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The Discrete-Analytical Solution Method for Investigation Dynamics of the Sphere with Inhomogeneous Initial Stresses

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Abstract: The paper deals with a development of the discrete-analytical method for the solution of the dynamical problems of a hollow sphere with inhomogeneous initial stresses. The examinations are made with respect to the problem on the natural vibration of the hollow sphere the initial stresses in which is caused by internal and external uniformly distributed pressure. The initial stresses in the sphere are determined within the scope of the exact equations of elastostatics. It is assumed that after appearing this static initial stresses the sphere gets a dynamical excitation and mechanical behavior of the sphere caused by this excitation is described with the so-called three-dimensional linearized equations of elastic wave propagation in initially stressed bodies. For the solution of these equations, which have variable coefficients, the discrete analytical solution method is developed and applied. In particular, it is established that the convergence of the numerical results with respect to the number of discretization is very acceptable and applicable for the considered type dynamical problems. Numerical results on the influence of the initial stresses on the values of the natural frequencies of the hollow sphere are also presented and these results are discussed.

Keywords: Discrete-analytical solution method, initial stress, hollow sphere, natural frequency, dynamical problem.

1 Introduction

Lamb started the investigations on the natural vibrations of a sphere [Lamb (1882)] in which it was assumed that the material of the sphere is homogeneous and isotropic. Note that in this work the mathematical procedures have been made by employing of the Cartesian coordinates and it was established that the sphere has two types of uncoupled

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free vibrations the first (the second) of which is the torsional (spheroidal) vibration. As noted by Love [Love (1944)], in the historical aspect this investigation by Lamb and its developing by other researches were associated originally with interest in the oscillations of the earth. Later on many researchers, such as Chree [Chree (1889)], Sato et al. [Sato and Usami (1962a, 1962b)] and Sato et al. [Sato, Usami and Ewing (1962)] have developed the results by Lamb and in the works of these researches a detailed analysis of the natural frequencies and vibration modes of the homogeneous isotropic solid sphere were performed and tabulated. At the same time, these results were discussed in the monograph by Eringen et al. [Eringen and Suhubi (1975)].

It was also investigated the natural vibration of the homogeneous isotropic elastic hollow sphere (or spherical shell). As the examples of such investigations can be taken the studies made in Shah [Shah, Ramakrishnan and Datta (1969a, 1969b)] within the scope of the three-dimensional exact equations of the linear theory of elastodynamics. Note that in these studies numerical results on the natural frequencies are given in graphical form for varies values of the geometrical parameters of the hollow sphere. Moreover, note that in the book by Lapwood et al. [Lapwood and Usami (1981)] the aforementioned and other related results were associated with the free oscillation of the earth.

The papers by Hasheminejad et al. [Hasheminejad and Mirzaei (2011)] and Sharma et al. [Sharma, Sharma and Dhaliwai (2013)] and others listed therein studied more complicated problems on the vibration of the solid and hollow spheres and the mentioned complication are caused with geometries and material properties of the spheres. It should be also noted that, in general, the recent years the investigations regarding the dynamics of the structural elements made of FGM, piezoelectric materials etc. are developed intensively. As an example for such investigations, it can be taken the papers [Asemi, Salehi and Sadighi (2014); Wang, Xu and Ding (2010); Asgari and Akhlaghi (2011); Ilhan and Koç (2015); Ye, Jin and Su (2014)] and others listed therein.

It should be noted that one of the main questions under investigations of the vibration of the hollow spheres is the accuracy of the theories (for instance, the accuracy of the various type shell theories) which are applied for these investigations. The benchmark results for testing the mentioned accuracy are the results obtained within the scope of the 3D exact equations and relations of the elastodynamics. Such testing was made in the paper Grigorenko et al. [Grigorenko and Kilina (1989)] and it is established that (as it can be predicted) the accuracy of the approximate shell theories decreases with increasing of the ratio h/R, where h is the thickness and R is the radius of the middle surface of the sphere. Moreover, the results obtained in the paper [Jiang, Young and Dickinson (1996)], in which the natural vibration of the three-layered hollow sphere was studied by utilizing the 3D exact equations of elastodynamics, can be taken as benchmark ones for the corresponding results obtained within the scope of various approximate shell theories.

The 3D exact equations of elastodynamics are also employed in the paper Chen et al. [Chen and Ding (2001)] for investigation of the vibration of the spherically isotropic (a special case of transversal isotropic materials) layered hollow sphere and numerical results are presented and discussed for the three-layered case, according to which, it is established the influence of the layers' materials anisotropy on the natural frequencies and vibration modes of the hollow sphere.

As noted above, the study of the vibration of the sphere originally was associated with the investigations of the Earth's oscillation and for acceptable applicability of these studies, it is necessary to use the modern Earth's real models and the modern real models on the bowels of the earth. Such models are detailed in many references (see, for instance, the monograph [Anderson (2007)] related to the modern Earth's theories. According to these theories, it is established that the mechanical properties, such as the modulus of elasticity and density of the mantle material increase continuously from the crust to the core. Namely, this statement allows researchers of the papers by Akbarov et al. [Akbarov, Guliyev and Yahnioglu (2016, 2017)] to use the FGM model for describing the mantle material and to consider within this framework the vibration problems of the layered solid and hollow spheres as the Earth's model and to investigate the natural frequencies of these spheres.

The other particularity of the modern Earth theories is to take into consideration of the reference properties of the bowels of the earth one of which is the inhomogeneous initial stresses appearing because of the Earth's gravitation. Consequently, this statement requires investigating the dynamics of the hollow and solid spheres with initial inhomogeneous stresses.

