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Topological Characterization of Book Graph and Stacked Book Graph

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Abstract: Degree based topological indices are being widely used in computer-aided modeling, structural activity relations, and drug designing to predict the underlying topological properties of networks and graphs. In this work, we compute the certain important degree based topological indices like Randic index, sum connectivity index, ABC index, ABC_4 index, GA index and GA_5 index of Book graph B_n and Stacked book graph $B_{m,n}$. The results are analyzed by using edge partition, and the general formulas are derived for the above-mentioned families of graphs.

Keywords: Topological indices, Book graph B_n, Stacked book graph $B_{m,n}$, network.

1 Introduction

Graph theory is used as a tool for designing and modeling chemical structures, complex network, and modeling of daily-life problems. In recent years, chemical structures and pharmaceutical techniques have been rapidly developed. In this period of rapid technological development, a huge number of new crystalline materials, nanomaterial, and drugs are designed using computer-aided modeling techniques. Researchers have found the topological index to be an influential and valuable tool in the narrative of molecular or network structure. A non-empirical scientific amount which quantitates the molecular structure and its branching pattern is described as a topological index of the associated graph. The chemical graph theory put on the tools from graph theory to mathematical models of molecular singularities, which is helpful for the study of molecular modeling and molecular structure. This theory plays a vigorous role in the field of theoretical chemical sciences.

In this paper all molecular graphs are considered to be connected, finite, loopless and deprived of parallel edges. Let F be a graph with n vertices and m edges. The degree of a vertex is the number of vertices adjacent to q and is signified as $d(q)$. By these terminologies, certain topological indices are well-defined in the following way.

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The Randic index is the oldest degree based topological index and is signified as $\chi(F)$ and presented by Randic [Randic (1975)]. He proposed this index for calculating the degree of branching of the carbon-atom skeleton of saturated hydrocarbons. Li et al. [Li and Shi (2008)] gave a comprehensive survey of the Randic index.

Definition 1.1 For any molecular graph F, the Randic index is defined as

$$\chi(F) = \sum_{qr \in E(F)} \frac{1}{\sqrt{d_q d_r}}. \tag{1}$$

A variation of Randic connectivity index is the sum connectivity index [Zhou and Trinajstic (2009)].

Definition 1.2. For a molecular graph F, the sum connectivity index is defined as

$$S(F) = \sum_{qr \in E(F)} \frac{1}{\sqrt{d_q + d_r}}. \tag{2}$$

Estrada et al. [Estrada, Torres and Rodriguez (1998)] proposed a degree based topological index of graphs, which is said to be the atom-bond connectivity index. Further, he proposed the atom-bond connectivity index of branched alkanes [Estrada (2008)]. For the atom-bond connectivity index several upper bounds for graphs are established and also studied in the context of the connected graph, and bicyclic chemical graphs [Chen, Liu and Guo (2012); Chen and Guo (2012); Xing, Zhou and Dong (2011)].

Definition 1.3. Let F be a molecular graph; then ABC index is defined as

$$ABC(F) = \sum_{qr \in E(F)} \sqrt{\frac{d_q + d_r - 2}{d_q d_r}}. \tag{3}$$

The geometric-arithmetic index is associated with a variation of physiochemical properties. It can be used as a possible tool for QSPR/QSAR research. Vukicevic et al. [Vukicevic and Furtula (2009)] introduced the geometric-arithmetic (GA) index.

Definition 1.4. Let F be a molecular graph, then geometric-arithmetic index is defined as

$$GA(F) = \sum_{qr \in E(F)} \frac{2\sqrt{d_q d_r}}{d_q + d_r} \tag{4}$$

Ghorbani et al. [Ghorbani and Hosseinzadeh (2010)] presented the fourth ABC index.

Definition 1.5. Let F be a molecular graph; then ABC_4 index is defined as

$$ABC_4(F) = \sum_{qr \in E(F)} \sqrt{\frac{S_q + S_r - 2}{S_q S_r}} \tag{5}$$

where S_q is the summation of degrees of all the neighbors of a vertex q in F.

Recently Graovac et al. [Graovac, Ghorbani and Hosseinzadeh (2011)] proposed the fifth GA index, which is defined below.

Definition 6. Let F be a molecular graph; then GA_5 index is defined as

$$GA_5(F) = \sum_{qr \in E(F)} \frac{2\sqrt{S_q S_r}}{(S_q + S_r)}. \tag{6}$$

Degree based topological indices are rigorously studied for nanotubes, computer networks and many other chemical graphs, for recent development in literature [Idrees, Said, Rauf et al. (2017); Gao, Wu, Siddiqui et al. (2018); Idrees, Hussain and Sadiq (2018); Imran, Baig Rehman et al. (2018); Joan (2019)]. Some other interesting results about network analysis using topological indices can be found in Hayat et al. [Hayat and Imran (2014); Javaid and Cao (2018)].

