# Some Geometric Invariants of Pseudo-Spherical Evolutes in the Hyperbolic 3-Space 

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#### Abstract

In this paper, we study the pseudo-spherical evolutes of curves in three dimensional hyperbolic space. We use techniques from singularity theory to investigate the singularities of pseudo-spherical evolutes and establish some relationships between singularities of these curves and geometric invariants of curves under the action of the Lorentz group. Besides, we defray with illustration some computational examples in support our main results.


Keywords: Pseudo-spherical evolutes, evolute curves, hyperbolic 3-space.

## 1 Introduction

The study of the extrinsic differential geometry of submanifolds in hyperbolic space is of special interest in relativity theory. On the other hand, the evolute of a space curve in Euclidean differential geometry is defined to be the locus of the center of osculating circles of the curve. The principal tools for the study of evolutes are the Frenet-Serret formulae and the distance squared functions on curves. In our case, we adopt a special pseudoorthogonal frame in $H_{+}^{3}(-1)$. We also define hyperbolic height functions on hyperbolic space curves. With the aid of a bit of singularity theory of hyperbolic height functions, we study singularities of evolutes and establish the relation between these singularities and hyperbolic invariants of the original curve. Torii studied other objects related to hyperbolic plane curves by using a similar framework and method as used [Torii (1999)]. Here, for convenience, we concentrate only on the hyperbolic evolutes of space curves. Similar descriptions for Euclidean plane curves are found in Bruce et al. [Bruce and Giblin (1992)]. For a curve $\gamma \in H_{+}^{3} \subset E_{1}^{4}$, we choose the unit tangent vector field $\mathbf{T}(s)$ and another normal vector fields $\mathbf{E}_{1}(s)$ and $\mathbf{E}_{2}(s)$ along $\gamma(s)$. As a result, we construct a pseudoorthonormal frame $\left\{\gamma(s), \mathbf{T}(s), \mathbf{E}_{1}(s), \mathbf{E}_{2}(s)\right\}$ along the curve $\gamma$. Also, we define two families of functions on a curve which are a timelike height function $H^{T}$ and a spacelike height function $H^{S}$. Differentiating these functions, we obtain two new invariants $\sigma_{H}$ and $\sigma_{D}$, whose properties are characterized by some conditions of derivation of $H^{T}$ and $H^{S}$. For instance,

[^0]$\frac{\partial H^{T}}{\partial t}=\frac{\partial^{2} H^{T}}{\partial t^{2}}=\frac{\partial^{3} H^{T}}{\partial t^{3}}=\frac{\partial^{4} H^{T}}{\partial t^{4}}=0$,
if and only if $\sigma_{H}=0$ and some other conditions [Bruce and Giblin (1992)].
Consider the surface $M=H_{+}^{3}(-1)$, in this case, we define two important curves; $h_{\gamma}$ in the hyperbolic space and $d_{\gamma}$ in de Sitter space by observing the conditions of first and second derivation of $H^{T}$ and $H^{S}$, respectively. We call $h_{\gamma}$, a hyperbolic evolute of $\gamma$ relative to $M$ and $d_{\gamma}$, a de Sitter evolute of $\gamma$ relative to $M$. We show that the hyperbolic evolute $h_{\gamma}$ is constant if and only if $\sigma_{H}=0$. In this case, the curve $\gamma$ is a special curve on the surface $M$, which is called a hyperbolic-slice (or an H-slice) of $M$. Also, we show that the de Sitter evolute $d_{\gamma}$ is constant if and only if $\sigma_{D}=0$ and define a special curve on the surface $M$ called a de Sitter-slice (or a D-slice) of $M$. We consider H -slice and D-slice of $M$ which is the model curves on the surface $M$ [Sato (2012)]. As an application of the theory of unfolding of functions, we give a classification of singularities of both the hyperbolic evolute and the de Sitter evolute in Theorem 4.2, which is one of the main results of this work.
The curves and their frames play an important role in differential geometry and in many branches of Science such as Mechanics and Physics, so we are interested here in studying one of these curves which has many applications in Computer Aided Design (CAD), Computer Aided Geometric Design (CAGD) and mathematical modeling. Also, these curves can be used in the discrete model and equivalent model which are usually adopted for the design and mechanical analysis of grid structures. So, we are looking forward to see that our results will be helpful to the researchers who are specialized in this field.

## 2 Basic concepts

In this section, we introduce some definitions and basic facts which are needed in the subsequent sections (for more details see [Izumiya, Pei and Torii (2004); Liu (2014)]).
Let $R^{4}$ denotes the four-dimensional vector space. For any
$\mathbf{x}=\left(x_{1}, x_{2}, x_{3}, x_{4}\right), \mathbf{y}=\left(y_{1}, y_{2}, y_{3}, y_{4}\right) \in R^{4}$, the pseudo-scalar product of $\mathbf{x}$ and $\mathbf{y}$ is defined by $\langle\mathbf{x}, \mathbf{y}\rangle=-x_{1} y_{1}+x_{2} y_{2}+x_{3} y_{3}+x_{4} y_{4}$.

We call $\left(R^{4},\langle\rangle,\right)$, the Minkowski 4 -space and denoted by $E_{1}^{4}$. We say that a vector $\mathbf{x} \in E_{1}^{4}$ is spacelike, lightlike or timelike if $\left\langle x_{1}, x_{2}\right\rangle>0,\left\langle x_{1}, x_{2}\right\rangle=0$ or $\left\langle x_{1}, x_{2}\right\rangle<0$, respectively. The norm of the vector $\mathbf{x} \in E_{1}^{4}$ is defined by $\|\mathbf{x}\|=\sqrt{|\langle\mathbf{x}, \mathbf{x}\rangle|}$. For a nonzero vector $v \in E_{1}^{4}$ and a real number $c$, we define a space with pseudo normal $v$ by $S(v, c)=\left\{x \in E_{1}^{4} \mid\langle x, v\rangle=c\right\}$.
We call $S(v, c)$ a spacelike space, a timelike space or a lightlike space if $v$ is timelike, spacelike or lightlike, respectively.

Now, we define a hyperbolic space by
$H_{+}^{3}(-1)=\left\{x \in E_{1}^{4} \mid\langle x, x\rangle=-1, x_{1}>0\right\}$,
and de Sitter 3-space by
$S_{1}^{3}=\left\{x \in E_{1}^{4} \mid\langle x, x\rangle=1\right\}$.
For any $\mathbf{x}=\left(x_{1}, x_{2}, x_{3}, x_{4}\right), \mathbf{y}=\left(y_{1}, y_{2}, y_{3}, y_{4}\right)$ and $\mathbf{z}=\left(z_{1}, z_{2}, z_{3}, z_{4}\right) \in E_{1}^{4}$, the pseudo vector product of $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ is defined as follows:

$$
\begin{aligned}
\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} & =\left|\begin{array}{cccc}
-i & j & k & l \\
x_{1} & x_{2} & x_{3} & x_{4} \\
y_{1} & y_{2} & y_{3} & y_{4} \\
z_{1} & z_{2} & z_{3} & z_{4}
\end{array}\right| \\
& =\left(-\left|\begin{array}{lll}
x_{2} & x_{3} & x_{4} \\
y_{2} & y_{3} & y_{4} \\
z_{2} & z_{3} & z_{4}
\end{array}\right|,-\left|\begin{array}{lll}
x_{1} & x_{3} & x_{4} \\
y_{1} & y_{3} & y_{4} \\
z_{1} & z_{3} & z_{4}
\end{array}\right|,\left|\begin{array}{lll}
x_{1} & x_{2} & x_{4} \\
y_{1} & y_{2} & y_{4} \\
z_{1} & z_{2} & z_{4}
\end{array}\right|,-\left|\begin{array}{lll}
x_{1} & x_{2} & x_{3} \\
y_{1} & y_{2} & y_{3} \\
z_{1} & z_{2} & z_{3}
\end{array}\right|\right) .
\end{aligned}
$$

We now prepare some basic facts of curves in hyperbolic 3-space.
Let $\gamma: I \rightarrow H_{+}^{3} \subset E_{1}^{4} ; \quad \gamma(t)=\left(x_{1}(t), x_{2}(t), x_{3}(t), x_{4}(t)\right)$ be a smooth regular curve in $H_{+}^{3}$ for any $t \in I$ where $I$ is an open interval.
Here, we construct the explicit differential geometry on curves in $H_{+}^{3}(-1)$.
Let $\gamma: I \rightarrow H_{+}^{3}(-1)$ be a regular curve. Since $H_{+}^{3}(-1)$ is a Riemannian manifold, then we can reparameterize $\gamma$ by the arc-length. Hence, we may assume that $\gamma(s)$ is a unit speed curve. So, we have the tangent vector $\mathbf{T}(s)=\gamma^{\prime}(s)$ with $\|\mathbf{T}\|=1$.