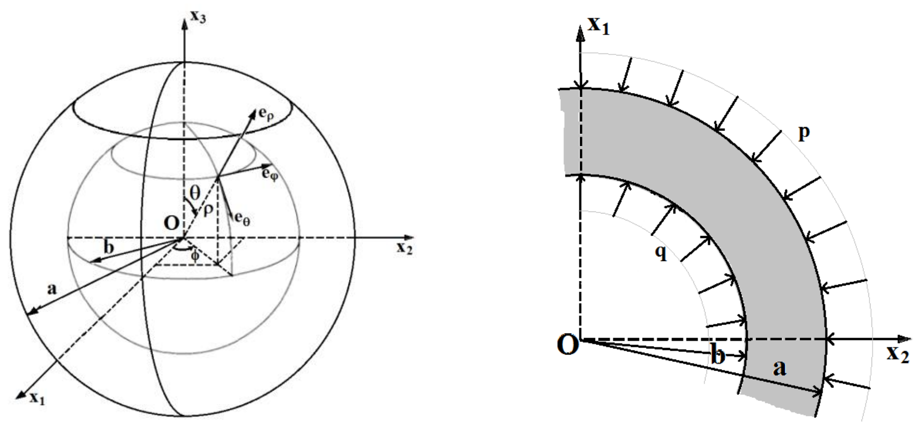
It should be noted that the first attempt on the investigation the dynamics of the solid sphere with initial stresses has been made in the papers [Guz (1985a, 1985b)] in which it is considered the natural vibration of the compressible [Guz (1985a] and incompressible [Guz (1985b] homogeneous isotropic solid sphere with homogeneous initial stresses caused by the initial overall compression of that. The mathematical modelling of the problems was made by utilizing the 3D linearized theory of elastic waves in bodies with initial stresses and it was established that in the qualitative sense Lamb's result on the types of natural vibration of the sphere occurs also for the initially stressed cases.

At the same time, in the works of Stevanovic et al. [Stevanovic, Wodicka, Bourland et al. (1995); Li and Luo (2010); Piacsek, Abdul-Wahid and Taylor (2012)] and others listed therein, the attempts have been made within the scope of the experimental or FEM methods for study of the influence of the internal pressure acting in the interior of the hollow sphere to its dynamic response to the action of the external dynamical forces.

However, up to now, there are not any investigations on the influence of the inhomogeneous initial stresses on the dynamic response of the layered hollow and solid spheres to the external dynamical loading. The mentioned inhomogeneous initial stresses can be caused by the internal or external overall compressions, as well as with the gravitational forces. In connection with this, in the present paper, we attempt to purpose the discrete-analytical approach for solution corresponding problems in the cases where the initial stresses distribution has central symmetry and these stresses are determined within the scope of the classical linear theory of elasticity. However, the dynamical behavior of the sphere is described within the scope of the 3D linearized theory of elastic waves in initially stressed bodies. For simplicity, all the mathematical procedures are made for the single-layer hollow sphere made of a homogeneous isotropic material in the case where in the initial state on the inner and outer face-surfaces of this sphere the uniformly distributed normal forces act.

2 Formulation of the problem

We consider a hollow sphere with internal radius b and external radius a and assume that this sphere is loaded with the uniformly distributed normal forces with intensities p and q on the outer and inner surfaces of that, respectively. The sketch of the sphere and external forces is shown in Fig. 1. We associate the Cartesian coordinate system $Ox_1x_2x_3$ and spherical coordinate system $Or\theta\phi$ with the center of the sphere (Fig. 1). It is known that (see, for instance, the reference Timoshenko et al. [Timoshenko and Goodier (1970)]) in the case under consideration, the stress state in the sphere is determined by the following expressions.



(a) Hollow sphere

(b) A cross-section of the hollow sphere at $x_3 = 0$

Figure 1: The sketch of the hollow sphere and initial external forces

$$\sigma_{rr}^{(0)} = -p\frac{a^3(r^3-b^3)}{(a^3-b^3)r^3} - q\frac{b^3(a^3-r^3)}{(a^3-b^3)r^3} \ ,$$

$$\sigma_{\theta\theta}^{(0)} = \sigma_{\varphi\varphi}^{(0)} = -p\frac{a^3(2r^3+b^3)}{2(a^3-b^3)r^3} + q\frac{b^3(2r^3+a^3)}{2(a^3-b^3)r^3} \,,$$

$$\sigma_{r\theta}^{(0)} = \sigma_{r\varphi}^{(0)} = \sigma_{\theta\varphi}^{(0)} = 0 \,, \tag{1}$$

where upper index (0) indicates that the corresponding quantity belongs to the initial stress state.

Note that the stress state determined by the expressions in (1) is called the initial stress state in the sphere in the case under consideration. It is assumed that the sphere (with these initial stresses) gets dynamic time-harmonic excitation and it is required to determine how these initial stresses influence on the dynamical behavior (for instance, on the natural frequencies) of the sphere. In order to take into consideration this influence, it is necessary to make the mathematical formulation of the related problem within the scope of the three-dimensional linearized theory of elastic waves in initially stressed bodies. Note that these equations are obtained from the linearization of the corresponding geometrically non-linear equations.

Thus, we consider the mathematical formulation of the problem on the dynamics of the hollow sphere with the initial stresses given in (1) within the scope of the aforementioned linearized field equations. According to the references [Eringen and Suhubi (1975); Guz (2004)], these equations in the spherical coordinates (r,θ,ϕ) can be presented as follows.

Equations of motion:

$$\frac{\partial t_{rr}}{\partial r}+\frac{1}{r}\frac{\partial t_{\phi r}}{\partial \phi}+\frac{1}{r\sin\phi}\frac{\partial t_{\theta r}}{\partial \theta}+\frac{1}{r}\left(2t_{rr}-t_{\phi\phi}-t_{\theta\theta}+t_{\phi r}\cot\phi\right)=\rho\frac{\partial^2 u_r}{\partial t^2},$$

$$\frac{\partial t_{r\phi}}{\partial r}+\frac{1}{r}\frac{\partial t_{\phi\phi}}{\partial \phi}+\frac{1}{r\sin\phi}\frac{\partial t_{\theta\phi}}{\partial \theta}+\frac{1}{r}\left(2t_{r\phi}+t_{\phi r}+\left(t_{\phi\phi}-t_{\theta\theta}\right)\cot\phi\right)=\rho\frac{\partial^2 u_\phi}{\partial t^2},$$

$$\frac{\partial t_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial t_{\phi\theta}}{\partial \phi}+\frac{1}{r\sin\phi}\frac{\partial t_{\theta\theta}}{\partial \theta}+\frac{1}{r}\left(2t_{r\theta}+t_{\theta r}+\left(t_{\phi\theta}+t_{\theta\phi}\right)\cot\phi\right)=\rho\frac{\partial^2 u_\theta}{\partial t^2}. \tag{2}$$