2 Main results for Book graph

Book graph B_n is obtained by taking cartesian product of star graph S_{n+1} with a path of length two P_2, i.e., $B_n := S_{n+1} \square P_2$, as shown in Fig. 1. The degree based topological indices like Randic index, sum connectivity index, atom-bond connectivity index, geometric-arithmetic index, fourth atom-bond connectivity index, GA_5 index for Book graph B_n are computed in this section.

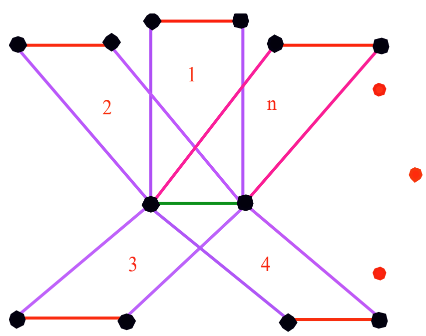

Figure 1: A representation of Book graph B_n

Table 1: Partition of edge created by the sum of adjacent vertices of every line

Edge of the form E_{d_q,d_r}	Sum of edges
$E_{(2,2)}$	n
$E_{(2,n+1)}$	$2n$
$E_{(n+1,n+1)}$	1

Theorem 2.1

Let B_n be the book graph. Then

i) The Randic index of Book graph is $\chi(B_n) = \frac{n}{2} + \frac{2n}{\sqrt{2(n+1)}} + \frac{1}{n+1}$.

ii) The Sum-connectivity index of Book graph (B_n) is $S(B_n) = \frac{n}{2} + \frac{2n}{\sqrt{3+n}} + \frac{1}{\sqrt{2(n+1)}}$.

iii) The Atom bond connectivity index of Book graph is $ABC(B_n) = 3n\sqrt{\frac{1}{2}} + \frac{\sqrt{2n}}{n+1}$.

iv) The geometric-arithmetic index (GA) of Book graph is $GA(B_n) = n+1+\frac{4n}{3+n}\sqrt{2(n+1)}$.

Proof. For the Book graph B_n , we divider the edges of B_n into edges of the form E_{d_q,d_r}, where qr is an edge. We develop the edges of the form $E_{(2,2)}$, $E_{(2,n+1)}$ and $E_{(n+1,n+1)}$. In Fig. 1, $E_{(2,2)}$, $E_{(2,n+1)}$ and $E_{(n+1,n+1)}$ are colored in red, lavender and green, respectively. The number of edges of these forms are given in the Tab. 1.

i) By Eq. (1), we have $\chi(B_n) = \sum_{qr\in E(F)} \frac{1}{\sqrt{d_q d_r}}$.

$$\chi(B_n) = |E_{(2,2)}| \sum_{qr\in E_{(2,2)}} \frac{1}{\sqrt{d_q d_r}} + |E_{(2,n+1)}| \sum_{qr\in E_{(2,n+1)}} \frac{1}{\sqrt{d_q d_r}} + |E_{(n+1,n+1)}| \sum_{qr\in E_{(n+1,n+1)}} \frac{1}{\sqrt{d_q d_r}}$$

$$= n\frac{1}{\sqrt{4}} + 2n\frac{1}{\sqrt{2(n+1)}} + \frac{1}{\sqrt{(n+1)^2}}$$

$$\chi(B_n) = \frac{n}{2} + \frac{2n}{\sqrt{2(n+1)}} + \frac{1}{n+1}.$$

ii) Using Eq. (2), $S(B_n) = \sum_{qr\in E(F)} \frac{1}{\sqrt{d_q + d_r}}$.

$$S(B_n) = |E_{(2,2)}| \sum_{qr\in E_{(2,2)}} \frac{1}{\sqrt{d_q + d_r}} + |E_{(2,n+1)}| \sum_{qr\in E_{(2,n+1)}} \frac{1}{\sqrt{d_q + d_r}} + |E_{(n+1,n+1)}| \sum_{qr\in E_{(n+1,n+1)}} \frac{1}{\sqrt{d_q + d_r}}$$

$$= n\frac{1}{\sqrt{4}} + 2n\frac{1}{\sqrt{2+n+1}} + \frac{1}{\sqrt{n+1+n+1}}$$

iii) We have, from Eq. (3), $ABC(B_n) = \sum_{qr\in E(F)} \sqrt{\frac{d_q + d_r - 2}{d_q d_r}}$.