If $\left\langle\mathbf{T}^{\prime}(s), \mathbf{T}^{\prime}(s)\right\rangle \neq-1$, then we have a unit vector $\mathbf{E}_{1}(s)=\frac{\mathbf{T}^{\prime}(s)-\gamma(s)}{\left\|\mathbf{T}^{\prime}(s)-\gamma(s)\right\|}$.
Moreover, define $\mathbf{E}_{2}(s)=\gamma(s) \wedge \mathbf{T}(s) \wedge \mathbf{E}_{1}(s)$, we have a pseudo orthonormal frame $\left\{\gamma(s), \mathbf{T}(s), \mathbf{E}_{1}(s), \mathbf{E}_{2}(s)\right\}$ of $E_{1}^{4}$ along $\gamma$. By standard arguments, under the assumption that $\left\langle\mathbf{T}^{\prime}(s), \mathbf{T}^{\prime}(s)\right\rangle \neq-1$, we have the following Frenet formulae:

$$
\left\{\begin{array}{l}
\gamma^{\prime}(s)=\mathbf{T}(s), \\
\mathbf{T}^{\prime}(s)=\gamma(s)+\kappa_{g} \mathbf{E}_{1}(s), \\
\mathbf{E}_{1}^{\prime}(s)=-\kappa_{g} \mathbf{T}(s)+\tau_{g} \mathbf{E}_{2}(s), \\
\mathbf{E}_{2}^{\prime}(s)=-\tau_{g} \mathbf{E}_{1}(s) .
\end{array}\right.
$$

or in the matrix form:

$$
\left[\begin{array}{c}
\gamma^{\prime}(s) \\
\mathbf{T}^{\prime}(s) \\
\mathbf{E}_{1}^{\prime}(s) \\
\mathbf{E}_{2}^{\prime}(s)
\end{array}\right]=\left[\begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & \kappa_{g} & 0 \\
0 & -\kappa_{g} & 0 & \tau_{g} \\
0 & 0 & -\tau_{g} & 0
\end{array}\right]\left[\begin{array}{c}
\gamma(s) \\
\mathbf{T}(s) \\
\mathbf{E}_{1}(s) \\
\mathbf{E}_{2}(s)
\end{array}\right],
$$

where

$$
\left\{\begin{array}{l}
\kappa_{g}=\left\|\mathbf{T}^{\prime}(s)-\gamma(s)\right\|, \\
\tau_{g}=-\frac{\operatorname{det}\left(\gamma(s), \gamma(s)^{\prime}, \gamma(s)^{\prime \prime}, \gamma(s)^{\prime \prime \prime}\right)}{\left(\kappa_{g}(s)\right)^{2}},
\end{array}\right.
$$

are the geodesic curvature and geodesic torsion of the curve $\gamma$ in $H_{+}^{3}(-1)$, respectively (see [Liu (2014)]).
Since
$\left\langle\mathbf{T}^{\prime}(s)-\gamma(s), \mathbf{T}^{\prime}(s)-\gamma(s)\right\rangle=\left\langle\mathbf{T}^{\prime}(s), \mathbf{T}^{\prime}(s)\right\rangle+1$,
therefore, the condition $\left\langle\mathbf{T}^{\prime}(s), \mathbf{T}^{\prime}(s)\right\rangle \neq-1$ is equivalent to the condition $\kappa_{g}(s) \neq 0$. Moreover, we can show that the curve $\gamma(s)$ satisfies the condition $\kappa_{g}(s) \equiv 0$ if and only if there exists a lightlike vector $c$ such that $\gamma(s)-c$ is a geodesic. Such a curve is called an equidistant curve.

## 3 Height functions

In the following, we introduce two families of functions on a curve $\gamma: I \rightarrow H_{+}^{3}$ lying on a spacelike surface $M$. Suppose that $\left\|\mathbf{T}^{\prime}(s)\right\| \neq 0$, we can define these functions as follows:

### 3.1 Hyperbolic time-like height function

We call $H^{T}$; the time-like height function of $\gamma$ on $M=H_{+}^{3}(-1)$. We denote $h_{v}^{T}(s)=H^{T}(s, v)$ for any fixed $v \in H_{+}^{3}(-1)$. We have
$H^{T}: I \wedge H_{+}^{3}(-1) \rightarrow R ;(s, v) \mapsto(\gamma(s), v)$,

From Frenet formulae, we have

$$
\frac{\partial H^{T}}{\partial s}=\left\langle\gamma^{\prime}(s), v\right\rangle=\langle\mathbf{T}, v\rangle=0 .
$$

Since $v \in H_{+}^{3}(-1)$, there are $\lambda, \mu, v \in R$ such that $v=\lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+v \mathbf{E}_{2}(s)$, therefore $\langle v, v\rangle=-1$, hence
$-\lambda^{2}+\mu^{2}+v^{2}=-1$, if $\frac{\partial H^{T}}{\partial s}=0$, then

$$
\begin{aligned}
\frac{\partial^{2} H^{T}}{\partial s^{2}} & =\left\langle\mathbf{T}^{\prime}, v\right\rangle \\
& =\left\langle\gamma(s)+\kappa_{g} \mathbf{E}_{1}(s), \lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+\nu \mathbf{E}_{2}(s)\right\rangle \\
& =-\lambda+\kappa_{g} \mu=0,
\end{aligned}
$$

which implies $\lambda=\kappa_{g} \mu$, and

$$
\begin{aligned}
\frac{\partial^{3} H^{T}}{\partial s^{3}} & =\left\langle\mathbf{T}^{\prime \prime}, v\right\rangle \\
& =\left\langle\gamma^{\prime}(s)+\kappa_{g}^{\prime} \mathbf{E}_{1}(s)+\kappa_{g} \mathbf{E}_{1}^{\prime}(s), \lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+\nu \mathbf{E}_{2}(s)\right\rangle \\
& =\left\langle\mathbf{T}+\kappa_{g}^{\prime} \mathbf{E}_{1}(s)+\kappa_{g}\left(-\kappa_{g} \mathbf{T}(s)+\tau_{g} \mathbf{E}_{2}(s)\right), \lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+\nu \mathbf{E}_{2}(s)\right\rangle \\
& \left.=\left\langle\left(1-\kappa_{g}^{2}\right) \mathbf{T}(s)+\kappa_{g}^{\prime} \mathbf{E}_{1}(s)+\kappa_{g} \tau_{g} \mathbf{E}_{2}(s)\right), \lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+\nu \mathbf{E}_{2}(s)\right\rangle \\
& =\kappa_{g}^{\prime} \mu+\kappa_{g} \tau_{g} \nu=0,
\end{aligned}
$$

then, we have

$$
\mu=-\frac{\kappa_{g} \tau_{g}}{\kappa_{g}^{\prime}} \nu \text { and } \lambda=-\frac{\kappa_{g}^{2} \tau_{g}}{\kappa_{g}^{\prime}} \nu
$$

therefore

$$
\frac{\partial H^{T}}{\partial s}=\frac{\partial^{2} H^{T}}{\partial s^{2}}=\frac{\partial^{3} H^{T}}{\partial s^{3}}=0,
$$

if and only if $v=\lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+\nu \mathbf{E}_{2}(s),-\lambda^{2}+\mu^{2}+\nu^{2}=-1$ and $\lambda=\kappa_{g} \mu$, it means that

$$
\begin{align*}
& \lambda= \pm \frac{\kappa_{g}}{\sqrt{\left.\| \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1 \right\rvert\,}},  \tag{3.2}\\
& \mu= \pm \frac{1}{\sqrt{\left.\| \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1 \right\rvert\,}},  \tag{3.3}\\
& \nu= \pm \frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g} \sqrt{\| \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1}} . \tag{3.4}
\end{align*}
$$

Since $\lambda, \mu, \nu \in R$, therefore we can consider under the condition

$$
\begin{aligned}
& \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\boldsymbol{\kappa}_{g} \tau_{g}}\right)^{2}>1, \text { that } \\
& v= \pm \frac{1}{\sqrt{\kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right) .
\end{aligned}
$$

Moreover, we obtain

$$
\left\langle\frac{\partial H^{T}}{\partial s}, v\right\rangle=\left\langle\frac{\partial^{2} H^{T}}{\partial s^{2}}, v\right\rangle=\left\langle\frac{\partial^{3} H^{T}}{\partial s^{3}}, v\right\rangle=\left\langle\frac{\partial^{(4)} H^{T}}{\partial s^{(4)}}, v\right\rangle=0,
$$

and then

$$
\begin{equation*}
\left\langle\frac{d}{d s}\left[\left(1-\kappa_{g}^{2}\right) \mathbf{T}(s)+\kappa_{g}^{\prime} \mathbf{E}_{1}(s)+\kappa_{g} \tau_{g} \mathbf{E}_{2}(s)\right] \lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+\nu \mathbf{E}_{2}(s)\right\rangle=0 . \tag{3.5}
\end{equation*}
$$

Differentiating Eq. (3.5) and using Eqs. (3.2)-(3.4), we get

$$
\begin{equation*}
\kappa_{g}^{\prime \prime \prime} \kappa_{g} \tau_{g}-\kappa_{g}^{\prime 2} \tau_{g}-\kappa_{g} \kappa_{g}^{\prime} \tau_{g}^{\prime}-\kappa_{g}^{\prime 2} \tau_{g}-\kappa_{g}^{2} \tau_{g}^{3}=0 \tag{3.6}
\end{equation*}
$$

Since, $\kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}>1$, therefore Eq. (3.6) leads to
$\left(\frac{\kappa_{g}^{\prime}}{\boldsymbol{\kappa}_{g} \tau_{g}}\right)^{\prime}-\tau_{g}\left[\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}+1\right]=0$.
Now, we define $\sigma_{H}$ as

$$
\sigma_{H}=\left(\frac{\kappa_{g}^{\prime}}{\boldsymbol{\kappa}_{g} \tau_{g}}\right)^{\prime}-\tau_{g}\left[\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}+1\right]
$$

