $w = \left(\frac{T_1}{\sum_{r=1}^{n} T_r}, \frac{T_2}{\sum_{r=1}^{n} T_r}, \ldots, \frac{T_n}{\sum_{r=1}^{n} T_r} \right)^T$, $T_r = \prod_{k=1}^{r-1} S(d_r)$, $T_1 = 1$ and $S(d_r)$, respectively, indicate the weights of the prioritized attributes and the comprehensive score of $d_r = (\mu_r, v_r)$ $(r = 1, 2, \ldots, n)$.

Theorem 2: Assume that $p, q > 0$, $d_j = (\mu_j, v_j)$ $(j = 1, 2, \ldots, n)$ is a Fermatean hesitant fuzzy set and the corresponding weight of the prioritized attributes is $w = \left(\frac{T_1}{\sum_{r=1}^{n} T_r}, \frac{T_2}{\sum_{r=1}^{n} T_r}, \ldots, \frac{T_n}{\sum_{r=1}^{n} T_r} \right)^T$, then

$FHFPHM(d_1, d_2, \ldots, d_n)$

$$= \cup_{\xi_i \in \mu_i, \xi_j \in \mu_j, \eta_i \in v_i, \eta_j \in v_j} \left\{ \left\{ \left[\left(1 - \prod_{i=1, j=i}^{n} \left(1 - \left(1 - (1 - \xi_i^3)^{\frac{T_i}{\sum_{r=1}^n T_r}} \right)^p \left(1 - (1 - \xi_j^3)^{\frac{T_j}{\sum_{r=1}^n T_r}} \right)^q \right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right]^{1/3} \right\}, \right.$$

$$\left. \left\{ \left[1 - \left(1 - \prod_{i=1, j=i}^{n} \left(1 - \left(1 - \eta_i^{\frac{3T_i}{\sum_{r=1}^n T_r}} \right)^p \left(1 - \eta_j^{\frac{3T_j}{\sum_{r=1}^n T_r}} \right)^q \right) \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right]^{1/3} \right\} \right\} \tag{27}$$

Proof: The proof process is similar to that of Theorem 1.

The FHFPHM operator still satisfies the following properties: idempotency, monotonicity, invariance, boundedness, and so on.

Property 5 (Idempotency): Assume $d_j = (\mu_j, v_j)$ $(j = 1, 2, \ldots, n)$ be a Fermatean hesitant fuzzy set. If $d_j = d = \cup_{\xi \in \mu, \eta \in v} \{\{\xi\}, \{\eta\}\}$ always holds true, $p > 0$, $q > 0$, then

$$FHFPHM(d_1, d_2, \ldots, d_n) = FHFPHM(d, d, \ldots, d) = d \tag{28}$$

Property 6 (Monotonicity): Let $d_i = \cup_{\xi_i \in \mu_i, \eta_i \in v_i} \{\{\xi_i\}, \{\eta_i\}\}$ $(i = 1, 2, \ldots, n)$ and $k_i = \cup_{\alpha_i \in s_i, \beta_i \in t_i} \{\{\alpha_i\}, \{\beta_i\}\}$ $(i = 1, 2, \ldots, n)$ be two Fermatean hesitant fuzzy sets. If $\xi_i^3 \leq \alpha_i^3$ and $\eta_i^3 \geq \beta_i^3$ are constant, then

$$FHFPHM(h_1, h_2, \ldots, h_n) \leq FHFPHM(k_1, k_2, \ldots, k_n) \tag{29}$$

Proof: Refer to the proof process of Property 2.

Property 7 (Invariance): Let $d_i = \cup_{\xi_i \in \mu_i, \eta_i \in v_i} \{\{\xi_i\}, \{\eta_i\}\}$ $(i = 1, 2, \ldots, n)$ be a Fermatean hesitant fuzzy set, then

$$FHFPHM\left(\dot{d}_1, \dot{d}_2, \ldots, \dot{d}_n\right) = FHFPHM(d_1, d_2, \ldots, d_n) \tag{30}$$

where $\left(\dot{d}_1, \dot{d}_2, \ldots, \dot{d}_n\right)$ is a permutation of $(d_1, d_2, \ldots, d_n)$.