where

$$t_{rr}=\sigma_{rr}+\sigma_{rr}^{(0)}\frac{\partial u_r}{\partial r},\quad t_{\phi\phi}=\sigma_{\phi\phi}+\sigma_{\phi\phi}^{(0)}\left(\frac{u_r}{r}+\frac{\partial u_\phi}{r\partial\phi}\right),\ t_{r\phi}=\sigma_{r\phi}+\sigma_{rr}^{(0)}\frac{\partial u_\phi}{\partial r},$$

$$t_{\theta\theta}=\sigma_{\theta\theta}+\sigma_{\theta\theta}^{(0)}\left(\frac{u_r}{r}+\cot\phi\frac{u_\phi}{r}+\frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial\theta}\right),\ t_{\phi r}=\sigma_{\phi r}+\sigma_{\phi\phi}^{(0)}\left(\frac{\partial u_r}{r\partial\phi}-\frac{u_\phi}{r}\right),$$

$$t_{r\theta}=\sigma_{r\theta}+\sigma_{rr}^{(0)}\frac{\partial u_\theta}{\partial r},\ t_{\theta r}=\sigma_{\theta r}+\sigma_{\theta\theta}^{(0)}\left(\frac{1}{r\sin\phi}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right),\ t_{\phi\theta}=\sigma_{\phi\theta}+\sigma_{\phi\phi}^{(0)}\frac{\partial u_\theta}{r\partial\phi},$$

$$t_{\theta\phi}=\sigma_{\theta\phi}+\sigma_{\theta\theta}^{(0)}\left(\frac{1}{r\sin\phi}\frac{\partial u_\phi}{\partial\theta}-\cot\phi\frac{u_\theta}{r}\right). \tag{3}$$

Elasticity relations:

$$\sigma_{rr}=\lambda(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\phi\phi})+2\mu\varepsilon_{rr},\ \sigma_{\theta\theta}=\lambda(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\phi\phi})+2\mu\varepsilon_{\theta\theta},$$

$$\sigma_{\varphi\varphi}=\lambda(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\phi\phi})+2\mu\varepsilon_{\phi\phi},\ \sigma_{r\theta}=2\mu\varepsilon_{r\theta};\ \sigma_{\theta\phi}=2\mu\varepsilon_{\theta\phi};$$

$$\sigma_{r\phi}=2\mu\varepsilon_{r\phi}. \tag{4}$$

Strain-displacement relations:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \ , \ \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}u_r \ ; \ \varepsilon_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{1}{r}u_r + \frac{1}{r}u_\theta \cot\theta \ ;$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right); \ \varepsilon_{\theta\phi}^{(k)} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\phi^{(k)}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\theta^{(k)}}{\partial \phi} - \frac{u_\phi^{(k)}}{r}\cot\theta\right);$$

$$\varepsilon_{r\phi} = \frac{1}{2}\left(\frac{\partial u_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \theta} - \frac{u_\phi}{r}\right). \tag{5}$$

In (2) and (3) the notation t_{rr} ,..., $t_{\theta\phi}$ indicates the component of the Kirchhoff stress tensor in the spherical coordinate system, however, the other notation used in (1)-(5) is the conventional one.

It must be added to the field Eqs. (2)-(5) the corresponding boundary conditions that are satisfied on the hollow sphere's inner and outer face surfaces. As an example of these conditions, we consider the case where the following homogeneous ones are satisfied.

$$t_{rr}\big|_{r=a} = 0 , \ t_{r\theta}\big|_{r=a} = 0 , \ t_{r\phi}\big|_{r=a} = 0 , \ t_{rr}\big|_{r=b} = 0 , \ t_{r\theta}\big|_{r=b} = 0 , \ t_{r\phi}\big|_{r=b} = 0 . \tag{6}$$

It is evident that it can be formulated the nonhomogeneous boundary conditions instead of the homogeneous boundary conditions given in (6). Moreover, it can also be formulated corresponding initial conditions for the non-stationary dynamical problems.

This completes the formulation of the problem and according to expressions given in (1) for the initial stresses, this formulation contains the system of partial differential equations with variable coefficients the analytical solution of which, in general, is impossible. Namely for such cases below a method is developed, according to which, the solution to these equations is reduced to the solutions to the series corresponding equations which have analytical solutions.

Note that in the case where the initial stresses in the sphere are absent, i.e. in the case where $p = q = 0$ the foregoing formulation coincides with the corresponding one made within the scope of the classical linear elastodynamics.

3 Method of solution

According to expressions in (1), the system of equations in (2)-(5) are the equations with variable coefficients the analytical solution of which, in general, is very difficult and in many cases is impossible. Therefore, in many cases for solutions, the problems formulated through the Eqs. (2)-(5) are solved numerically with employing various numerical methods described for instance in the works Babuscu Yesil et al. [Babuscu Yesil (2017); Wei, Chen and Chen (2015)] and others listed therein. However, in the case under consideration there is the following particularity with respect to the variable coefficients of the Eqs. (2)-(5): these coefficients depend only on the coordinate r . Namely, this particularity of the coefficients allows us to employ the discrete-analytical method developed and employed in the works by Akbarov et al. [Akbarov (2006); Akbarov (2015); Akbarov and Panakhli (2015, 2017); Akbarov, Guliyev and Yahnioglu (2016, 2017)] for solution of the Eqs. (2)-(5).

Thus, we attempt to develop and employ the aforementioned discrete-analytical method to solve the Eqs. (2)-(5) and illustrate what kinds of difficulties appear under this employing and how these difficulties are overcome.

3.1 Discretization of the solution domain and obtaining the equations for the functions which enter into the classical Lame decomposition