If we calculate the fifth derivative of $H^{T}$, we can show that the above conditions and

$$
\frac{d}{d s}\left(\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{\prime}-\tau_{g}\left[\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}+1\right]\right)=0
$$

are equivalent to the conditions $\sigma_{H}(s)=0$ and $\sigma_{H}^{\prime}(s)=0$. As a consequence, we have the following proposition.
Proposition 3.1 Suppose that $\mathbf{T}^{\prime}(s) \neq 0$. For any $(s, v) \in I \times H_{+}^{3}(-1)$, we have

1) $\left(h_{v}^{T}\right)^{\prime}(s)=\left(h_{v}^{T}\right)^{\prime \prime}(s)=0 \quad$ if and only if $v=\lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+\nu \mathbf{E}_{2}(s)$ where $\lambda, \mu, v \in R$ such that $-\lambda^{2}+\mu^{2}+v^{2}=-1$.
2) $\left(h_{v}^{T}\right)^{\prime}(s)=\left(h_{v}^{T}\right)^{\prime \prime}(s)=\left(h_{v}^{T}\right)^{\prime \prime \prime}(s)=0$ if and only if

$$
\begin{aligned}
& v= \pm \frac{1}{\sqrt{\| \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right) \\
& \\
& \qquad\left(\frac{\xi_{g}^{\prime}}{\ddagger g_{g}^{\prime} \delta_{g}^{\prime}}\right)^{2} \odot 1
\end{aligned}
$$

3) $\left(h_{v}^{T}\right)^{\prime}(s)=\left(h_{v}^{T}\right)^{\prime \prime}(s)=\left(h_{v}^{T}\right)^{\prime \prime \prime}(s)=\left(h_{v}^{T}\right)^{(4)}(s)=0$ if and only if

$$
\begin{aligned}
& v= \pm \frac{1}{\sqrt{\left\lvert\, \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1\right.}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right) \\
& \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}>1 \text { and } \sigma_{H}=0 .
\end{aligned}
$$

4) $\left(h_{v}^{T}\right)^{\prime}(s)=\left(h_{v}^{T}\right)^{\prime \prime}(s)=\left(h_{v}^{T}\right)^{\prime \prime \prime}(s)=\left(h_{v}^{T}\right)^{(4)}(s)=\left(h_{v}^{T}\right)^{(5)}(s)=0$ if and only if

$$
\begin{aligned}
& v= \pm \frac{1}{\sqrt{\left\lvert\, \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1\right.}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right) \\
& \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}>1, \sigma_{H}=0 \text { and }\left(\sigma_{H}\right)^{\prime}=0 .
\end{aligned}
$$

In the light of Proposition 3.1, we have the invariant $\sigma_{H}$. So, we define the curve $h_{\gamma}: I \rightarrow H_{+}^{3}(-1)$ as follows

$$
\begin{equation*}
h_{\gamma}(s)=\frac{1}{\sqrt{\| \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right), \tag{3.7}
\end{equation*}
$$

and we call it a hyperbolic evolute of $\gamma$ relative to $M$.
By straightforward calculations, we have $h_{\gamma}^{\prime}(s)=0$, if and only if $\sigma_{H}(s) \equiv 0$. Also, $h_{\gamma}=v_{0}$ is constant if and only if $\sigma_{H}(s) \equiv 0$.

From Proposition 3.1, we have $h_{v_{0}}^{T}$ is constant, that is, there is a real number $c \in R$ such that $\left\langle\gamma(s), v_{\mathrm{o}}=c\right\rangle$. It means that $\operatorname{Im} \gamma=P\left(v_{\mathrm{o}}, c\right) \cap M$. It suggests that curves of the form $P(v, c) \cap M$ for $v \in H_{+}^{3}(-1)$ are the candidates of model curves on $M$.
These curves play a similar role to curves in Euclidean space and call them (hyperbolicslices or H -slices) of $M$.

### 3.2 De Sitter space-like height function

We call $H^{S}$; the spacelike height function of $\gamma$ on $M$, where
$H^{S}: I \wedge S_{1}^{3} \rightarrow R ;(s, v) \mapsto(\gamma(s), v)$.
We denote $h_{v}^{S}(s)=H^{S}(s, v)$ for any fixed $v \in S_{1}^{3}$, and by using Frenet-Serret formulae, we get
$\frac{\partial H^{s}}{\partial s}=\left\langle\gamma^{\prime}(s), v\right\rangle=\langle\mathbf{T}, v\rangle=0$.
Since $v \in S_{1}^{3}$, therefore there exist $\lambda, \mu, v \in R$, such that $v=\lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+\nu \mathbf{E}_{2}(s)$,
where $v \in S_{1}^{3}$, then $\langle v, v\rangle=1$, so we have
$-\lambda^{2}+\mu^{2}+v^{2}=1$,
and if $\frac{\partial H^{s}}{\partial s}=0$, we get

$$
\begin{aligned}
\frac{\partial^{2} H^{s}}{\partial s^{2}} & =\left\langle\mathbf{T}^{\prime}, v\right\rangle \\
& =\left\langle\gamma(s)+\kappa_{g} \mathbf{E}_{1}(s), \lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+\nu \mathbf{E}_{2}(s)\right\rangle \\
& =\lambda+\kappa_{g} \mu=0,
\end{aligned}
$$

this leads to
$\lambda=-\kappa_{g} \mu$.
Also, we have

$$
\begin{aligned}
\frac{\partial^{3} H^{s}}{\partial s^{3}} & =\left\langle\mathbf{T}^{\prime \prime}, v\right\rangle \\
& =\left\langle\gamma^{\prime}(s)+\kappa_{g}^{\prime} \mathbf{E}_{1}(s)+\kappa_{g} \mathbf{E}_{1}^{\prime}(s), \lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+\nu \mathbf{E}_{2}(s)\right\rangle \\
& =\left\langle\mathbf{T}+\kappa_{g}^{\prime} \mathbf{E}_{1}(s)+\kappa_{g}\left(-\kappa_{g} \mathbf{T}(s)+\tau_{g} \mathbf{E}_{2}(s)\right), \lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+\nu \mathbf{E}_{2}(s)\right\rangle \\
& \left.=\left\langle\left(1-\kappa_{g}^{2}\right) \mathbf{T}(s)+\kappa_{g}^{\prime} \mathbf{E}_{1}(s)+\kappa_{g} \tau_{g} \mathbf{E}_{2}(s)\right), \lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+\nu \mathbf{E}_{2}(s)\right\rangle \\
& =\kappa_{g}^{\prime} \mu+\kappa_{g} \tau_{g} v=0,
\end{aligned}
$$

therefore
$\mu=-\frac{\kappa_{g} \tau_{g}}{\kappa_{g}^{\prime}} \nu$ and $\lambda=\frac{\kappa_{g}^{2} \tau_{g}}{\kappa_{g}^{\prime}} \nu$.
Hence, the following is satisfied
$\frac{\partial H^{s}}{\partial s}=\frac{\partial^{2} H^{s}}{\partial s^{2}}=\frac{\partial^{3} H^{s}}{\partial s^{3}}=0$,
if and only if $v=\lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+\mathbf{E}_{2}(s),-\lambda^{2}+\mu^{2}+\nu^{2}=1$ and $\lambda=-\kappa_{g} \mu$, it means that

$$
\lambda= \pm \frac{\kappa_{g}}{\sqrt{\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-\kappa_{g}^{2}+1}},
$$

$\mu= \pm \frac{1}{\sqrt{\left.\|\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-\kappa_{g}^{2}+1 \right\rvert\,}}$,
$\nu= \pm \frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g} \sqrt{\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-\kappa_{g}^{2}+1}}$.
Since $\lambda, \mu, \nu \in R$, therefore we can consider the condition $\kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}<1$, so we have $v= \pm \frac{1}{\sqrt{\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-\kappa_{g}^{2}+1}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right)$.
Moreover,
$\left\langle\frac{\partial H^{S}}{\partial s}, v\right\rangle=\left\langle\frac{\partial^{2} H^{S}}{\partial s^{2}}, v\right\rangle=\left\langle\frac{\partial^{3} H^{S}}{\partial s^{3}}, v\right\rangle=\left\langle\frac{\partial^{(4)} H^{S}}{\partial s^{(4)}}, v\right\rangle=0$,
then
$\left.\left\langle\frac{d}{d s}\left[\left(1-\kappa_{g}^{2}\right) \mathbf{T}(s)+\kappa_{g}^{\prime} \mathbf{E}_{1}(s)+\kappa_{g} \tau_{g} \mathbf{E}_{2}(s)\right)\right], \lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+\nu \mathbf{E}_{2}(s)\right\rangle=0$,
After using the values of $\lambda, \mu$ and $v$ in Eq. (3.8), we obtain
$\kappa_{g}^{\prime \prime} \kappa_{g} \tau_{g}-\kappa_{g} \kappa_{g}^{\prime} \tau_{g}^{\prime}-\kappa_{g}^{\prime 2} \tau_{g}-2 \kappa_{g}^{2} \tau_{g}+2 \kappa_{g}^{4} \tau_{g}+\kappa_{g}^{2} \tau_{g}^{3}=0$.
Under the condition; $\kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}<1$, Eq. (3.9) leads to
$\left(\frac{\kappa_{g}^{\prime}}{\boldsymbol{\kappa}_{g} \tau_{g}}\right)^{\prime}-\tau_{g}\left[1+2\left(\frac{\kappa_{g}}{\tau_{g}}\right)^{2}-\left(\frac{2}{\tau_{g}^{2}}\right)\right]=0$. Now, we define $\sigma_{D}$ as
$\sigma_{D}=\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{\prime}-\tau_{g}\left[1+2\left(\frac{\kappa_{g}}{\tau_{g}}\right)^{2}-\left(\frac{2}{\tau_{g}^{2}}\right)\right]$,
therefore, we can show that the above conditions with the extra condition:

$$
\frac{d}{d s}\left[\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{\prime}-\tau_{g}\left[1+2\left(\frac{\kappa_{g}}{\tau_{g}}\right)^{2}-\left(\frac{2}{\tau_{g}^{2}}\right)\right]\right]=0
$$

are equivalent to the conditions $\sigma_{D}(s)=0$ and $\sigma_{D}^{\prime}(s)=0$. As a consequence, we have the following proposition.
Proposition 3.2 If $\mathbf{T}^{\prime}(s) \neq 0$. Then, for any $(s, v) \in I \times S_{1}^{2}$, we have