Using Tab. 1, we get

$$ABC(B_n) = |E_{(2,2)}| \sum_{qr\in E_{(2,2)}} \sqrt{\frac{d_q + d_r - 2}{d_q d_r}} + |E_{(2,n+1)}| \sum_{qr\in E_{(2,n+1)}} \sqrt{\frac{d_q + d_r - 2}{d_q d_r}} + |E_{(n+1,n+1)}| \sum_{qr\in E_{(n+1,n+1)}} \sqrt{\frac{d_q + d_r - 2}{d_q d_r}}$$

$$= n\sqrt{\frac{2+2-2}{4}} + 2n\sqrt{\frac{2+n+1-2}{2n+2}} + \sqrt{\frac{n+1+n+1-2}{(n+1)^2}}$$

$$= n\sqrt{\frac{1}{2}} + 2n\sqrt{\frac{1}{2}} + \frac{\sqrt{2n}}{n+1}$$

$$ABC(B_n) = 3n\sqrt{\frac{1}{2}} + \frac{\sqrt{2n}}{n+1}.$$

iv) We know that $GA(B_n) = \sum_{qr\in E(F)} \frac{2\sqrt{d_q d_r}}{d_q + d_r}$.

$$GA(B_n) = |E_{(2,2)}| \sum_{qr\in E_{(2,2)}} 2\frac{\sqrt{d_q d_r}}{d_q + d_r} + |E_{(2,n+1)}| \sum_{qr\in E_{(2,n+1)}} 2\frac{\sqrt{d_q d_r}}{d_q + d_r} + |E_{(n+1,n+1)}| \sum_{qr\in E_{(n+1,n+1)}} 2\frac{\sqrt{d_q d_r}}{d_q + d_r}$$

$$= n\frac{2\sqrt{4}}{4} + 2n\frac{2\sqrt{2(n+1)}}{2+n+1} + \frac{2\sqrt{(n+1)^2}}{n+1+n+1}$$

$$= n + \frac{4n}{3+n}\sqrt{2(n+1)} + \frac{2(n+1)}{2(n+1)}$$

$$GA(B_n) = n + 1 + \frac{4n}{3+n}\sqrt{2(n+1)}.$$

Theorem 2.2

i) The fourth atom bond connectivity index (ABC_4) Book graph is

$$ABC_4(B_n) = n\frac{\sqrt{2(n+2)}}{n+3} + 2n\sqrt{\frac{2(2n+1)}{(n+3)(3n+1)}} + \frac{\sqrt{6n}}{3n+1}.$$

ii) The Fifth geometric arithmetic index (GA_5) of Book graph is

$$GA_5(B_n) = n + 1 + \frac{n\sqrt{(n+3)(3n+1)}}{n+1}$$

Proof. Consider the Book graph B_n. The edges of B_n can be divided into edges of form E_{d_q,d_r}, where qr is an edge. We develop the edges of the form $E_{(n+3,n+3)}$, $E_{(n+3,3n+1)}$ and $E_{(3n+1,3n+1)}$ that are shown in Tab. 2 given below, by evaluating sum of degrees of neighboring vertices.

Table 2: Partition of edges created by the sum of degrees of neighbors of the head-to-head vertices of every edge

Edge of the form E_{s_q, s_r}	Sum of edges
$E_{(n+3,n+3)}$	n
$E_{(n+3,3n+1)}$	$2n$
$E_{(3n+1,3n+1)}$	1

By Eq. (5), we have $ABC_4(B_n) = \sum_{qr\in E(F)} \sqrt{\frac{S_q + S_r - 2}{S_q S_r}}$.

$$ABC_4(B_n) = \left|E_{(n+3,n+3)}\right| \sum_{qr\in E_{(n+3,n+3)}} \sqrt{\frac{S_q+S_r-2}{S_qS_r}} + \left|E_{(n+3,3n+1)}\right| \sum_{qr\in E_{(n+3,3n+1)}} \sqrt{\frac{S_q+S_r-2}{S_qS_r}}$$
$$+ \left|E_{(3n+1,3n+1)}\right| \sum_{qr\in E_{(3n+1,3n+1)}} \sqrt{\frac{S_q+S_r-2}{S_qS_r}}$$

From Tab. 2, we get

$$ABC(B_n) = \sqrt{\frac{n+3+n+3-2}{(n+3)^2}} + 2n\sqrt{\frac{n+3+3n+1-2}{(n+3)(3n+1)}} + \sqrt{\frac{3n+1+3n+1-2}{(3n+1)^2}}$$
$$= n\frac{\sqrt{2n+4}}{n+3} + 2n\sqrt{\frac{4n+2}{(n+3)(3n+1)}} + \frac{\sqrt{6n}}{3n+1}$$
$$ABC_4(B_n) = n\frac{\sqrt{2(n+2)}}{n+3} + 2n\sqrt{\frac{2(2n+1)}{(n+3)(3n+1)}} + \frac{\sqrt{6n}}{3n+1}.$$

ii) We know that $GA_5(F) = \sum_{qr\in E(F)} \frac{2\sqrt{S_qS_r}}{S_q+S_r}$.

$$GA_5(B_n) = \left|E_{(n+3,n+3)}\right| \sum_{qr\in E_{(n+3,n+3)}} \frac{2\sqrt{S_qS_r}}{S_q+S_r} + \left|E_{(n+3,3n+1)}\right| \sum_{qr\in E_{(n+3,3n+1)}} \frac{2\sqrt{S_qS_r}}{S_q+S_r} + \left|E_{(3n+1,3n+1)}\right| \sum_{qr\in E_{(3n+1,3n+1)}} \frac{2\sqrt{S_qS_r}}{S_q+S_r}$$