First, for employing the discrete-analytical method, it is necessary to divide of the interval $r \in [a,b]$ a certain number subintervals. For describing this dividing we introduce notation $R_1 = b$ and $R_2 = a$ and divide the interval $[R_1, R_2]$ into the subintervals $[R_{1k}, R_{2k}]$ where $\bigcup_{k=1}^{N} [R_{1k}, R_{2k}] = [R_1, R_2]$, $R_{1k} = R_1 + (k-1)(R_2 - R_1)/N$ and $R_{2k} = R_1 + k(R_2 - R_1)/N$. Within the framework of each subinterval $[R_{1k}, R_{2k}]$ the functions $\sigma_{rr}^{(0)}(r)$, $\sigma_{\theta\theta}^{(0)}(r)$ and $\sigma_{\phi\phi}^{(0)}(r)$ are taken as constant ones which are equal to their value at $r_k = R_{1k} + (R_{2k} - R_{1k})/2$, i.e. it is written $\sigma_{rr}^{(0)}(r_k)$, $\sigma_{\theta\theta}^{(0)}(r_k)$ and $\sigma_{\phi\phi}^{(0)}(r_k)$ instead of the corresponding functions. After this discretization of the solution interval, the complete system of Eqs. (2)-(5) are satisfied within each subintervals $[R_{1k}, R_{2k}]$ separately under which the variable coefficients $\sigma_{rr}^{(0)}(r)$, $\sigma_{\theta\theta}^{(0)}(r)$ and $\sigma_{\phi\phi}^{(0)}(r)$ in the system of Eqs. (2) and (3) are replaced with constants $\sigma_{rr}^{(0)}(r_k)$, $\sigma_{\theta\theta}^{(0)}(r_k)$ and $\sigma_{\phi\phi}^{(0)}(r_k)$ where $r_k = R_{1k} + (R_{2k} - R_{1k})/2$. On the interface surfaces between the subintervals the continuity conditions for force and displacement vectors are satisfied. Denoting the thickness of each sublayer through $h_k = (R_1 - R_2)/N$ and introducing the additional upper index (k) for the quantities related to the $k-th$ sublayer these continuity conditions can be written as follows

$$t_{rr}^{(1)}\Big|_{r=R_1} = 0,\ t_{r\theta}^{(1)}\Big|_{r=R_1} = 0,\ t_{r\phi}^{(1)}\Big|_{r=R_1} = 0,$$

$$t_{rr}^{(1)}\Big|_{r=R_1+h} = t_{rr}^{(2)}\Big|_{r=R_1+h},\ t_{r\theta}^{(1)}\Big|_{r=R_1+h} = t_{r\theta}^{(2)}\Big|_{r=R_1+h},\ t_{r\phi}^{(1)}\Big|_{r=R_1+h} = t_{r\phi}^{(2)}\Big|_{r=R_1+h},$$

$$u_{r}^{(1)}\Big|_{r=R_1+h} = u_{r}^{(2)}\Big|_{r=R_1+h},\ u_{\theta}^{(1)}\Big|_{r=R_1+h} = u_{\theta}^{(2)}\Big|_{r=R_1+h},\ u_{\phi}^{(1)}\Big|_{r=R_1+h} = u_{\phi}^{(2)}\Big|_{r=R_1+h},$$

....…,

$$t_{rr}^{(N-1)}\Big|_{r=R_1+(N-1)h}=t_{rr}^{(N)}\Big|_{r=R_1+(N-1)h},\ t_{r\theta}^{(N-1)}\Big|_{r=R_1+(N-1)h}=t_{r\theta}^{(N)}\Big|_{r=R_1+(N-1)h},$$

$$t_{r\phi}^{(N-1)}\Big|_{r=R_1+(N-1)h}=t_{r\phi}^{(N)}\Big|_{r=R_1+(N-1)h},\ u_{r}^{(N-1)}\Big|_{r=R_1+(N-1)h}=u_{r}^{(N)}\Big|_{r=R_1+(N-1)h},$$

$$u_{\theta}^{(N-1)}\Big|_{r=R_1+(N-1)h}=u_{\theta}^{(N)}\Big|_{r=R_1+(N-1)h},\ u_{\phi}^{(N-1)}\Big|_{r=R_1+(N-1)h}=u_{\phi}^{(N)}\Big|_{r=R_1+(N-1)h},$$

$$t_{rr}^{(N)}\Big|_{r=R_2}=0,\ t_{r\theta}^{(1)}\Big|_{r=R_2}=0,\ t_{r\phi}^{(N)}\Big|_{r=R_2}=0. \tag{7}$$

At the same time, within each subdomain $[R_{1k},R_{2k}]$ the system of Eqs. (2)-(5) are satisfied separately, i.e. within each subdomain $[R_{1k},R_{2k}]$ we have the following system of equations.

Equations of motion.

$$\frac{\partial t_{rr}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial t_{\phi r}^{(k)}}{\partial \phi}+\frac{1}{r\sin\phi}\frac{\partial t_{\theta r}^{(k)}}{\partial \theta}+\frac{1}{r}\left(2t_{rr}^{(k)}-t_{\phi\phi}^{(k)}-t_{\theta\theta}^{(k)}+t_{\phi r}^{(k)}\cot\phi\right)=\rho\frac{\partial^2 u_r^{(k)}}{\partial t^2},$$

$$\frac{\partial t_{r\phi}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial t_{\phi\phi}^{(k)}}{\partial \phi}+\frac{1}{r\sin\phi}\frac{\partial t_{\theta\phi}^{(k)}}{\partial \theta}+\frac{1}{r}\left(2t_{r\phi}^{(k)}+t_{\phi r}^{(k)}+\left(t_{\phi\phi}^{(k)}-t_{\theta\theta}^{(k)}\right)\cot\phi\right)=\rho\frac{\partial^2 u_\phi^{(k)}}{\partial t^2},$$

$$\frac{\partial t_{r\theta}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial t_{\phi\theta}^{(k)}}{\partial \phi}+\frac{1}{r\sin\phi}\frac{\partial t_{\theta\theta}^{(k)}}{\partial \theta}+\frac{1}{r}\left(2t_{r\theta}^{(k)}+t_{\theta r}^{(k)}+\left(t_{\phi\theta}^{(k)}+t_{\theta\phi}^{(k)}\right)\cot\phi\right)=\rho\frac{\partial^2 u_\theta^{(k)}}{\partial t^2}, \tag{8}$$

where

$$t_{rr}^{(k)}=\sigma_{rr}^{(k)}+\sigma_{rr}^{(0)}(r_k)\frac{\partial u_{rr}^{(k)}}{\partial r},\ t_{\phi\phi}^{(k)}=\sigma_{\phi\phi}^{(k)}+\sigma_{\phi\phi}^{(0)}(r_k)\left(\frac{u_r^{(k)}}{r}+\frac{\partial u_\phi^{(k)}}{r\partial\phi}\right),$$