Property 8 (Boundedness): Let $d_i = \cup_{\xi_i \in \mu_i, \eta_i \in v_i} \{\{\xi_i\}, \{\eta_i\}\}$ $(i = 1, 2, \ldots, n)$ be a Fermatean hesitant fuzzy set, and

$$d^- = \cup_{\xi_i \in \mu_i, \eta_i \in v_i} \{\{\min(\xi_i)\}, \{\max(\eta_i)\}\}$$

$$d^+ = \cup_{\xi_i \in \mu_i, \eta_i \in v_i} \{\{\max(\xi_i)\}, \{\min(\eta_i)\}\} \tag{31}$$

Then

$$d^- \leq FHFPHM(h_1, h_2, \ldots, h_n) \leq d^+ \tag{32}$$

Proof: Refer to the proof process of Property 4.

5 MADM Based on Fermatean Hesitant Information and Its Application

5.1 MADM Based on Fermatean Hesitant Information

Based on the FHFPHM operator, this section presents a Fermatean hesitant fuzzy decision-making method with attribute priorities. Let $A = (A_1, A_2, \ldots, A_m)$ be the alternatives and $G = \{G_1, G_2, \cdots, G_n\}$ be the attributes with priority relationship. If $G_1 \succ G_2 \succ \cdots \succ G_n$, then the priority relationship between attributes is decreasing. The expert group is denoted by $E = \{e_1, e_2, \cdots, e_p\}$, and they need to make an anonymous evaluation of the scheme $A_i \in A$ in terms of $G_j \in G$, which includes the degree of membership and non-membership. The Fermatean hesitant fuzzy matrix $d_{ij} = (\mu_{ij}, v_{ij})(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ can be obtained. Assume $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of the attribute G_j of the scheme A_i. The detailed calculation steps are as follows:

Step 1: The comprehensive score of the scheme $A_i \in A$ regarding $G_j \in G$ is calculated by Definition 5, then the attribute weights with priority relationships are calculated by Definition 12.

$$w = \left(\frac{T_1}{\sum\limits_{r=1}^{n} T_r}, \frac{T_2}{\sum\limits_{r=1}^{n} T_r}, \ldots, \frac{T_n}{\sum\limits_{r=1}^{n} T_r} \right)^T, \ T_r = \prod_{k=1}^{r-1} S(d_r), \ T_1 = 1 \tag{33}$$

Step 2: Using the FHFPHM operator, the Fermatean hesitant fuzzy preference information of alternative A_i is calculated as

$FHFPHM(d_{i1}, d_{i2}, \ldots, d_{in})$

$$= \bigcup_{\xi_i \in \mu_i, \xi_j \in \mu_j, \eta_i \in v_i, \eta_j \in v_j} \left\{ \left\{ \left[\left(1 - \prod_{i=1, j=i}^{n} \left(1 - \left(1 - \left(1 - \xi_i^3 \right)^{\frac{T_i}{\sum_{r=1}^{n} T_r}} \right)^p \left(1 - \left(1 - \xi_j^3 \right)^{\frac{T_j}{\sum_{r=1}^{n} T_r}} \right)^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right]^{1/3} \right\}, \right.$$

$$\left. \left\{ \left[1 - \left(1 - \prod_{i=1, j=i}^{n} \left(1 - \left(1 - \eta_i^{\frac{3T_i}{\sum_{r=1}^{n} T_r}} \right)^p \left(1 - \eta_j^{\frac{3T_j}{\sum_{r=1}^{n} T_r}} \right)^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right]^{1/3} \right\} \right\} \tag{34}$$

Step 3: The composite score of alternatives is calculated according to Definition 3.

$$S(d_i) = \frac{1 + \mu_i^3 - v_i^3}{2} \tag{35}$$

The rankings of alternatives are generated by the descending of the composite scores.

5.2 Numerical Example

This section applied the proposed Fermatean hesitant fuzzy MADM approach to the multi-sensor electronic reconnaissance decision-making problem. In this problem, the paper uses the multi-attribute information of the enemy's pulse repetition frequency, carrier frequency, power and pulse width measured by the Radar, ESM, infrared and other sensors to determine the type of the enemy's airborne platforms. However, in the complex electromagnetic environment, electronic reconnaissance equipment frequently experiences interference from airborne clutter and enemy signals during the measurement process, which leads to certain information uncertainty. This conforms to the characteristics of Fermatean hesitant fuzzy numbers. Therefore, it is appropriate to select FHFS to make relevant decisions.

Assume that there are four types of airborne platforms reported to the fusion center by each electronic reconnaissance sensor, and each of them has four types of attributes: G_1: Power; G_2: Carrier frequency; G_3: Pulse width; G_4: Pulse repetition frequency. The priority order of attributes is as follows: $C_1 \succ C_2 \succ C_3 \succ C_4$. According to the above attributes, the fusion center needs to make a multi-attribute judgment on these airborne platforms under an uncertain environment. Further, the electronic reconnaissance device measures the values of each attribute $G_{ij} \in G$ and forms a Fermatean hesitant fuzzy decision matrix in Table 1.