1) $\left(h_{v}^{S}\right)^{\prime}(s)=\left(h_{v}^{S}\right)^{\prime \prime}(s)=0$ if and only if $v=\lambda \gamma(s)+\mu \mathbf{E}_{1}(s)+\nu \mathbf{E}_{2}(s)$, where $\lambda, \mu, v \in R ;-\lambda^{2}+\mu^{2}+v^{2}=1$.
2) $\left(h_{\nu}^{S}\right)^{\prime}(s)=\left(h_{\nu}^{S}\right)^{\prime \prime}(s)=\left(h_{\nu}^{S}\right)^{\prime \prime \prime}(s)=0$ if and only if

$$
\begin{aligned}
& v= \pm \frac{1}{\sqrt{\left.\|\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-\kappa_{g}^{2}+1 \right\rvert\,}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right) \\
& \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}<1
\end{aligned}
$$

3) $\left(h_{v}^{S}\right)^{\prime}(s)=\left(h_{v}^{S}\right)^{\prime \prime}(s)=\left(h_{v}^{S}\right)^{\prime \prime \prime}(s)=\left(h_{v}^{S}\right)^{(4)}(s)=0$ if and only if

$$
\begin{aligned}
& v= \pm \frac{1}{\sqrt{\left.\|\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-\kappa_{g}^{2}+1 \right\rvert\,}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right) \\
& \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}<1 \text { and } \sigma_{D}=0 .
\end{aligned}
$$

4) $\left(h_{v}^{S}\right)^{\prime}(s)=\left(h_{v}^{S}\right)^{\prime \prime}(s)=\left(h_{v}^{S}\right)^{\prime \prime \prime}(s)=\left(h_{v}^{S}\right)^{(4)}(s)=\left(h_{v}^{S}\right)^{(5)}(s)=0$, if and only if

$$
\begin{aligned}
& v= \pm \frac{1}{\sqrt{\left.\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-\kappa_{g}^{2}+1 \right\rvert\,}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right) \\
& \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}<1, \sigma_{D}=0 \text { and }\left(\sigma_{D}\right)^{\prime}=0 .
\end{aligned}
$$

According to this proposition, we have the invariant of $\sigma_{D}$. So, we define the curve

$$
\begin{aligned}
d_{\gamma}: I & \rightarrow S_{1}^{3} \text { by } \\
d_{\gamma}(s)= & \frac{1}{\sqrt{\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-\kappa_{g}^{2}+1}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right),
\end{aligned}
$$

and we call $d_{\gamma}$; a de Sitter evolute of $\gamma$ relative to $M$.
By straightforward calculations, we have $d_{\gamma}^{\prime}(s)=0$ if and only if $\sigma_{D}(s)=0$. Therefore, $d_{\gamma}=v_{o}$ is constant if and only if $\sigma_{D}(s) \equiv 0$. Using Proposition 3.2, we have $h_{\nu_{0}}^{T}$ is constant, that is, there is a real number $c \in R$ such that $\left\langle\gamma(s), v_{0}=c\right\rangle$. It means that $\operatorname{Im} \gamma=P\left(v_{0}, c\right) \cap M$. It suggests that curves of the form $P(v, c) \cap M$ for $v \in S_{1}^{3}$ are the candidates of model curves on $M$. Also, these curves play a similar role to curves in Euclidean space and we call them (de Sitter-slices or D-slices) of $M$.

### 3.3 Hyperbolic (De Sitter) invariants of curves

In this section, we study the geometric properties of the hyperbolic evolute of a curve in $H_{+}^{3}$. For any $r \in R$ and $v_{0} \in H_{+}^{3}$ or $v_{0} \in S_{1}^{3}$, we denote
$P S^{1}\left(v_{0}, r\right)=\left\{v \in H_{+}^{3} \mid\left\langle v, v_{o}\right\rangle=r\right\}$, and we call it a pseudo-sphere in $H_{+}^{3}$ with center $v_{0}$.
Proposition 3.3 If $\gamma: I \rightarrow H_{+}^{3}$ be a unit speed curve with $\kappa_{g}^{\prime} \neq 0$. Then, $\tau_{g} \equiv 0$ if and only if

$$
v_{o}= \pm \frac{1}{\sqrt{\| \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right),
$$

and $\gamma$ is a part of a pseudo-sphere in $H_{+}^{3}$ whose center is $v_{0}$.
Proof. We denote

$$
P_{ \pm}(s)= \pm v_{o}= \pm \frac{1}{\sqrt{\left\lvert\, \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1\right.}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right),
$$

then, we have

$$
P_{ \pm}(s)^{\prime}=\mp \frac{1}{\left(\left|\kappa_{g}^{4} \tau_{g}^{2}-\kappa_{g}^{\prime 2}-\kappa_{g}^{2} \tau_{g}^{2}\right|\right)^{\frac{3}{2}}}\left(\begin{array}{c}
\left(\kappa_{g}^{2} \kappa_{g}^{\prime} \kappa_{g}^{\prime \prime} \tau_{g}-2 \kappa_{g} \kappa_{g}^{\prime 3} \tau_{g}-\kappa_{g}^{2} \kappa_{g}^{\prime 2} \tau_{g}^{\prime}-\kappa_{g}^{3} \kappa_{g}^{\prime} g_{g}^{3}\right) \gamma(s) \\
+\left(\kappa_{g} \kappa_{g}^{\prime} \kappa_{g}^{\prime \prime} \tau_{g}-2 \kappa_{g}^{\prime 3} \tau_{g}-\kappa_{g} \kappa_{g}^{2} \tau_{g}^{-}-\kappa_{g}^{\prime} \kappa_{g}^{\prime} \tau_{g}^{3}\right) \mathbf{E}_{1}(s) \\
+\left(\kappa_{g}^{4} \kappa_{g}^{\prime} \tau_{g}^{2}-2 \kappa_{g}^{3} \xi_{g}^{2} \tau_{g}^{3}-\kappa_{g}^{5} \tau_{g}^{4}-\kappa_{g}^{4} \kappa_{g}^{\prime} \tau_{g} \tau_{g}^{\prime}+\kappa_{g} \kappa_{g}^{\prime 2} \tau_{g}^{2}\right. \\
\left.\left.-\kappa_{g}^{2} \kappa_{g}^{\prime \prime} \tau_{g}^{2}+\kappa_{g} \kappa_{g}^{2} \tau_{g}^{2}+\kappa_{g}^{2} \kappa_{g}^{\prime} \tau_{g} \tau_{g}^{\prime}+\kappa_{g}^{3} \tau_{g}^{4}\right) \mathbf{E}_{2}(s)\right]
\end{array}\right) .
$$

Therefore, $P_{ \pm}(s)^{\prime} \equiv 0$ if and only if $\tau_{g} \equiv 0$. Under this condition, we put

$$
r=\mp \frac{\kappa_{g}}{\sqrt{\left|\kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1\right|}},
$$

and

$$
v_{\circ}=\mp \frac{\kappa_{g}}{\sqrt{\| \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right)
$$

Thus, it is easy to show that $\gamma(s)$ is a part of the pseudo-sphere $P S^{1}\left(v_{\mathrm{o}}, r\right)$.
Let $\gamma: I \rightarrow H_{+}^{3}$ be a unit speed curve with $\kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}>1$. For any $s_{o} \in I$, we consider the pseudo-sphere $P S^{1}\left(v_{o}, r_{\circ}^{ \pm}\right)$where $v_{0}=h_{\gamma}(s)$ and
$r_{\circ}=-\frac{\kappa_{g}}{\sqrt{\left|\kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1\right|}}$.
Therefore, we can give the following proposition.
Proposition 3.4 Under the above assumptions, $\gamma$ and $P S^{1}\left(v_{o}, r_{o}\right)$ have at least 4-points contact at $\gamma\left(s_{\mathrm{o}}\right)$.
Proof. We assume that $P S^{1}\left(v_{\mathrm{o}}, r_{\mathrm{o}}\right) \subset H_{+}^{3}$. In this case, we consider the hyperbolic timelike height function $H^{T}$. By definition, we have $P S^{1}\left(v_{0}, r_{\mathrm{o}}\right)=\left(h_{v_{0}}^{T}\right)^{-1}\left(r_{\mathrm{o}}\right)$. Proposition 3.1(2) means that $\gamma$ and $P S^{1}\left(v_{\mathrm{o}}, r_{\mathrm{o}}\right)$ have at least a 4-point contact at $\gamma\left(S_{\mathrm{o}}\right)$ If $P S^{1}\left(v_{o}, r_{\mathrm{o}}\right) \subset S_{1}^{3}$, then we adopt the hyperbolic spacelike height function $H^{S}$, and the assertion follows from exactly the same arguments as those of the previous case.
We call $P S^{1}\left(v_{0}, r_{\mathrm{o}}\right)$ in Proposition 3.4, the osculating pseudo-sphere (or, the pseudo-
sphere of geodesic curvature); its center $v_{0}$ is called the center of geodesic curvature. So, the hyperbolic evolute is the locus of the center of geodesic curvature.