$$t_{r\phi}^{(k)}=\sigma_{r\phi}^{(k)}+\sigma_{rr}^{(0)}(r_k)\frac{\partial u_\phi^{(k)}}{\partial r},\ t_{\theta\theta}^{(k)}=\sigma_{\theta\theta}^{(k)}+\sigma_{\theta\theta}^{(0)}(r_k)\left(\frac{u_r^{(k)}}{r}+\cot\phi\frac{u_\phi^{(k)}}{r}+\frac{1}{r\sin\phi}\frac{\partial u_\theta^{(k)}}{\partial\theta}\right),$$

$$t_{\phi r}^{(k)}=\sigma_{\phi r}^{(k)}+\sigma_{\phi\phi}^{(0)}(r_k)\left(\frac{\partial u_r^{(k)}}{r\partial\phi}-\frac{u_\phi^{(k)}}{r}\right),\ t_{r\theta}^{(k)}=\sigma_{r\theta}^{(k)}+\sigma_{rr}^{(0)}(r_k)\frac{\partial u_\theta^{(k)}}{\partial r},$$

$$t_{\theta r}^{(k)}=\sigma_{\theta r}^{(k)}+\sigma_{\theta\theta}^{(0)}(r_k)\left(\frac{1}{r\sin\phi}\frac{\partial u_r^{(k)}}{\partial\theta}-\frac{u_\theta^{(k)}}{r}\right),\ t_{\phi\theta}^{(k)}=\sigma_{\phi\theta}^{(k)}+\sigma_{\phi\phi}^{(0)}(r_k)\frac{\partial u_\theta^{(k)}}{r\partial\phi},$$

$$t_{\theta\phi}^{(k)} = \sigma_{\theta\phi}^{(k)} + \sigma_{\theta\theta}^{(0)}(r_k)\left(\frac{1}{r\sin\phi}\frac{\partial u_{\phi}^{(k)}}{\partial\theta} - \cot\phi\frac{u_{\theta}^{(k)}}{r}\right). \tag{9}$$

Elasticity relations:

$$\sigma_{rr}^{(k)} = \lambda(\varepsilon_{rr}^{(k)} + \varepsilon_{\theta\theta}^{(k)} + \varepsilon_{\phi\phi}^{(k)}) + 2\mu\varepsilon_{rr}^{(k)} \ , \ \sigma_{\theta\theta}^{(k)} = \lambda(\varepsilon_{rr}^{(k)} + \varepsilon_{\theta\theta}^{(k)} + \varepsilon_{\phi\phi}^{(k)}) + 2\mu\varepsilon_{\theta\theta}^{(k)} ,$$

$$\sigma_{\phi\phi}^{(k)} = \lambda(\varepsilon_{rr}^{(k)} + \varepsilon_{\theta\theta}^{(k)} + \varepsilon_{\phi\phi}^{(k)}) + 2\mu\varepsilon_{\phi\phi}^{(k)} , \ \sigma_{r\theta}^{(k)} = 2\mu\varepsilon_{r\theta}^{(k)} ,$$

$$\sigma_{\theta\phi}^{(k)} = 2\mu\varepsilon_{\theta\phi}^{(k)}, \quad \sigma_{r\phi}^{(k)} = 2\mu\varepsilon_{r\phi}^{(k)} . \tag{10}$$

Strain-displacement relations:

$$\varepsilon_{rr}^{(k)} = \frac{\partial u_r^{(k)}}{\partial r} \ , \ \varepsilon_{\theta\theta}^{(k)} = \frac{1}{r}\frac{\partial u_{\theta}^{(k)}}{\partial\theta} + \frac{1}{r}u_r^{(k)} ;$$

$$\varepsilon_{\phi\phi}^{(k)} = \frac{1}{r\sin\theta}\frac{\partial u_{\phi}^{(k)}}{\partial\phi} + \frac{1}{r}u_r^{(k)} + \frac{1}{r}u_{\theta}^{(k)}\cot\theta ;$$

$$\varepsilon_{r\theta}^{(k)} = \frac{1}{2}\left(\frac{\partial u_{\theta}^{(k)}}{\partial r} + \frac{1}{r}\frac{\partial u_r^{(k)}}{\partial\theta} - \frac{u_{\theta}^{(k)}}{r}\right);$$

$$\varepsilon_{\theta\phi}^{(k)} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_{\phi}^{(k)}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_{\theta}^{(k)}}{\partial\phi} - \frac{u_{\phi}^{(k)}}{r}\cot\theta\right);$$

$$\varepsilon_{r\phi}^{(k)} = \frac{1}{2}\left(\frac{\partial u_{\phi}^{(k)}}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_r^{(k)}}{\partial\theta} - \frac{u_{\phi}^{(k)}}{r}\right). \tag{11}$$

Thus, in this way the solution to the system of Eqs. (1)-(5) with the boundary conditions in (6) is reduced the solution to the system of Eqs. (8)-(11) with the boundary and contact conditions (7).

3.2 Solution to the system of equations (8-11)

For the solution to the system of Eqs. (8)-(11) for the $k-th$ sublayer, we use the following classical Lame decomposition [Eringen and Suhubi (1975)].

$$u_r^{(k)} = \frac{\partial \Phi^{(k)}}{\partial r} + \frac{\partial^2 (r\chi^{(k)})}{\partial r^2} - r\nabla^2 \chi^{(k)} \ ,$$

$$u_\theta^{(k)} = \frac{1}{r}\frac{\partial \Phi^{(k)}}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial \psi^{(k)}}{\partial \varphi} + \frac{1}{r}\frac{\partial^2 (r\chi^{(k)})}{\partial \theta \partial r}$$

$$u_\varphi^{(k)} = \frac{1}{r\sin\theta}\frac{\partial \Phi^{(k)}}{\partial \varphi} - \frac{\partial \psi^{(k)}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial^2 (r\chi^{(k)})}{\partial \varphi \partial r} \tag{12}$$

Substituting the expressions in (12) into the Eqs. (8)-(11) and doing cumbersome mathematical manipulations we obtain the following equations for the functions $\Phi^{(k)}$, $\psi^{(k)}$ and $\chi^{(k)}$.