Table 1: The Fermatean hesitant fuzzy decision matrix

	G_1	G_2	G_3	G_4
A_1	{{0.4, 0.5}, {0.6}}	{{0.7, 0.9}, {0.1}}	{{0.6}, {0.2, 0.3}}	{{0.5, 0.6}, {0.3}}
A_2	{{0.3, 0.4}, {0.5}}	{{0.6, 0.7}, {0.3}}	{{0.7, 0.8}, {0.2}}	{{0.6}, {0.3, 0.5}}
A_3	{{0.5}, {0.2, 0.3}}	{{0.2, 0.3, 0.4}, {0.6}}	{{0.8}, {0.1}}	{{0.7}, {0.2, 0.3}}
A_4	{{0.6, 0.7}, {0.3}}	{{0.6}, {0.4}}	{{0.3, 0.4}, {0.6}}	{{0.4, 0.6}, {0.2, 0.3}}

Step 1: Calculate the comprehensive scores by Definition 5, and then use Definition 12 to calculate the attribute weights with priority relationships, as shown in Table 2.

Table 2: Prioritized weight vectors

	G_1	G_2	G_3	G_4
A_1	0.5055	0.2220	0.1704	0.1021
A_2	0.5120	0.2357	0.1476	0.1047
A_3	0.5126	0.2838	0.1160	0.0876
A_4	0.4680	0.2931	0.1688	0.0700

Step 2: Using the FHFPHM operator, the preference information of alternatives $A = (A_1, A_2, \ldots, A_m)$ is obtained in Table 3.

Table 3: Scores after FHFPHM operator aggregation

	A_1	A_2	A_3	A_4
$p = 0.01, q = 10$	{0.5108, 0.6892 }	{0.4272, 0.7656}	{0.4143, 0.6594}	{0.4812, 0.7017}
$p = q = 2$	{0.4563, 0.7481}	{0.3925, 0.7774}	{0.3847, 0.7246}	{0.4398, 0.7604}
$p = 2.1, q = 7$	{0.4979, 0.7089}	{0.4168, 0.7651}	{0.4050, 0.6681}	{0.4778, 0.7084}
$p = 3, q = 5$	{0.4918, 0.7199}	{0.4125, 0.7656}	{0.4012, 0.6783}	{0.4731, 0.7169}
$p = q = 5$	{0.5001, 0.7087}	{0.4174, 0.7571}	{0.4048, 0.6554}	{0.4819, 0.6969}
$p = 10, q = 0.01$	{0.5178, 0.6693}	{0.4262, 0.7302}	{0.4117, 0.5715}	{0.5035, 0.6318}

Step 3: According to Definition 5, all the alternatives are ranked and the results are listed in Table 4.

When $p = q = 2$, the scoring functions of A_1 and A_3 should be further compared since $S(A_1) = S(A_3) \cdot p(A_1) = \mu_1^3 + \nu_1^3 = 0.5137, p(A_3) = \mu_3^3 + \nu_3^3 = 0.4374$, obviously $p(A_1) \succ p(A_3)$, so the scheme A_1 is better than A_3. Further, as can be seen from Tables 3 and 4, the comprehensive scores and function values of each scheme may be different for the different values p and q. By analyzing the specific scores, it can be further found that when $p \le 2$, the best solution is A_1 while when $p > 2$, the best solution is A_3.

Table 4: Composite scores and scenario rankings

	A_1	A_2	A_3	A_4	
$p = 0.01, q = 10$	0.4030	0.3146	0.3922	0.3830	$A_1 \succ A_3 \succ A_4 \succ A_2$
$p = q = 2$	0.3382	0.2953	0.3382	0.3227	$A_1 \succ A_3 \succ A_4 \succ A_2$
$p = 2.1, q = 7$	0.3836	0.3123	0.3841	0.3768	$A_3 \succ A_1 \succ A_4 \succ A_2$
$p = 3, q = 5$	0.3729	0.3107	0.3762	0.3687	$A_3 \succ A_1 \succ A_4 \succ A_2$
$p = q = 5$	0.3846	0.3194	0.3924	0.3867	$A_3 \succ A_4 \succ A_1 \succ A_2$
$p = 10, q = 0.01$	0.4195	0.3440	0.4416	0.4377	$A_3 \succ A_4 \succ A_1 \succ A_2$

The results show that when the parameters p and q select different values, the best scheme will also change, but with the gradual increase of p and q, the rankings tend to be stable. Therefore, the parameters p and q should not be too large, which can effectively reflect the pros and cons of the schemes and simplify the operation process.