## 4 Unfolding of functions of one variable

In order to investigate the singularities of pseudo-spherical evolutes, we apply the theory of unfolding of functions. First, we give a quick review on this theorem of one variable. Detailed descriptions are found in Bruce et al. [Bruce and Giblin (1992); Izumiya (2013)]. Let $F:\left(R \times R^{r},\left(s_{0}, x_{0}\right)\right) \rightarrow R$ be a smooth function defined around a specific point $\left(s_{0}, x_{0}\right)$. We call such a function germ at $\left(s_{0}, x_{0}\right)$. We call $F$ an $r$-parameter unfolding of $f$, where $f(s)=F_{x_{0}}\left(s_{0}, x_{0}\right)$. We say that $f$ has type $A_{k}$-singularity at $s_{0}$ if $f^{(p)}\left(s_{0}\right)=0$, for all $1 \leq p \leq k, f^{k+1}\left(s_{0}\right) \neq 0$. Let $F$ be an unfolding of $f$ and $f(s)$ has $A_{k}$ - singularity $(k \geq 1)$ at $s_{0}$. We denote the Taylor series of the partial derivative $\frac{\partial f}{\partial x_{i}}$ at $s_{0}$ up to $(k-1)$ terms by
$j^{k-1}\left(\frac{\partial F}{\partial x_{i}}\left(s, x_{0}\right)\right)\left(s_{0}\right)=\sum_{j=0}^{k-1} \alpha_{j i}\left(s-s_{0}\right)^{j}$,
for $i=1, \ldots, r$. Then, $F$ is called a $(p)$ versal unfolding if the $(k-1) \times r$ matrix of coefficients $\alpha_{(j i)}(j=1, \ldots, k-1)$ has rank $k-1 ;(k-1 \leq r)$. Under the same condition as the above, $F$ is called an $\mathfrak{R}$-versal unfolding if the $k \times r$ matrix of coefficients $\left(\alpha_{0 i}, \alpha_{j i}\right)$ has rank $k(k \leq r)$, where $\alpha_{0 i}=\frac{\partial F}{\partial x_{i}}\left(s_{0}, x_{0}\right)$.

Now, we introduce an important set concerning the unfolding relative to the above notions. The bifurcation set $B_{F}$ of $F$ is

$$
B_{F}=\left\{x \in R^{r} \mid \exists s ; \text { with } \frac{\partial F}{\partial s}(s, x)=\frac{\partial^{2} F}{\partial s^{2}}(s, x)=0\right\} .
$$

Therefore, we have the following fundamental result of the unfolding theory.
Theorem 4.1 Let $F:\left(R \times R^{r},\left(s_{o}, x_{\mathrm{o}}\right)\right) \rightarrow R$ be an $r$-parameter unfolding of $f$ which has the type $A_{k}$ at $s_{0}$. If $F$ is an $\mathfrak{R}^{+}$-versal unfolding, then we have

1) If $k=3$, then $B_{F}$ is locally diffeomorphic to $C \times R^{r-2}$,
2) If $k=4$, then $B_{F}$ is locally diffeomorphic to $S W \times R^{r-3}$,
where $C=\left\{\left(x_{1}, x_{2}\right) \mid x_{1}^{2}=x_{2}^{3}\right\}$ is the ordinary cusp (see Fig.1) and $S W=\left\{\left(x_{1}, x_{2}, x_{3}\right) \mid x_{1}=3 u^{4}+u^{2} v, x_{2}=4 u^{3}+2 u v, x_{3}=v\right\}$,
is the swallowtail surface (see Fig. 4).


Figure 1: The ordinary cusp
We consider that $H^{T}(s, v)$ (respectively, $H^{S}(s, v)$ ) is an unfolding of $h_{\nu_{0}}^{T}(s)$ (respectively, $h_{\nu_{o}}^{s}(s)$ ). So, we have the following proposition.

Proposition 4.1 Let $\gamma: I \rightarrow H_{+}^{3}$ be a unit speed curve with $\kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}>1$. Then, we have

1) If $h_{v_{0}}^{T}(s)$ has the $A_{4}$-singularity at $s_{o}$, then $H^{T}$ is the $\mathfrak{R}^{+}$versal unfolding of $h_{v_{o}}^{T}$,
2) If $h_{v_{o}}^{s}(s)$ has the $A_{4}$-singularity at $S_{o}$, then $H^{s}$ is the $\mathfrak{R}^{+}$versal unfolding of $h_{v_{o}}^{s}$.

Proof. We denote $\gamma(s)=\left(x_{1}(s), x_{2}(s), x_{3}(s), x_{4}(s)\right)$,
$v=\left(v_{1}, v_{2}, v_{3}, \sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}\right) \in H_{+}^{3}(-1)$ and
$H^{T}(s, v)=-x_{1}(s) v_{1}+x_{2}(s) v_{2}+x_{3}(s) v_{3}+x_{4}(s) \sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}$,
it follows that

$$
\left\{\begin{array} { l } 
{ \frac { \partial H ^ { T } } { \partial v _ { 1 } } = - x _ { 1 } + \frac { v _ { 1 } } { \sqrt { - 1 + v _ { 1 } ^ { 2 } - v _ { 2 } ^ { 2 } - v _ { 3 } ^ { 2 } } } x _ { 4 } , } \\
{ \frac { \partial H ^ { T } } { \partial v _ { 2 } } = x _ { 2 } - \frac { v _ { 2 } } { \sqrt { - 1 + v _ { 1 } ^ { 2 } - v _ { 2 } ^ { 2 } - v _ { 3 } ^ { 2 } } } x _ { 4 } , } \\
{ \frac { \partial H ^ { T } } { \partial v _ { 3 } } = x _ { 3 } - \frac { v _ { 3 } } { \sqrt { - 1 + v _ { 1 } ^ { 2 } - v _ { 2 } ^ { 2 } - v _ { 3 } ^ { 2 } } } x _ { 4 } , }
\end{array} \left\{\begin{array}{l}
\frac{\partial^{2} H^{T}}{\partial s \partial v_{1}}=-x_{1}^{\prime}+\frac{v_{1}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} x_{4}^{\prime}, \\
\frac{\partial^{2} H^{T}}{\partial s \partial v_{2}}=x_{2}^{\prime}-\frac{v_{2}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} x_{4}^{\prime}, \\
\frac{\partial^{2} H^{T}}{\partial s \partial v_{3}}=x_{3}^{\prime}-\frac{v_{3}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} x_{4}^{\prime}, \\
\left\{\begin{array} { l } 
{ \frac { \partial ^ { 3 } H ^ { T } } { \partial s ^ { 2 } \partial v _ { 1 } } = - x _ { 1 } ^ { \prime \prime } + \frac { v _ { 1 } } { \sqrt { - 1 + v _ { 1 } ^ { 2 } - v _ { 2 } ^ { 2 } - v _ { 3 } ^ { 2 } } } x _ { 4 } ^ { \prime \prime } , } \\
{ \frac { \partial ^ { 3 } H ^ { T } } { \partial s ^ { 2 } \partial v _ { 2 } } = x _ { 2 } ^ { \prime \prime } - \frac { v _ { 2 } } { \sqrt { - 1 + v _ { 1 } ^ { 2 } - v _ { 2 } ^ { 2 } - v _ { 3 } ^ { 2 } } } x _ { 4 } ^ { \prime \prime } , }
\end{array} \quad \left\{\begin{array}{l}
\frac{\partial^{4} H^{T}}{\partial s^{3} \partial v_{1}}=-x_{1}^{\prime \prime \prime}+\frac{v_{1}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} x_{4}^{\prime \prime \prime}, \\
\frac{\partial^{4} H^{T}}{\partial s^{3} \partial v_{2}}=x_{2}^{\prime \prime \prime}-\frac{v_{2}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} x_{4}^{\prime \prime \prime}, \\
\frac{\partial^{4} H^{T}}{\partial s^{3} \partial v_{3}}=x_{3}^{\prime \prime \prime}-\frac{v_{3}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} x_{4}^{\prime \prime \prime},
\end{array}\right.\right.
\end{array}\right.\right.
$$

From Proposition 3.1, $h_{v}^{T}\left(s_{\circ}\right)$ has a type $A_{4}$ at $S_{\circ}$ if and only if

1) $v= \pm \frac{1}{\sqrt{\kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right)$,
2) $\left(\kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}\right)>1, \sigma_{H}=0$ and $\left(\sigma_{H}\right)^{\prime}=0$.