$$\mu\nabla^2 \begin{Bmatrix} \psi^{(k)} \\ \chi^{(k)} \end{Bmatrix} + \sigma_{rr}^{(0)}(r_k)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\begin{Bmatrix} \psi^{(k)} \\ \chi^{(k)} \end{Bmatrix}\right)\right] +$$

$$\sigma_{\theta\theta}^{(0)}(r_k)\left[\frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta}\begin{Bmatrix} \psi^{(k)} \\ \chi^{(k)} \end{Bmatrix} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\begin{Bmatrix} \psi^{(k)} \\ \chi^{(k)} \end{Bmatrix} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\begin{Bmatrix} \psi^{(k)} \\ \chi^{(k)} \end{Bmatrix}\right] = \rho\frac{\partial^2}{\partial t^2}\begin{Bmatrix} \psi^{(k)} \\ \chi^{(k)} \end{Bmatrix}$$

$$(\lambda + 2\mu)\nabla^2 \Phi^{(\mathrm{k})} + \sigma_{rr}^{(0)}(r_k)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\Phi^{(\mathrm{k})}\right)\right] +$$

$$\sigma_{\theta\theta}^{(0)}(r_k)\left[\frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta}\Phi^{(\mathrm{k})} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\Phi^{(\mathrm{k})} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\Phi^{(\mathrm{k})}\right] = \rho\frac{\partial^2}{\partial t^2}\Phi^{(\mathrm{k})}, \tag{13}$$

where

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \Delta_{\theta,\phi} \ , \ \Delta_{\theta,\phi} = \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}. \tag{14}$$

Note that under obtaining the equations in (13) the equality $\sigma_{\phi\phi}^{(0)}(r_k) = \sigma_{\theta\theta}^{(0)}(r_k)$ is taken into consideration. Moreover, note that in the case where $\sigma_{rr}^{(0)}(r_k) = \sigma_{\theta\theta}^{(0)}(r_k) = \sigma_{\phi\phi}^{(0)}(r_k) = 0$ the equations in (13) coincide with the corresponding classical ones given, for instance, in the monograph [Eringen and Suhubi (1975)]. Consequently, the equations in (13) are new ones and as will be shown below, allow to obtain the analytical solutions and to investigate a certain class dynamical problems for the sphere with initial stresses determined through the expressions in (1).

3.3 Obtaining the analytical solutions for the equations for the functions which enter into the Lame decomposition for the natural vibration problem

According to the well-known physico-mechanical considerations, first, the functions $\psi^{(k)}$, $\chi^{(k)}$ and $\Phi^{(k)}$ are represented as follows

$$\phi^{(k)}(r,\theta,\varphi,t) = F_{\phi}^{(k)}(n,r)P_n^m(\cos\theta)\cos m\varphi e^{i\omega t},$$

$$\psi^{(k)}(r,\theta,\varphi,t) = F_{\psi}^{(k)}(n,r)P_n^m(\cos\theta)\sin m\varphi e^{i\omega t},$$

$$\chi^{(k)}(r,\theta,\varphi,t) = F_{\chi}^{(k)}(n,r)P_n^m(\cos\theta)\cos m\varphi e^{i\omega t}, \tag{15}$$

where $P_n^m(\cos\theta)$ in the expression (15) denotes the associated Legendre functions with $m-$th order and with $n-$th harmonic.

Thus, substituting the expressions in (15) into the Eqs. (13) and (14), we obtain the following equations for the functions $F_{\phi}^{(k)}(n,r)$, $F_{\psi}^{(k)}(n,r)$ and $F_{\chi}^{(k)}(n,r)$

$$\frac{d^2F_{\Psi;\chi}^{(k)}(n,r)}{dr^2} + \frac{2}{r}\frac{dF_{\Psi;\chi}^{(k)}(n,r)}{dr} + \left((\lambda^{(k)})^2 - \frac{\nu_n^{(k)}(\nu_n^{(k)}+1)}{r^2}\right)F_{\Psi;\chi}^{(k)}(n,r) = 0,$$

$$\frac{d^2F_{\Phi}^{(k)}(n,r)}{dr^2} + \frac{2}{r}\frac{dF_{\Phi}^{(k)}(n,r)}{dr} + \left((\gamma^{(k)})^2 - \frac{\eta_n^{(k)}(\eta_n^{(k)}+1)}{r^2}\right)F_{\Phi}^{(k)}(n,r) = 0, \tag{16}$$

where

$$(\lambda^{(k)})^2 = \rho\omega^2 / (\mu + \sigma_{rr}^{(0)}(r_k)), \quad \nu_n^{(k)} = -\frac{1}{2} + \sqrt{\frac{1}{4} + \alpha^{(k)}n(n+1)},$$

$$(\gamma^{(k)})^2 = \rho\omega^2 / (\lambda + 2\mu + \sigma_{rr}^{(0)}(r_k)), \eta_n^{(k)} = -\frac{1}{2} + \sqrt{\frac{1}{4} + \beta^{(k)}n(n+1)},$$

$$\alpha^{(k)} = \frac{(\mu + \sigma_{\theta\theta}^{(0)}(r_k))}{(\mu + \sigma_{rr}^{(0)}(r_k))}, \ \beta^{(k)} = \frac{(\lambda + 2\mu + \sigma_{\theta\theta}^{(0)}(r_k))}{(\lambda + 2\mu + \sigma_{rr}^{(0)}(r_k))}. \tag{17}$$

Thus, the solutions to the Eq. (16) are presented through the spherical Bessel functions as follows.

$$F_{\Psi}^{(k)}(n,r) = C^{(k)} j_{\nu_n^{(k)}}(\lambda^{(k)}r) + D^{(k)} y_{\nu_n^{(k)}}(\lambda^{(k)}r),$$

$$F_{\chi}^{(k)}(n,r)=E^{(k)}j_{\nu_n^{(k)}}(\lambda^{(k)}r)+G^{(k)}y_{\nu_n^{(k)}}(\lambda^{(k)}r),$$

$$F_{\Phi}^{(k)}(n,r)=A^{(k)}j_{\eta_n^{(k)}}(\gamma^{(k)}r)+B^{(k)}y_{\eta_n^{(k)}}(\gamma^{(k)}r), \tag{18}$$

where

$$j_{\alpha}(cr)=\left(\frac{\pi}{2cr}\right)^{\frac{1}{2}}J_{\alpha+1/2}(cr),\ y_{\alpha}(cr)=\left(\frac{\pi}{2cr}\right)^{\frac{1}{2}}Y_{\alpha+1/2}(cr). \tag{19}$$

In (19) the functions $J_{\alpha+1/2}(cr)$ and $Y_{\alpha+1/2}(cr)$ are the Bessel functions of the first and the second kind with non-integer order, respectively.