Substituting the values from Tab. 2,

$$GA_5(B_n) = n\,\frac{2\sqrt{(n+3)^2}}{n+3+n+3} + 2n\,\frac{2\sqrt{(n+3)(3n+1)}}{n+3+3n+1} + \frac{2\sqrt{(3n+1)^2}}{3n+1+3n+1}$$
$$= \frac{2n(n+3)}{2(n+3)} + \frac{4n\sqrt{(n+3)(3n+1)}}{4n+4} + \frac{2(3n+1)}{6n+2}$$

$$= n + \frac{n\sqrt{(n+3)(3n+1)}}{n+1} + 1$$

$$GA_5(B_n) = n + 1 + \frac{n\sqrt{(n+3)(3n+1)}}{n+1}$$

and we get the desired result.

3 Main results for Stacked book graph

The Stacked book graph of order (m, n), denoted by $B_{m,n}$ is the Cartesian product $S_{m+1} \square P_n$ of graphs, where S_m is a star graph and P_n is the path graph on n points. It is therefore the graph resultant to the edges of n copies of an m-page book stacked one on top of another and is a generalization of the book graph. The degree based topological indices like Randic, sum, atom-bond, geometric-arithmetic, fourth atom-bond, fifth geometric-arithmetic connectivity index for Stacked book graph $B_{m,n}$ are computed in this section. These graph invariants are computed by edge partition based on degrees of end vertices of edges as given in Tab. 3 below.

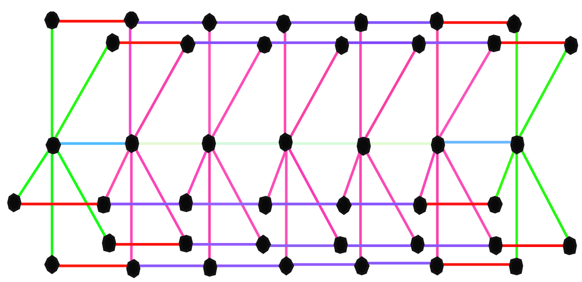

Figure 2: A representation of Stacked book graph $B_{5,7}$

Table 3: Edge partition created by sum of adjacent vertices of every line

Edge of the form E_{d_q,d_r}	Sum of edges
$E_{(2,3)}$	$2m$
$E_{(2,m+1)}$	$2m$
$E_{(3,3)}$	$m(n-3)$
$E_{(3,m+2)}$	$m(n-2)$
$E_{(m+1,m+2)}$	2
$E_{(m+2,m+2)}$	$(n-3)$

Theorem 3.1

i) The Randic connectivity index of Stacked book graph is

$$\chi(B_{m,n}) = 2m(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2(m+1)}}) + \frac{mn}{3} - m + \frac{m(n-2)}{\sqrt{3(m+2)}} + \frac{2}{\sqrt{(m+1)(m+2)}} + \frac{n-3}{m+2},$$

where $n \geq 6$.

ii) The sum connectivity index of Stacked book graph is $S(B_{m,n}) = 2m\left(\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{3+m}}\right) + \frac{m(n-3)}{\sqrt{6}} + \frac{m(n-2)}{\sqrt{5+m}} + \frac{2}{\sqrt{2m+3}} + \frac{n-3}{\sqrt{2m+4}}$, where $n \geq 6$.

iii) The Atom bond connectivity index (ABC) of Stacked book graph is

$$ABC(B_{m,n}) = 4m\sqrt{\frac{1}{2}} + \frac{2mn}{3} - 2m + m(n-2)\sqrt{\frac{3+m}{3(m+2)}} + 2\sqrt{\frac{2m+1}{(m+1)(m+2)}} + (n-3)\sqrt{\frac{2(m+1)}{(m+2)}}$$

iv) The Geometric-Arithmetic index (GA) o stacked book graph is

$$GA(B_{m,n}) = \frac{4m\sqrt{6}}{5} + \frac{4m\sqrt{2(m+1)}}{3+m} + m(n-3) + \frac{2m(n-2)}{(m+5)}\sqrt{2m+6} + \frac{4\sqrt{(m+1)(m+2)}}{2m+3} + (n-3)$$

Proof. Consider the Stacked book graph $B_{m,n}$. The edges of $B_{m,n}$ can be partitioned into edges of the form E_{d_q,d_r}, where qr is an edge. In $B_{m,n}$, We develop the edges of the form $E_{(2,3)}$, $E_{(2,m+1)}$, $E_{(3,3)}$, $E_{(3,m+2)}$, $E_{(m+1,m+2)}$ and $E_{(m+2,m+2)}$. In Fig. 2, $E_{(2,3)}$, $E_{(2,m+1)}$, $E_{(3,3)}$, $E_{(3,m+2)}$, $E_{(m+1,m+2)}$ and $E_{(m+2,m+2)}$ are colored by red, bright green, lavender, pink, navy blue, and silver. The sum of edges of these forms is given in the Tab. 3.