For this purpose, the following matrix

$$
A=\left[\begin{array}{ccc}
-x_{1}^{\prime}+\frac{v_{1}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} x_{4}^{\prime} & x_{2}^{\prime}-\frac{v_{2}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} x_{4}^{\prime} & x_{3}^{\prime}-\frac{v_{3}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} x_{4}^{\prime} \\
-x_{1}^{\prime \prime}+\frac{v_{1}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} x_{4}^{\prime \prime} & x_{2}^{\prime \prime}-\frac{v_{2}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} x_{4}^{\prime \prime} & x_{3}^{\prime \prime}-\frac{v_{3}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} x_{4}^{\prime \prime} \\
-x_{1}^{\prime \prime \prime}+\frac{v_{1}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} x_{4}^{\prime \prime \prime} & x_{2}^{\prime \prime \prime}-\frac{v_{2}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} x_{4}^{\prime \prime \prime} & x_{3}^{\prime \prime \prime}-\frac{v_{3}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} x_{4}^{\prime \prime \prime}
\end{array}\right],
$$

must be non-singular. Therefore, we calculate the determinant of it as follows

$$
\begin{aligned}
\operatorname{det} A & =\left(-x_{1}^{\prime} x_{2}^{\prime \prime} x_{3}^{\prime \prime \prime}+x_{1}^{\prime} x_{2}^{\prime \prime \prime} x_{3}^{\prime \prime}+x_{1}^{\prime \prime} x_{2}^{\prime} x_{3}^{\prime \prime \prime}-x_{1}^{\prime \prime \prime} x_{2}^{\prime} x_{3}^{\prime \prime}-x_{1}^{\prime \prime} x_{2}^{\prime \prime \prime} x_{3}^{\prime \prime}+x_{1}^{\prime \prime \prime} x_{2}^{\prime \prime} x_{3}^{\prime}\right) \\
& +\left(x_{2}^{\prime \prime} x_{3}^{\prime \prime \prime} x_{4}^{\prime}-x_{2}^{\prime \prime \prime} x_{3}^{\prime \prime} x_{4}^{\prime}-x_{2}^{\prime} x_{3}^{\prime \prime \prime} x_{4}^{\prime \prime}+x_{2}^{\prime} x_{3}^{\prime \prime} x_{4}^{\prime \prime \prime}+x_{2}^{\prime \prime \prime} x_{3}^{\prime} x_{4}^{\prime \prime}-x_{2}^{\prime \prime} x_{3}^{\prime} x_{4}^{\prime \prime \prime}\right)\left(\frac{v_{1}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}}\right) \\
& +\left(x_{1}^{\prime} x_{3}^{\prime \prime \prime} x_{4}^{\prime \prime}-x_{1}^{\prime} x_{3}^{\prime \prime} x_{4}^{\prime \prime \prime}-x_{1}^{\prime \prime} x_{3}^{\prime \prime \prime} x_{4}^{\prime}+x_{1}^{\prime \prime \prime} x_{3}^{\prime \prime} x_{4}^{\prime}+x_{1}^{\prime \prime} x_{3}^{\prime} x_{4}^{\prime \prime \prime}-x_{1}^{\prime \prime \prime} x_{3}^{\prime} x_{4}^{\prime \prime}\right)\left(\frac{v_{2}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}}\right) \\
& +\left(x_{1}^{\prime} x_{2}^{\prime \prime} x_{4}^{\prime \prime \prime}-x_{1}^{\prime} x_{2}^{\prime \prime \prime} x_{4}^{\prime \prime}-x_{1}^{\prime \prime} x_{2}^{\prime} x_{4}^{\prime \prime \prime}+x_{1}^{\prime \prime \prime} x_{2}^{\prime} x_{4}^{\prime \prime}+x_{1}^{\prime \prime} x_{2}^{\prime \prime \prime} x_{4}^{\prime}-x_{1}^{\prime \prime \prime} x_{2}^{\prime \prime} x_{4}^{\prime}\right)\left(\frac{v_{3}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}}\right),
\end{aligned}
$$

or in another form

$$
\begin{aligned}
\operatorname{det} A & =\left(x_{1}^{\prime}\left(x_{2}^{\prime \prime} x_{3}^{\prime \prime \prime}-x_{2}^{\prime \prime \prime} x_{3}^{\prime \prime}\right)-x_{2}^{\prime}\left(x_{1}^{\prime \prime} x_{3}^{\prime \prime \prime}-x_{1}^{\prime \prime \prime} x_{3}^{\prime \prime}\right)+x_{3}^{\prime}\left(x_{1}^{\prime \prime} x_{2}^{\prime \prime \prime}-x_{1}^{\prime \prime \prime} x_{2}^{\prime \prime}\right)\right. \\
& , x_{2}^{\prime}\left(x_{3}^{\prime \prime} x_{4}^{\prime \prime \prime}-x_{3}^{\prime \prime \prime} x_{4}^{\prime \prime}\right)-x_{3}^{\prime}\left(x_{2}^{\prime \prime} x_{4}^{\prime \prime \prime}-x_{2}^{\prime \prime \prime} x_{4}^{\prime \prime}\right)+x_{4}^{\prime}\left(x_{2}^{\prime \prime} x_{3}^{\prime \prime \prime}-x_{2}^{\prime \prime \prime} x_{3}^{\prime \prime}\right) \\
& ,-x_{1}^{\prime}\left(x_{3}^{\prime \prime} x_{4}^{\prime \prime \prime}-x_{3}^{\prime \prime \prime} x_{4}^{\prime \prime}\right)+x_{3}^{\prime}\left(x_{1}^{\prime \prime} x_{4}^{\prime \prime \prime}-x_{1}^{\prime \prime \prime} x_{4}^{\prime \prime}\right)-x_{4}^{\prime}\left(x_{1}^{\prime \prime} x_{3}^{\prime \prime \prime}-x_{1}^{\prime \prime \prime} x_{3}^{\prime \prime}\right) \\
& \left., x_{1}^{\prime}\left(x_{2}^{\prime \prime} x_{4}^{\prime \prime \prime}-x_{2}^{\prime \prime \prime} x_{4}^{\prime \prime}\right)-x_{2}^{\prime}\left(x_{1}^{\prime \prime} x_{4}^{\prime \prime \prime}-x_{1}^{\prime \prime \prime} x_{4}^{\prime \prime}\right)+x_{4}^{\prime}\left(x_{1}^{\prime \prime} x_{2}^{\prime \prime \prime}-x_{1}^{\prime \prime \prime} x_{2}^{\prime \prime t r}\right)\right) \Omega^{t r}
\end{aligned}
$$

where
$\Omega^{t r}=\left(-1 \frac{v_{1}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} \quad \frac{v_{2}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}} \frac{v_{3}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}}\right)$,
and $\Omega^{t r}$ is the transpose of a matrix $\Omega$. Then we get
$\operatorname{det} A=\left(\left|\begin{array}{lll}x_{1}^{\prime} & x_{2}^{\prime} & x_{3}^{\prime} \\ x_{1}^{\prime \prime} & x_{2}^{\prime \prime} & x_{3}^{\prime \prime} \\ x_{1}^{\prime \prime \prime} & x_{2}^{\prime \prime \prime} & x_{3}^{\prime \prime \prime}\end{array}\right|, \begin{array}{ccc}x_{2}^{\prime} & x_{3}^{\prime} & x_{4}^{\prime} \\ x_{2}^{\prime \prime} & x_{3}^{\prime \prime} & x_{4}^{\prime \prime} \\ x_{2}^{\prime \prime \prime} & x_{3}^{\prime \prime \prime} & x_{4}^{\prime \prime \prime}\end{array}| | \begin{array}{ccc}x_{1}^{\prime} & x_{3}^{\prime} & x_{4}^{\prime} \\ x_{1}^{\prime \prime} & x_{3}^{\prime \prime} & x_{4}^{\prime \prime} \\ x_{1}^{\prime \prime \prime} & x_{3}^{\prime \prime \prime} & x_{4}^{\prime \prime \prime}\end{array}\left|,-\left|\begin{array}{lll}x_{1}^{\prime} & x_{2}^{\prime} & x_{4}^{\prime} \\ x_{1}^{\prime \prime} & x_{2}^{\prime \prime} & x_{4}^{\prime \prime} \\ x_{1}^{\prime \prime \prime} & x_{2}^{\prime \prime \prime} & x_{4}^{\prime \prime \prime}\end{array}\right|\right) \Omega^{t r}\right.$, therefore, we
have
$\operatorname{det} A=\left(\left|\begin{array}{cccc}-i & j & k & l \\ x_{1}^{\prime} & x_{2}^{\prime} & x_{3}^{\prime} & x_{4}^{\prime} \\ x_{1}^{\prime \prime} & x_{2}^{\prime \prime} & x_{3}^{\prime \prime} & x_{4}^{\prime \prime} \\ x_{1}^{\prime \prime \prime} & x_{2}^{\prime \prime \prime} & x_{3}^{\prime \prime \prime} & x_{4}^{\prime \prime \prime}\end{array}\right|\right) \Omega^{t r}$,
by using Frenet formulae, we can write