Note that in the case where $\sigma_{rr}^{(0)}(r_k)=\sigma_{\theta\theta}^{(0)}(r_k)=\sigma_{\phi\phi}^{(0)}(r_k)=0$, according to which, $\alpha^{(k)}=\beta^{(k)}=1$ the expressions in (18) coincide with the corresponding ones obtained in the classical case given for instance in Akbarov et al. [Akbarov, Guliyev and Yahnioglu (2016, 2017); Eringen and Suhubi (1975)] and other related references.

Thus, using the relations (18), (15). (12), (11) and (10) we obtain expressions for the displacements and for the components of the stress tensor. For simplification of writing the obtained expressions, we introduce two sets of complete orthogonal functions in $[0,\pi]$ determined as follows:

$$X_{nm}(\theta)=P_n^m(\cos\theta)\ ,\ Y_{nm}(\theta)=n\cot\theta P_n^m(\cos\theta)-\frac{n+m}{\sin\theta}P_{n-1}^m(\cos\theta). \tag{20}$$

Using the notation in (20) we can write the following expressions for the sought values:

$$u_r^{(k)}=\frac{1}{r}\left\{A^{(k)}u_{11}^{(k)}+B^{(k)}u_{12}^{(k)}+E^{(k)}u_{31}^{(k)}+G^{(k)}u_{32}^{(k)}\right\}X_{nm}(\theta)\cos m\varphi e^{i\omega t},$$

$$u_{\theta}^{(k)}=\frac{1}{r}\left\{\left[A^{(k)}v_{11}^{(k)}+B^{(k)}v_{12}^{(k)}+E^{(k)}v_{31}^{(k)}+G^{(k)}v_{32}^{(k)}\right]Y_{nm}(\theta)+\right.$$

$$\left.(C^{(k)}v_{21}^{(k)}+D^{(k)}v_{22}^{(k)})\frac{m}{\sin\theta}X_{nm}(\theta)\right\}\cos m\varphi e^{i\omega t},$$

$$u_{\varphi}^{(k)}=\frac{1}{r}\left\{\left[A^{(k)}v_{11}^{(k)}+B^{(k)}v_{12}^{(k)}+E^{(k)}v_{31}^{(k)}+G^{(k)}v_{32}^{(k)}\right]\frac{-m}{\sin\theta}X_{nm}(\theta)+\right.$$

$$\left.(-C^{(k)}v_{21}^{(k)}-D^{(k)}v_{22}^{(k)})Y_{nm}(\theta)\right\}\sin m\varphi e^{i\omega t},$$

$$\sigma_{rr}^{(k)}=\frac{2\mu^{(k)}}{r^2}\left[A^{(k)}T_{111}^{(k)}+B^{(k)}T_{112}^{(k)}+E^{(k)}T_{131}^{(k)}+G^{(k)}T_{132}^{(k)}\right]X_{nm}(\theta)\cos m\varphi e^{i\omega t},$$

$$\sigma_{r\theta}^{(k)} = \frac{2\mu^{(k)}}{r^2}\left\{\left[A^{(k)}T_{411}^{(k)} + B^{(k)}T_{412}^{(k)} + E^{(k)}T_{431}^{(k)} + G^{(k)}T_{432}^{(k)}\right]Y_{nm}(\theta) +\right.$$

$$\left.\left[-C^{(k)}T_{421}^{(k)} - D^{(k)}T_{422}^{(k)}\right]\frac{m}{\sin\theta}X_{nm}(\theta)\right\}\cos m\varphi e^{i\omega t},$$

$$\sigma_{r\varphi}^{(k)} = \frac{2\mu^{(k)}}{r^2}\left\{\left[A^{(k)}T_{411}^{(k)} + B^{(k)}T_{412}^{(k)} + E^{(k)}T_{431}^{(k)} + G^{(k)}T_{432}^{(k)}\right]\frac{m}{\sin\theta}X_{nm}(\theta) +\right.$$

$$\left.\left[-C^{(k)}T_{421}^{(k)} - D^{(k)}T_{422}^{(k)}\right]Y_{nm}(\theta)\right\}\cos m\varphi e^{i\omega t}, \tag{21}$$

where

$$u_{11}^{(k)} = \eta_n^{(k)} j_{\eta_n^{(k)}}(\gamma^{(k)}r) - \gamma^{(k)} r j_{\eta_n^{(k)}+1}(\gamma^{(k)}r),$$

$$u_{12}^{(k)} = \eta_n^{(k)} y_{\eta_n^{(k)}}(\gamma^{(k)}r) - \gamma^{(k)} r y_{\eta_n^{(k)}+1}(\gamma^{(k)}r),$$

$$u_{31}^{(k)} = \nu_n^{(k)}(\nu_n^{(k)}+1) j_{\nu_n^{(k)}}(\lambda^{(k)}r),\ u_{32}^{(k)} = \nu_n^{(k)}(\nu_n^{(k)}+1)\,\mathrm{y}_{\nu_n^{(k)}}(\lambda^{(k)}r),$$

$$v_{11}^{(k)} = j_{\eta_n^{(k)}}(\gamma^{(k)}r),\ v_{12}^{(k)} = y_n(\gamma^{(k)}r),\ v_{21}^{(k)} = j_{\nu_n^{(k)}}(\lambda^{(k)}r),\ v_{22}^{(k)} = y_{\nu_n^{(k)}}(\lambda^{(k)}r),$$

$$v_{31}^{(k)} = (\nu_n^{(k)}+1) j_{\nu_n^{(k)}}(\lambda^{(k)}r) - \lambda^{(k)} r j_{\nu_n^{(k)}+1}(\lambda^{(k)}r),$$

$$v_{32}^{(k)} = (\nu_n^{(k)}+1) y_{\nu_n^{(k)}}(\lambda^{(k)}r) - \lambda^{(k)} r y_{\nu_n^{(k)}+1}(\lambda^{(k)}r),$$

$$T_{111}^{(k)} = ((\eta_n^{(k)})^2 - \eta_n^{(k)} - \frac{1}{2}(\gamma^{(k)})^2 r^2) j_{\eta_n^{(k)}}(\gamma^{(k)}r) + 2\gamma^{(k)} r j_{\eta_n^{(k)}+1}(\gamma^{(k)}r),$$