We know that $\chi(F) = \sum_{qr \in E(F)} \frac{1}{\sqrt{d_q d_r}}$.

$$\chi(B_{m,n}) = \left|E_{(2,3)}\right| \sum_{qr \in E_{(2,3)}} \frac{1}{\sqrt{d_q d_r}} + \left|E_{(2,m+1)}\right| \sum_{qr \in E_{(2,m+1)}} \frac{1}{\sqrt{d_q d_r}} + \left|E_{(3,3)}\right| \sum_{qr \in E_{(3,3)}} \frac{1}{\sqrt{d_q d_r}}$$

$$+ \left|E_{(3,m+2)}\right| \sum_{qr \in E_{(3,m+2)}} \frac{1}{\sqrt{d_q d_r}} + \left|E_{(m+1,m+2)}\right| \sum_{qr \in E_{(m+1,m+2)}} \frac{1}{\sqrt{d_q d_r}} + \left|E_{(m+2,m+2)}\right| \sum_{qr \in E_{(m+2,m+2)}} \frac{1}{\sqrt{d_q d_r}}$$

Substituting the values from Tab. 3, we get,

$$\chi(B_{m,n}) = \frac{2m}{\sqrt{2.3}} + \frac{2m}{\sqrt{2(m+1)}} + \frac{m(n-3)}{\sqrt{3.3}} + \frac{m(n-2)}{\sqrt{3(m+2)}} + \frac{2}{\sqrt{(m+1)(m+2)}} + \frac{n-3}{\sqrt{(m+2)(m+2)}}$$

$$=\frac{2m}{\sqrt{6}}+\frac{2m}{\sqrt{2(m+1)}}+\frac{m(n-3)}{\sqrt{9}}+\frac{m(n-2)}{\sqrt{3(m+2)}}+\frac{2}{\sqrt{(m+1)(m+2)}}+\frac{n-3}{\sqrt{(m+2)^2}}$$

$$\chi(B_{m,n})=2m\left(\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{2(m+1)}}\right)+\frac{mn}{3}-m+\frac{m(n-2)}{\sqrt{3(m+2)}}+\frac{2}{\sqrt{(m+1)(m+2)}}+\frac{n-3}{m+2}.$$

ii) From Eq. (2), we have $S(B_{m,n})=\sum_{qr\in E(F)}\frac{1}{\sqrt{d_q+d_r}}$.

$$S(B_{m,n})=\left|E_{(2,3)}\right|\sum_{qr\in E_{(2,3)}}\frac{1}{\sqrt{d_q+d_r}}+\left|E_{(2,m+1)}\right|\sum_{qr\in E_{(2,m+1)}}\frac{1}{\sqrt{d_q+d_r}}+\left|E_{(3,3)}\right|\sum_{qr\in E_{(3,3)}}\frac{1}{\sqrt{d_q+d_r}}$$
$$+\left|E_{(3,m+2)}\right|\sum_{qr\in E_{(3,m+2)}(F)}\frac{1}{\sqrt{d_q+d_r}}+\left|E_{(m+1,m+2)}\right|\sum_{qr\in E_{(m+1,m+2)}(F)}\frac{1}{\sqrt{d_q+d_r}}+\left|E_{(m+2,m+2)}\right|\sum_{qr\in E_{(m+2,m+2)}(F)}\frac{1}{\sqrt{d_q+d_r}}$$

Substituting the values from Tab. 3, we get

$$S(B_{m,n})=\frac{2m}{\sqrt{2+3}}+\frac{2m}{\sqrt{2+m+1}}+\frac{m(n-3)}{\sqrt{3+3}}+\frac{m(n-2)}{\sqrt{3+m+2}}+\frac{2}{\sqrt{m+1+m+2}}+\frac{n-3}{\sqrt{m+2+m+2}}$$
$$=\frac{2m}{\sqrt{5}}+\frac{2m}{\sqrt{3+m}}+\frac{m(n-3)}{\sqrt{6}}+\frac{m(n-2)}{\sqrt{5+m}}+\frac{2}{\sqrt{2m+3}}+\frac{n-3}{\sqrt{2m+4}}$$
$$S(B_{m,n})=2m\left(\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{3+m}}\right)+\frac{m(n-3)}{\sqrt{6}}+\frac{m(n-2)}{\sqrt{5+m}}+\frac{2}{\sqrt{2m+3}}+\frac{n-3}{\sqrt{2m+4}}.$$

iii) We know that $ABC(F)=\sum_{qr\in E(F)}\sqrt{\frac{d_q+d_r-2}{d_qd_r}}$.