$$
\begin{aligned}
\operatorname{det} A & =\left(\mathbf{T} \wedge\left(\mathbf{T}^{\prime} \wedge \mathbf{T}^{\prime \prime}\right)\right) \Omega^{t r} \\
& =\left(\left(\mathbf{T} \wedge\left(\gamma(s)+\kappa_{g} \mathbf{E}_{1}(s)\right) \wedge\left(\left(1-\kappa_{g}^{2}\right) \mathbf{T}(s)+\kappa_{g}^{\prime} \mathbf{E}_{1}(s)-\kappa_{g} \tau_{g} \mathbf{E}_{2}(s)\right)\right) \Omega^{t r}\right. \\
& =\frac{-1}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}}\left(\kappa_{g}^{2} \tau_{g} \gamma(s)+\kappa_{g} \tau_{g} \mathbf{E}_{1}(s)+\kappa_{g}^{\prime} \mathbf{E}_{2}(s)\right)\left(\begin{array}{c}
-v_{1} \\
v_{2} \\
v_{3} \\
\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}
\end{array}\right) \\
& =\left(\frac{-1}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}}\right)\left\langle\kappa_{g}^{2} \tau_{g} \gamma(s)+\kappa_{g} \tau_{g} \mathbf{E}_{1}(s)+\kappa_{g}^{\prime} \mathbf{E}_{2}(s), v\right\rangle,
\end{aligned}
$$

where

$$
v= \pm \frac{1}{\sqrt{\left.\| \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1 \right\rvert\,}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right)
$$

then we have

$$
\begin{aligned}
& \operatorname{det} A=\left(\frac{-1}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}}\right)\left( \pm \frac{-\kappa_{g}^{3} \tau_{g}+\kappa_{g} \tau_{g}+\frac{\kappa_{g}^{\prime 2}}{\kappa_{g} \tau_{g}}}{\sqrt{\| \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1}}\right) \\
& =\left(\frac{-1}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}}\right)\left( \pm \frac{-\kappa_{g} \tau_{g}\left(\kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1\right)}{\sqrt{\left.\| \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1 \right\rvert\,}}\right) \\
& =\left(\mp \frac{\kappa_{g} \tau_{g} \sqrt{\left|\kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1\right|}}{\sqrt{-1+v_{1}^{2}-v_{2}^{2}-v_{3}^{2}}}\right) \neq 0 .
\end{aligned}
$$

By the same way as the above, if we consider the spacelike height function $H^{S}$, we can prove Proposition 4.1(2).

Theorem 4.2 Let $\gamma: I \rightarrow H_{+}^{3}(-1)$ be a regular curve such that $\left\|\mathbf{T}^{\prime}(s)\right\| \neq 0$. Then we have the following assertions:
(A1) The hyperbolic evolute at $h_{\gamma}\left(s_{0}\right)$ is regular if $\sigma_{H}\left(s_{0}\right) \neq 0$.
(A2) The following conditions are equivalent:

1. The germ of the hyperbolic evolute at $h_{\gamma}\left(s_{0}\right)$ is diffeomorphic to a Swallowtail surface,
2. $\quad \sigma_{H}\left(s_{0}\right)=0$ and $\sigma_{H}^{\prime}\left(s_{0}\right) \neq 0$,
3. $\gamma(s)$ and the pseudo sphere $P S^{1}\left(v_{0}, r_{0}\right)$ have contact of order four where $v_{0}=h_{\gamma}\left(s_{0}\right)$.
(B1) The de Sitter evolute at $d_{\gamma}\left(s_{0}\right)$ is regular if $\sigma_{D}\left(s_{0}\right) \neq 0$.
(B2) The following conditions are equivalent:
4. The germ of the de Sitter evolute at $d_{\gamma}\left(s_{0}\right)$ is diffeomorphic to a Swallowtail surface,
5. $\quad \sigma_{D}\left(s_{0}\right)=0$ and $\sigma_{D}^{\prime}\left(s_{0}\right) \neq 0$,
6. $\gamma(s)$ and the pseudo sphere $P S^{1}\left(v_{0}, r_{0}\right)$ have contact of order four where $v_{0}=d_{\gamma}\left(s_{0}\right)$.

Proof. (A1) By the assertion of Proposition 3.1, we have $h_{\gamma}^{\prime}(s)=0$ if $\sigma_{H}(s)=0$. It means that the hyperbolic evolute at $h_{\gamma}(s) \neq 0$ is regular if $\sigma_{H}(s) \neq 0$.
(A2) By Proposition 3.1, the bifurcation set of $H^{T}$ is
$B_{H^{\tau}}=\left(\left.h_{\gamma}(s)=\frac{1}{\sqrt{\| \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}-1}}\left(\kappa_{g} \gamma(s)+\mathbf{E}_{1}(s)+\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}} \mathbf{E}_{2}(s)\right) \right\rvert\, \kappa_{g}^{2}-\left(\frac{\kappa_{g}^{\prime}}{\kappa_{g} \tau_{g}}\right)^{2}>1\right)$.
By Theorem 4.1 and Proposition 4.1, the germ of the bifurcation set is diffeomorphic to a Swallowtail surface if $\sigma_{H}=0$ and $\sigma_{H}^{\prime} \neq 0$. Moreover, we have other equivalences from Proposition 3.3 and Proposition 3.4. This completes the proof of (A1) and (A2). If we consider the spacelike height function $H^{S}$, then we can prove the remaining assertions of the theorem.

## 5 Computational examples

In this section, we consider some illustrative examples to explain the evolute curve on a
surface of graphical representation in $H_{+}^{2}(-1)$ and $H_{+}^{3}(-1)$. From Theorem 4.1, the evolute curve in $H_{+}^{2}(-1)$ at $h_{\gamma}\left(s_{0}\right)$ is regular if $\sigma_{H} \neq 0$, and is the ordinary cusp locally if $\sigma_{H}=0$, and $\sigma_{H}^{\prime} \neq 0$, see Example 5.1 and Example 5.2.
Also, by Proposition 3.1, $\left(h_{v}^{T}\right)^{\prime}(s)=\left(h_{v}^{T}\right)^{\prime \prime}(s)=\left(h_{v}^{T}\right)^{\prime \prime \prime}(s)=\left(h_{v}^{T}\right)^{(4)}(s)=\left(h_{v}^{T}\right)^{(5)}(s)=0$,
if and only if $h_{\gamma}\left(s_{0}\right)$ is the evolute curve in $H_{+}^{3}(-1)$ and $\sigma_{H}=\sigma_{H}^{\prime}=0$, see Example 5.3.
Example 5.1 In $H_{+}^{2}(-1)$, we have the hyperbolic Frenet formulae of a curve $\gamma$ :
$\left\{\begin{array}{l}\gamma^{\prime}(s)=\mathbf{T}(s), \\ \mathbf{T}^{\prime}(s)=\gamma(s)+\kappa_{g}(s) \mathbf{E}(s), \\ \mathbf{E}^{\prime}(s)=-\kappa_{g}(s) \mathbf{T}(s),\end{array}\right.$
where $\kappa_{g}$ is the geodesic curvature of $\gamma$ in $H_{+}^{2}$, which is given by
$\kappa_{g}(s)=\operatorname{det}\left(\gamma(s) \mathbf{T}(s) \mathbf{T}^{\prime}(s)\right)$.
The hyperbolic evolute of $\gamma$ is given by
$h_{\gamma}(s)=\frac{1}{\sqrt{\left|\kappa_{g}^{2}(s)-1\right|}}\left(\kappa_{g}(s) \gamma(s)+\mathbf{E}(s)\right)$,
Here, we have

$$
\sigma_{H}=\frac{-\kappa_{g}^{\prime}}{\kappa_{g}^{2}} \text { and } \sigma_{H}^{\prime}=\frac{-\kappa_{g}^{\prime \prime} \kappa_{g}+2 \kappa_{g}^{\prime 2}}{\kappa_{g}^{3}}[\text { Sato (2012) }]
$$

Now, suppose that $X(u, v)=\left(u^{2}, u^{3}, v\right)$ be a cuspidal edge surface, see Fig. 2. We investigate the following curve

$$
\begin{equation*}
\gamma(u)=\left(u^{2}, u^{3}, 1\right), \tag{5.3}
\end{equation*}
$$

From Eq. (5.3), we have

$$
\begin{equation*}
\mathbf{T}=\frac{\gamma^{\prime}}{\left\|\gamma^{\prime}\right\|}=\frac{1}{\sqrt{9 u^{2}+4}}(2,3 u, 0), \tag{5.4}
\end{equation*}
$$

Since $\mathbf{E}=\gamma \wedge \mathbf{T}$, therefore we get
$\mathbf{E}=\frac{1}{\sqrt{9 u^{2}+4}}\left|\begin{array}{ccc}-i & j & k \\ u^{2} & u^{3} & 1 \\ 2 & 3 u & 0\end{array}\right|$,
therefore, we have

$$
\begin{align*}
& \mathbf{E}=\frac{1}{\sqrt{9 u^{2}+4}}\left(3 u, 2, u^{3}\right),  \tag{5.6}\\
& \mathbf{T}^{\prime}=\frac{1}{\left(9 u^{2}+4\right)^{3 / 2}}(-18 u, 12,0), \tag{5.7}
\end{align*}
$$

from Eqs. (5.3), (5.4) and (5.7), we get

$$
\begin{align*}
\kappa_{g} & =\left|\begin{array}{ccc}
u^{2} & \frac{u^{3}}{} & 1 \\
\frac{3 u}{\sqrt{u^{2}+4}} & \frac{3}{\sqrt{9 u^{2}+4}} & 0 \\
\frac{-18 u}{\left(9 u^{2}+4\right)^{\frac{3}{2}}} & \frac{12}{\left(9 u^{2}+4\right)^{\frac{3}{2}}} & 0
\end{array}\right|  \tag{5.8}\\
& =\frac{6}{\left(9 u^{2}+4\right)} .
\end{align*}
$$