5.3 Comparative Analysis

To illustrate the rationality of the FHFPHM operator, this paper further compares it with the Fermatean fuzzy weighted average (FFWA) operator and the Fermatean fuzzy weighted power average (FFWPA) operator proposed by Senapati et al. [28], and the dual hesitant fuzzy geometric weighted Heronian mean (DHFGHM) operator developed by Qu et al. [61].

Definition 13 [28]: Let $d_j = (\mu_j, \nu_j)$ $(j = 1, 2, \ldots, n)$ be a Fermatean fuzzy set, then

$$FFWA(d_1, d_2, \ldots, d_n) = \left\{ \sum_{i=1}^{n} w_i \mu_i, \sum_{i=1}^{n} w_i \nu_i \right\} \tag{36}$$

$$FFWPA(d_1, d_2, \ldots d_n) = \left\{ \left(\sum_{i=1}^{n} w_i \mu_i^3 \right)^{\frac{1}{3}}, \left(\sum_{i=1}^{n} w_i \nu_i^3 \right)^{\frac{1}{3}} \right\} \tag{37}$$

Since the evaluation data in Table 1 also abides by the characteristics of dual hesitant fuzzy sets, the dual hesitant fuzzy geometric weighted Heronian mean (DHFGWHM) operator proposed by Qu et al. [61] can also be used for calculation. The formula is given as follows:

Definition 14 [61]: Let $d_i = \cup_{\xi_i \in h_i, \eta_i \in g_i} \{\{\xi_i\}, \{\eta_i\}\}$ $(i = 1, 2, \ldots, n)$ be a dual hesitant fuzzy set, then $DHFGWHM(d_1, d_2, \ldots, d_n)$

$$= \cup_{\xi_i \in h_i, \xi_j \in h_j, \eta_i \in g_i, \eta_j \in g_j} \left\{ \left\{ 1 - \left(1 - \prod_{i=1, j=i}^{n} \left(1 - (1 - \xi_i^{w_i})^p \left(1 - \xi_j^{w_j} \right)^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right\}, \right.$$
$$\left. \left\{ \left(1 - \prod_{i=1, j=i}^{n} \left(1 - (1 - (1 - \eta_i)^{w_i})^p \left(1 - (1 - \eta_j)^{w_j} \right)^q \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} \right\} \right\} \tag{38}$$

References [28] and [61] assume that the attribute weights are known. To facilitate the comparative analysis, the attribute weights calculated by the method proposed in this paper (as shown in Table 2) are used for calculation here. Based on the FFWA, FFWPA, and DHFGWHM operators, the information aggregation results for each scheme are obtained in Table 5.

Table 5: Comprehensive scores and their rankings under different types of aggregation operators

Operators	A_1	A_2	A_3	A_4	
FFWA	0.5578	0.5332	0.5426	0.5679	$A_4 \succ A_1 \succ A_3 \succ A_2$
FFWPA	0.5513	0.5473	0.5455	0.5705	$A_4 \succ A_1 \succ A_2 \succ A_3$
DHFGWHM (p = 0.01, q = 0)	0.4631	0.4784	0.4159	0.4840	$A_4 \succ A_2 \succ A_1 \succ A_3$
DHFGWHM (p = 2, q = 2)	0.5782	0.4776	0.4393	0.4886	$A_1 \succ A_4 \succ A_2 \succ A_3$
DHFGWHM (p = 2.1, q = 7)	0.4756	0.4769	0.4179	0.4846	$A_4 \succ A_2 \succ A_1 \succ A_3$
DHFGWHM (p = 3, q = 5)	0.4950	0.4761	0.4210	0.4855	$A_1 \succ A_4 \succ A_2 \succ A_3$
DHFGWHM (p = 5, q = 5)	0.4593	0.4740	0.4161	0.4843	$A_4 \succ A_2 \succ A_1 \succ A_3$
DHFGWHM (p = 10, q = 0.01)	0.4728	0.4686	0.4033	0.4814	$A_4 \succ A_1 \succ A_2 \succ A_3$

It can be seen from Table 5 that the results obtained by the method proposed in this paper are different from the results obtained in reference [28]. The main reason may lie in whether or not the interactions between the attributes are considered. Additionally, the results obtained in this paper are basically consistent with those in reference [61]. This paper not only considered the problem of attribute correlations during the ranking process but also used the score function and accuracy function to compare different parameters, which makes the results comprehensive and scientific. However, compared with dual hesitant fuzzy sets, the application range of the obtained method is much broader since it allows the cubic sum of the membership degree and non-membership degree to be less than 1. References [28] and [61] both assume that the attribute weights are known, but when solving practical problems, it is difficult to determine the weights of attributes, while the priority relationship between attributes is relatively easier to obtain. Therefore, the proposed method is consistent with actual situations, reasonable, and comprehensive.