$$ABC(B_{m,n})=\left|E_{(2,3)}\right|\sum_{qr\in E_{(2,3)}}\sqrt{\frac{d_q+d_r-2}{d_qd_r}}+\left|E_{(2,m+1)}\right|\sum_{qr\in E_{(2,m+1)}}\sqrt{\frac{d_q+d_r-2}{d_qd_r}}+\left|E_{(3,3)}\right|\sum_{qr\in E_{(3,3)}}\sqrt{\frac{d_q+d_r-2}{d_qd_r}}$$
$$+\left|E_{(3,m+2)}\right|\sum_{qr\in E_{(3,m+2)}}\sqrt{\frac{d_q+d_r-2}{d_qd_r}}+\left|E_{(m+1,m+2)}\right|\sum_{qr\in E_{(m+1,m+2)}}\sqrt{\frac{d_q+d_r-2}{d_qd_r}}+\left|E_{(m+2,m+2)}\right|\sum_{qr\in E_{(m+2,m+2)}}\sqrt{\frac{d_q+d_r-2}{d_qd_r}}$$

From Tab. 3, we get,

$$ABC(B_{m,n})=2m\sqrt{\frac{2+3-2}{2.3}}+2m\sqrt{\frac{2+m+1-2}{2(m+1)}}+m(n-3)\sqrt{\frac{3+3-2}{3.3}}+m(n-2)\sqrt{\frac{3+m+2-2}{3(m+2)}}$$
$$+2\sqrt{\frac{m+1+m+2-2}{(m+1)(m+2)}}+(n-3)\sqrt{\frac{m+2+m+2-2}{(m+2)(m+2)}}.$$

$$=2m\sqrt{\frac{1}{2}}+2m\sqrt{\frac{1}{2}}+m(n-3)\frac{2}{3}+m(n-2)\sqrt{\frac{3+m}{3(m+2)}}+2\sqrt{\frac{2m+1}{(m+1)(m+2)}}+(n-3)\sqrt{\frac{2(m+1)}{(m+2)^2}}$$

$$ABC(B_{m,n})=4m\sqrt{\frac{1}{2}}+\frac{2mn}{3}-2m+m(n-2)\sqrt{\frac{3+m}{3(m+2)}}+2\sqrt{\frac{2m+1}{(m+1)(m+2)}}+(n-3)\sqrt{\frac{2(m+1)}{(m+2)}}$$

iv) We know that $GA(F)=\sum_{qr\in E(F)}\frac{2\sqrt{d_q d_r}}{d_q+d_r}$.

Using edge partition given in Tab. 3, we have

$$GA(B_{m,n})=|E_{(2,3)}|\sum_{qr\in E_{(2,3)}}\frac{2\sqrt{d_q d_r}}{d_q+d_r}+|E_{(2,m+1)}|\sum_{qr\in E_{(2,m+1)}}\frac{2\sqrt{d_q d_r}}{d_q+d_r}+|E_{(3,3)}|\sum_{qr\in E_{(3,3)}}\frac{2\sqrt{d_q d_r}}{d_q+d_r}$$

$$+|E_{(3,m+2)}|\sum_{qr\in E_{(3,m+2)}(F)}\frac{2\sqrt{d_q d_r}}{d_q+d_r}+|E_{(m+1,m+2)}|\sum_{qr\in E_{(m+1,m+2)}}\frac{2\sqrt{d_q d_r}}{d_q+d_r}+|E_{(m+2,m+2)}|\sum_{qr\in E_{(m+2,m+2)}}\frac{2\sqrt{d_q d_r}}{d_q+d_r}.$$

$$=2m\frac{2\sqrt{2.3}}{2+3}+2m\frac{2\sqrt{2(m+1)}}{2+m+1}+m(n-3)\frac{2\sqrt{3.3}}{3+3}+m(n-2)\frac{2\sqrt{3(m+2)}}{3+m+2}$$

$$+2\frac{2\sqrt{(m+1)(m+2)}}{m+1+m+2}+(n-3)\frac{2\sqrt{(m+2)(m+2)}}{m+2+m+2}.$$

$$=2m\frac{2\sqrt{6}}{5}+2m\frac{2\sqrt{2m+2}}{3+m}+m(n-3)\frac{2\sqrt{9}}{6}+m(n-2)\frac{2\sqrt{3m+6}}{m+5}+2\frac{2\sqrt{(m+1)(m+2)}}{2m+3}+(n-3)\frac{2(m+2)}{2m+4}.$$

After simplification, we have

$$GA(B_{m,n})=\frac{4m}{5}\sqrt{6}+4m\frac{\sqrt{2m+2}}{3+m}+m(n-3)+\frac{2m(n-2)}{m+5}\sqrt{3m+6}+\frac{4\sqrt{(m+1)(m+2)}}{2m+3}+(n-3)$$