Then, we have

$$
\kappa_{g}^{\prime}=\frac{-108 u}{\left(9 u^{2}+4\right)^{2}},
$$

and so

$$
\kappa_{g}^{\prime \prime}=\frac{26244 u^{4}+7776 u^{2}-1728}{\left(9 u^{2}+4\right)^{4}}
$$

At the point $u=0$, we have

$$
\gamma=(0,0,1), \mathbf{T}=(1,0,0), \mathbf{E}=(0,1,0)
$$

$\kappa_{g}=\frac{3}{2}, \kappa_{g}^{\prime}=0, \kappa_{g}^{\prime \prime}=-\frac{27}{4}, \sigma_{H}=0, \sigma_{H}^{\prime}=3$.
Thus, the hyperbolic evolute of $\gamma$ has a cusp at the origin.
Now, we find the equation of the hyperbolic evolute curve of cuspidal edge surface. From Eqs. (5.2), (5.3), (5.6) and (5.8), we have

$$
\begin{equation*}
h_{\gamma}=\frac{1}{\sqrt{36-(9 u+4)^{2}}}\left(6 u^{2}+3 u \sqrt{9 u+4}, 6 u^{3}+2 \sqrt{9 u+4}, 6+u^{3} \sqrt{9 u+4}\right) \tag{5.9}
\end{equation*}
$$

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Figure 2: The curve $\gamma$, Cuspidal edge surface $\Psi$

Example 5.2 In this example, we consider a spacelike curve $\gamma(s)$ lying fully on an oriented ruled surface $\Omega$ in $H_{+}^{2}(-1)$, as follows
$\Omega(s, v)=(s, \sin (\sqrt{2} s), \cos (\sqrt{2} s))+v(1, \cos (\sqrt{2} s),-\sin (\sqrt{2} s))$,
(see Fig. 3). We investigate the following curve:
$\gamma(s)=(s, \sin (\sqrt{2} s), \cos (\sqrt{2} s))$,
it follows that
$\left\{\begin{array}{l}\gamma^{\prime}=(1, \sqrt{2} \cos (\sqrt{2} s),-\sqrt{2} \sin (\sqrt{2} s)), \\ \gamma^{\prime \prime}=(0,-2 \sin (\sqrt{2} s),-2 \cos (\sqrt{2} s)), \\ \gamma^{\prime \prime \prime}=(0,-2 \sqrt{2} \cos (\sqrt{2} s), 2 \sqrt{2} \sin (\sqrt{2} s)) .\end{array}\right.$
From Eq. (5.11), we get
$\mathbf{T}=(1, \sqrt{2} \cos (\sqrt{2} s),-\sqrt{2} \sin (\sqrt{2} s))$,
where $\mathbf{E}=\gamma \wedge \mathbf{T}$, and we get
$\mathbf{E}=\left|\begin{array}{ccc}-i & j & k \\ s & \sin (\sqrt{2} s) & \cos (\sqrt{2} s) \\ 1 & \sqrt{2} \cos (\sqrt{2} s) & -\sqrt{2} \sin (\sqrt{2} s)\end{array}\right|$,
therefore, we have
$\mathbf{E}=(\sqrt{2}, \sqrt{2} s \sin (\sqrt{2} s)+\cos (\sqrt{2} s), \sqrt{2} s \cos (\sqrt{2} s)-\sin (\sqrt{2} s))$,
$\mathbf{T}^{\prime}=(0,-2 \sin (\sqrt{2} s),-2 \cos (\sqrt{2} s))$,
using Eqs. (5.9), (5.11) and (5.13), we get
$\kappa_{g}=\left|\begin{array}{ccc}s & \sin (\sqrt{2} s) & \cos (\sqrt{2} s) \\ 1 & \sqrt{2} \cos (\sqrt{2} s) & -\sqrt{2} \sin (\sqrt{2} s) \\ 0 & -2 \sin (\sqrt{2} s) & -2 \cos (\sqrt{2} s)\end{array}\right|=-2 \sqrt{2} s$.
Then, we have $\kappa_{g}^{\prime}=-2 \sqrt{2}$, and then $\kappa_{g}^{\prime \prime}=0$. Therefore, we get $\sigma_{H} \neq 0$, this means that the hyperbolic evolute curve is regular curve.


Figure 3: A spacelike curve $\gamma(s)$ lying fully on an oriented ruled surface $\Omega$

Example 5.3 Suppose that $\Phi(u, v)=\left(3 u^{4}+u^{2} v, 4 u^{3}+2 u v, v\right)$ be a Swallowtail surface (see Fig. 4). In this example, we investigate the following curve $\beta$ in $H_{+}^{3}(-1)$ :

$$
\begin{equation*}
\beta(u)=\left(3 u^{4}+u^{2}, 4 u^{3}+2 u, 1,1\right), \tag{5.16}
\end{equation*}
$$

it follows that

$$
\begin{aligned}
\beta^{\prime} & =\left(12 u^{3}+2 u, 12 u^{2}+2,0,0\right), \\
\beta^{\prime \prime} & =\left(36 u^{2}+2,24 u, 0,0\right), \\
\beta^{\prime \prime \prime} & =(72 u, 24,0,0), \\
\beta^{(4)} & =(72,0,0,0), \\
\beta^{(5)} & =(0,0,0,0) .
\end{aligned}
$$

Therefore, we have
$\mathbf{T}=\frac{\beta^{\prime}}{\left\|\beta^{\prime}\right\|}=\frac{1}{\sqrt{-36 u^{6}+24 u^{4}+11 u^{2}+1}}\left(6 u^{3}+u, 6 u^{2}+1,0,0\right)$,
and

$$
\mathbf{T}^{\prime}=\frac{1}{\left(-36 u^{6}+24 u^{4}+11 u^{2}+1\right)^{3 / 2}}\left(\begin{array}{c}
-1296 u^{8}+576 u^{6}+336 u^{4}+40 u^{2}+1  \tag{5.18}\\
-1080 u^{7}+468 u^{5}+246 u^{3}+23 u \\
0 \\
0
\end{array}\right)
$$

from Eqs. (5.16) and (5.18), we get

$$
\mathbf{T}^{\prime}-\beta=\left(\begin{array}{c}
\frac{\left(-1296 u^{8}+576 u^{6}+336 u^{4}+40 u^{2}+1\right)}{\left(-36 u^{6}+24 u^{4}+11 u^{2}+1\right)^{3 / 2}}-\left(3 u^{4}+u^{2}\right)  \tag{5.19}\\
\frac{\left(-1080 u^{7}+468 u^{5}+246 u^{3}+23 u\right)}{\left(-36 u^{6}+24 u^{4}+11 u^{2}+1\right)^{3 / 2}}-\left(4 u^{3}+2 u\right) \\
-1 \\
-1
\end{array}\right),
$$

therefore

$$
\begin{equation*}
\left\|\mathbf{T}^{\prime}-\beta\right\|=\binom{\left(\frac{\left(-1296 u^{8}+576 u^{6}+336 u^{4}+40 u^{2}+1\right)}{\left(-36 u^{6}+24 u^{4}+11 u^{2}+1\right)^{3 / 2}}-\left(3 u^{4}+u^{2}\right)\right)^{2}}{+\left(\frac{\left(-1080 u^{7}+468 u^{5}+246 u^{3}+23 u\right)}{\left(-36 u^{6}+24 u^{4}+11 u^{2}+1\right)^{3 / 2}}-\left(4 u^{3}+2 u\right)\right)^{\frac{1}{2}}+2}^{2} . \tag{5.20}
\end{equation*}
$$

Then, from Eqs. (5.19) and (5.20), we get

$$
\mathbf{E}_{1}=\frac{\mathbf{T}^{\prime}-\beta}{\left\|\mathbf{T}^{\prime}-\beta\right\|}, \quad \kappa_{g}=\left\|\mathbf{T}^{\prime}-\beta\right\| .
$$

And we have

$$
\begin{aligned}
\tau_{g} & =-\frac{\operatorname{det}\left(\beta, \beta^{\prime}, \beta^{\prime \prime}, \beta^{\prime \prime \prime}\right)}{\kappa_{g}^{2}} \\
& =\frac{-1}{\kappa_{g}^{2}}\left|\begin{array}{cccc}
-3 u^{4}+u^{2} & 4 u^{3}+2 u & 1 & 1 \\
12 u^{3}+2 u & 12 u^{2}+2 & 0 & 0 \\
36 u^{2}+2 & 24 u & 0 & 0 \\
72 u & 24 & 0 & 0
\end{array}\right|=0 .
\end{aligned}
$$

So, at the point $u=0$, we obtain

$$
\beta=(0,0,1,1), \mathbf{T}=(0,1,0,0), \mathbf{E}_{1}=\left(\frac{1}{\sqrt{3}}, 0, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right),
$$

and
$\mathbf{E}_{2}=\left(0,0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \kappa_{g}=\sqrt{3}, \tau_{g}=0, \sigma_{H}=0, \sigma_{H}^{\prime}=0$.
Finally, from the previous calculations, we can find the equation of the evolute curve in hyperbolic 3 -space as given in Eq. (3.7).


Figure 4: The Swallowtail surface $\Phi(u, v)$

## 6 Conclusion

In the three dimensional hyperbolic space, the pseudo-spherical evolutes of curves are studied. Also, some relationships between singularities of these curves and geometric invariants under the action of the Lorentz group are obtained. Furthermore, some computational examples in support of main results are given and plotted.

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