6 Conclusion

In this paper, a novel approach is proposed to solve MADM problems, where the membership degree, non-membership degree and priority relationship are considered simultaneously. First, the paper introduced the Fermatean hesitant fuzzy Heronian mean (FHFHM) operator and Fermatean hesitant fuzzy weighted Heronian mean (FHFWHM) operator and discussed their related properties such as idempotency, monotonicity, boundedness. Then considering that the attribute priority is often easier to obtain than the attribute weight, the paper proposed a method of MADM based on the Fermatean hesitant fuzzy prioritized Heronian mean (FHFPHM) operator under the Fermatean fuzzy environment. This method effectively combines Fermatean fuzzy information and hesitant fuzzy information, improves the application scope of the membership degree and the non-membership degree, and reflects the personal preference and hesitancy degree of experts in the process of multi-attribute group decision-making. This makes the results of information aggregation accurate, scientific, and rational. Furthermore, the paper took the priority problems between attributes into consideration, which effectively improves the comprehensiveness of the results and makes them more scientific. Then a numerical example of multi-sensor electronic reconnaissance was implemented. The results demonstrate that the ordering of the alternatives may be different for the different values p and q in the FHFPHM operator. However, with the gradual increase of p and q, the rankings tend to be stable, thus p and q have some influence on the ranking results. To verify the rationality of

the FHFPHM operator, the paper conducted additional comparative analysis and compared it with the FFWA operator, the FFWPA operator, and the GHFGWHM operator. The reasons for obtaining different rankings using the different operators were briefly explained, which illustrate the effectiveness of the proposed method.

In future researches, researches can introduce more factors beyond the priority relationship based on the combination of FFS and HFS, such as distance measures, similarity measures, and correlation coefficient measures. Also, it is suggested to merge the MADM method with q-rung orthopair fuzzy sets, neutrosophic cubic numbers and soft rough sets to enrich and improve the theoretical system of FFSs. Additionally, we will focus on extending more types of aggregation operators and also apply the developed methodology to more fields, such as the performance evaluation of internal departments, supplier capability assessment in green supply chains and science-technology project assessment.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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A Appendix

IFS	Intuitionistic fuzzy set
PFS	Pythagorean fuzzy set
LDFS	Linear Diophantine fuzzy set
FFS	Fermatean fuzzy set
HFS	Hesitant fuzzy set
FHFS	Fermatean hesitant fuzzy set
VIKOR	VlseKriterijumska Optimizacija IKompromisno Resenje
FFWA	Fermatean fuzzy weighted average operator
FFWG	Fermatean fuzzy weighted geometric operator
MADM	Multi-attribute decision making
PA	Prioritized aggregation operator
FFHPWA	Fermatean fuzzy Hamacher priority weighted average operator
HHHPWG	Fermatean fuzzy Hamacher priority weighted geometric operator
HM	Heronian mean operator
FFHIWG	Fermatean fuzzy Hammas interactive weighted geometric operator
FFHIOWG	Fermatean fuzzy Hammas interactive ordered weighted geometric operator
FFHIHWG	Fermatean fuzzy Hammas interactive hybrid weighted geometric operator
PHFFDCIG	Probabilistic hesitant Fermatean fuzzy Dombi Choquet integral geometric operator
PHFFDCIA	Probabilistic hesitant Fermatean fuzzy Dombi Choquet integral average operator
TOPSIS	Technique for Order Preference by Similarity to an Ideal Solution

IVIFS	Interval-valued intuitionistic fuzzy set
CoCoSo	Combined compromise solution
NCN	Neutrosophic cubic number
FHFHM	Fermatean hesitant fuzzy Heronian mean operator
FHFWHM	Fermatean hesitant fuzzy weighted Heronian mean operator
FHFPHM	Fermatean hesitant fuzzy priority Heronian mean operator
FFWPA	Fermatean fuzzy weighted power average operator
DHFGWHM	Dual hesitant fuzzy geometric weighted Heronian mean operator