Theorem 3.2

The Fourth atom bond connectivity index (ABC_4) and fifth geometric-arithmetic index (GA_5) of Stacked book graph $B_{m,n}$ are given as

$$ABC_4(B_{m,n})=2m\left(\sqrt{\frac{2m+9}{(m+4)(m+7)}}+\sqrt{\frac{6m+8}{(m+7)(5m+3)}}+\sqrt{\frac{2m+13}{(m+7)(m+8)}}+2\sqrt{\frac{m+1}{(m+4)(3m+2)}}\right)$$

$$+2\left(\sqrt{\frac{8m+3}{(3m+2)(5m+3)}}+\sqrt{\frac{10m+4}{(5m+2)(5m+4)}}\right)+m(n-5)\frac{\sqrt{2m+14}}{m+8}$$

$$+m(n-4)\sqrt{\frac{6m+10}{(m+8)(5m+4)}}+n-5\frac{\sqrt{10m+6}}{5m+4}$$, where $n\geq 6$.

$$GA_5(B_{m,n})=4m\left(\frac{\sqrt{(m+4)(m+7)}}{2m+11}+\frac{\sqrt{(m+7)(m+8)}}{2m+15}\right)+2m\left(\frac{\sqrt{(m+4)(3m+2)}}{2m+3}+\frac{\sqrt{(m+7)(5m+3)}}{3m+5}\right)$$

$$+4\left(\frac{\sqrt{(3m+2)(5m+3)}}{8m+5}+\frac{\sqrt{(5m+3)(5m+4)}}{10m+7}\right)+m(n-5)+\frac{m(n-4)\sqrt{(m+8)(5m+4)}}{3m+6}$$
$+(n-5)$, where $n \geq 6$.

Proof. Consider the Stacked book graph $B_{m,n}$. The edges of $B_{m,n}$ can be partitioned into edges of the form E_{d_q,d_r}, where qr is an edge. In $B_{m,n}$. We develop the edges of the form $E_{(m+4,m+7)}$, $E_{(m+4,3m+2)}$, $E_{(m+7,5m+3)}$, $E_{(3m+2,5m+3)}$, $E_{(m+7,m+8)}$, $E_{(m+8,m+8)}$, $E_{(m+8,5m+4)}$, $E_{(5m+3,5m+4)}$ and $E_{(5m+4,5m+4)}$ that are shown in Tab. 4.

Table 4: Edge partition created by the sum of degrees of neighbors of the head-to-head vertices of every edge

Edge of the form E_{d_q,d_r}	Number of edges
$E_{(m+4,m+7)}$	$2m$
$E_{(m+4,3m+2)}$	$2m$
$E_{(m+7,5m+3)}$	$2m$
$E_{(3m+2,5m+3)}$	2
$E_{(m+7,m+8)}$	$2m$
$E_{(m+8,m+8)}$	$m(n-5)$
$E_{(m+8,5m+4)}$	$m(n-4)$
$E_{(5m+3,5m+4)}$	2
$E_{(5m+4,5m+4)}$	$(n-5)$

We know that $ABC_4(F)=\sum_{qr\in E(F)}\sqrt{\frac{S_q+S_r-2}{S_qS_r}}$.

Using the edge partition given in Tab. 4, we have

$$ABC_4(B_{m,n})=\left|E_{(m+4,m+7)}\right|\sum_{qr\in E_{(m+4,m+7)}}\sqrt{\frac{S_q+S_r-2}{S_qS_r}}+\left|E_{(m+4,3m+2)}\right|\sum_{qr\in E_{(m+4,3m+2)}}\sqrt{\frac{S_q+S_r-2}{S_qS_r}}$$

$$+\left|E_{(m+7,5m+3)}\right|\sum_{qr\in E_{(m+7,5m+3)}}\sqrt{\frac{S_q+S_r-2}{S_qS_r}}+\left|E_{(3m+2,5m+3)}\right|\sum_{qr\in E_{(3m+2,5m+3)}}\sqrt{\frac{S_q+S_r-2}{S_qS_r}}$$

$$+\left|E_{(m+7,m+8)}\right|\sum_{qr\in E_{(m+7,m+8)}}\sqrt{\frac{S_q+S_r-2}{S_qS_r}}+\left|E_{(m+8,m+8)}\right|\sum_{qr\in E_{(m+8,m+8)}}\sqrt{\frac{S_q+S_r-2}{S_qS_r}}$$

$$+\left|E_{(m+8,5m+4)}\right|\sum_{qr\in E_{(m+8,5m+4)}}\sqrt{\frac{S_q+S_r-2}{S_qS_r}}+\left|E_{(5m+3,5m+4)}\right|\sum_{qr\in E_{(5m+3,5m+4)}}\sqrt{\frac{S_q+S_r-2}{S_qS_r}}$$

$$+\left|E_{(5m+4,5m+4)}\right|\sum_{qr\in E_{(5m+4,5m+4)}}\sqrt{\frac{S_q+S_r-2}{S_qS_r}}.$$

$$=2m\sqrt{\frac{m+4+m+7-2}{(m+4)(m+7)}}+2m\sqrt{\frac{m+4+3m+2-2}{(m+4)(3m+2)}}+2m\sqrt{\frac{m+7+5m+3-2}{(m+7)(5m+3)}}$$

$$+2\sqrt{\frac{3m+2+5m+3-2}{(3m+2)(5m+3)}}+2m\sqrt{\frac{m+7+m+8-2}{(m+7)(m+8)}}+m(n-5)\sqrt{\frac{m+8+m+8-2}{(m+8)(m+8)}}$$

$$+m(n-4)\sqrt{\frac{m+8+5m+4-2}{(m+8)(5m+4)}}+2\sqrt{\frac{5m+2+5m+4-2}{(5m+2)(5m+4)}}+n-5\sqrt{\frac{5m+4+5m+4-2}{(5m+4)(5m+4)}}.$$

After further simplification, we get

$$ABC_4(B_{m,n})=2m\left(\sqrt{\frac{2m+9}{(m+4)(m+7)}}+\sqrt{\frac{6m+8}{(m+7)(5m+3)}}+\sqrt{\frac{2m+13}{(m+7)(m+8)}}+2\sqrt{\frac{m+1}{(m+4)(3m+2)}}\right)$$

$$+2\left(\sqrt{\frac{8m+3}{(3m+2)(5m+3)}}+\sqrt{\frac{10m+4}{(5m+2)(5m+4)}}\right)+m(n\text{-}5)\frac{\sqrt{2m+14}}{m+8}$$

$$+m(n\text{-}4)\sqrt{\frac{6m+10}{(m+8)(5m+4)}}+n\text{-}5\frac{\sqrt{10m+6}}{5m+4}.$$

which yields the required result.

We know that $GA_5(F)=\sum_{qr\in E(F)}\frac{2\sqrt{S_qS_r}}{S_q+S_r}$.

$$GA_5(B_{m,n})=\left|E_{(m+4,m+7)}\right|\sum_{qr\in E_{(m+4,m+7)}}\frac{2\sqrt{S_qS_r}}{S_q+S_r}+\left|E_{(m+4,3m+2)}\right|\sum_{qr\in E_{(m+4,3m+2)}}\frac{2\sqrt{S_qS_r}}{S_q+S_r}$$

$$+\left|E_{(m+7,5m+3)}\right|\sum_{qr\in E_{(m+7,5m+3)}}\frac{2\sqrt{S_qS_r}}{S_q+S_r}+\left|E_{(3m+2,5m+3)}\right|\sum_{qr\in E_{(3m+2,5m+3)}}\frac{2\sqrt{S_qS_r}}{S_q+S_r}+\left|E_{(m+7,m+8)}\right|\sum_{qr\in E_{(m+7,m+8)}}\frac{2\sqrt{S_qS_r}}{S_q+S_r}$$

$$+\left|E_{(m+8,m+8)}\right|\sum_{qr\in E_{(m+8,m+8)}}\frac{2\sqrt{S_qS_r}}{S_q+S_r}+\left|E_{(m+8,5m+4)}\right|\sum_{qr\in E_{(m+8,5m+4)}}\frac{2\sqrt{S_qS_r}}{S_q+S_r}+\left|E_{(5m+3,5m+4)}\right|\sum_{qr\in E_{(5m+3,5m+4)}}\frac{2\sqrt{S_qS_r}}{S_q+S_r}$$

$$+\left|E_{(5m+4,5m+4)}\right|\sum_{qr\in E_{(5m+4,5m+4)}}\frac{2\sqrt{S_qS_r}}{S_q+S_r}.$$

Again substituting the values from Tab. 4, we get

$$GA_5(B_{m,n})=4m\left(\frac{\sqrt{(m+4)(m+7)}}{2m+11}+\frac{\sqrt{(m+7)(m+8)}}{2m+15}\right)+2m\left(\frac{\sqrt{(m+4)(3m+2)}}{2m+3}+\frac{\sqrt{(m+7)(5m+3)}}{3m+5}\right)$$

$$+4\left(\frac{\sqrt{(3m+2)(5m+3)}}{8m+5}+\frac{\sqrt{(5m+3)(5m+4)}}{10m+7}\right)+m(n-5)+\frac{m(n-4)\sqrt{(m+8)(5m+4)}}{3m+6}$$

$$+(n-5).$$

4 Conclusion

In this work, we analyzed the graph-theoretic invariants of certain networks dependent upon connectivity of the nodes like ABC index, ABC_4 index, Randic connectivity index, sum connectivity index, GA index and GA_5 index of Book graph B_n and Stacked book graph $B_{m,n}$. The results can be applied to investigate the topological properties of the computer network and structure-activity relation where the graph correspond to book graph and stacked book graph. We derived the general formulas of various degree based topological indices and computed the results analytically for the above-mentioned families of the graph. These graph-theoretic invariants depend upon connectivity of the nodes of the graph. These results can be employed to further understand the topological properties of graphs with graph-theoretic properties.

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