$$T_{112}^{(k)} = ((\eta_n^{(k)})^2 - \eta_n^{(k)} - \frac{1}{2}(\gamma^{(k)})^2 r^2) y_{\eta_n^{(k)}}(\gamma^{(k)}r) + 2\gamma^{(k)} r y_{\eta_n^{(k)}+1}(\gamma^{(k)}r)$$

$$T_{131}^{(k)} = \nu_n^{(k)}(\nu_n^{(k)}+1)\left[(\nu_n^{(k)}-1) j_n(\lambda^{(k)}r) - \lambda^{(k)} r j_{\nu_n^{(k)}+1}(\lambda^{(k)}r)\right],$$

$$T_{132}^{(k)} = \nu_n^{(k)}(\nu_n^{(k)}+1)\left[(\nu_n^{(k)}-1) y_{\nu_n^{(k)}}(\lambda^{(k)}r) - \lambda^{(k)} r y_{\nu_n^{(k)}+1}(\lambda^{(k)}r)\right]$$

$$T_{411}^{(k)} = (\eta_n^{(k)}-1) j_{\eta_n^{(k)}}(\gamma^{(k)}r) - \gamma^{(k)} r j_{\eta_n^{(k)}+1}(\gamma^{(k)}r),$$

$$T_{412}^{(k)} = (\eta_n^{(k)}-1) y_{\eta_n^{(k)}}(\gamma^{(k)}r) - \gamma^{(k)} r y_{\eta_n^{(k)}+1}(\gamma^{(k)}r),$$

$$T_{421}^{(k)} = \frac{1}{2} r \left[(\nu_n^{(k)} - 1) j_{\nu_n^{(k)}}(\lambda^{(k)} r) - \lambda^{(k)} r j_{\nu_n^{(k)}+1}(\lambda^{(k)} r) \right],$$

$$T_{422}^{(k)} = \frac{1}{2} r \left[(\nu_n^{(k)} - 1) y_{\nu_n^{(k)}}(\lambda^{(k)} r) - \lambda^{(k)} r y_{\nu_n^{(k)}+1}(\lambda^{(k)} r) \right],$$

$$T_{431}^{(k)} = ((\nu_n^{(k)})^2 - 1 - \frac{1}{2}(\lambda^{(k)})^2 r^2) j_{\nu_n^{(k)}}(\lambda^{(k)} r) + \lambda^{(k)} r j_{\nu_n^{(k)}+1}(\lambda^{(k)} r),$$

$$T_{432}^{(k)} = ((\nu_n^{(k)})^2 - 1 - \frac{1}{2}(\lambda^{(k)})^2 r^2) y_{\nu_n^{(k)}}(\lambda^{(k)} r) + \lambda^{(k)} r y_{\nu_n^{(k)}+1}(\lambda^{(k)} r), \tag{22}$$

where $k = 1, 2, ..., N$.

As noted above, here N is the number of subintervals into which the solution region is divided with respect to the radial coordinate r and this number is determined according to the convergence of the numerical results.

Thus, substituting the expressions (21) and (22) into the boundary and contact conditions in (7) we obtain two uncoupled system of the homogeneous algebraic equations. The first (second) system contains the unknowns $A^{(k)}$, $B^{(k)}$, $E^{(k)}$ and $G^{(k)}$ ($C^{(k)}$ and $D^{(k)}$). Equating to zero the determinant of the coefficient matrix of the first and second group of the equations separately, the following equations for determination of the frequency of the natural vibration are obtained.

$$\det\left(\alpha_{q_1 q_2}\right) = 0 \quad q_1; q_2 = 1, 2, ..., 4N \text{ (for the spheroidal vibration)} \tag{23}$$

and

$$\det\left(\delta_{p_1 p_2}\right) = 0, \quad p_1; p_2 = 1, 2, ..., 2N \text{ (for the torsional vibration).} \tag{24}$$

Note that the expressions of the components of the matrixes $(\alpha_{q_1 q_2})$ and $(\delta_{p_1 p_2})$ can be easily determined from the expressions given in (21) and (22) and therefore these expressions are not given here.

4 Numerical examples

Numerical results are obtained through the solution of the Eqs. (23) (for the spheroidal mode) and (24) (for the torsional mode) and this solution is made numerically with employing the well-known bi-section method. The authors in MATLAB compose the algorithm and corresponding PC programs. Akbarov et al. [Akbarov, Guliyev and Yahnioglu (2016, 2017)] already complete the testing of this algorithm and programs in the works and therefore we here do not consider again this question.

Note that the results which will be discussed below relate to the dimensionless natural frequencies denoted as $\Omega = \omega a / \sqrt{\mu / \rho}$ obtained for various values of the ratios b / a, p / μ and q / μ. Note that the latter two ratios characterize the initial stresses in the hollow sphere. Moreover, numerical results will be distinguished with respect to the vibration harmonics and with respect to the sequences of the roots in each harmonics.

First, we examine how the convergence of the numerical results with respect to the number N, i.e. with respect to the number of sublayers into which the hollow sphere is divided. The results illustrating this convergence are given in Tab. 1 and are obtained for the case where $b/a=0.2$, $p/\mu=0.1$ and $q/\mu=0$. It follows from the table that the mentioned convergence significantly depends on the vibration modes and harmonics, as well as on the sequences of the roots. For instance, in the spheroidal vibration mode for the 3-rd root in the $n=0$ harmonic, for the 2-nd and 3-rd roots in the $n=2$ and 3 harmonics it is necessary to take $N=21$ in order to obtain the converged numerical results with the high accuracy. Taking this statement into account, all the results which will be discussed below are obtained in the case where $N=21$.

Table 1: The influence of the number of the sublayers N on the values of the natural frequencies $\Omega=\omega a\sqrt{\rho^{(1)}/\mu^{(1)}}$ obtained for homogeneous hollow sphere in the case where $b/a=0.2$, $q/\mu=0$ and $p/\mu=0.1$

n	N	Torsional vibration			Spheroidal vibration		
		Sequences of the roots					
		1	2	3	1	2	3
0	3	3.5965	6.4255	9.2395	3.5825	3.5965	6.4255
	5	3.5925	6.4175	9.2265	3.5825	3.5925	6.4175
	7	4.3015	6.9105	9.5785	3.5835	3.5905	6.4125
	11	3.5875	6.4075	9.2115	3.5845	3.5875	6.4075
	15	3.5865	6.4045	9.2065	3.5845	3.5865	6.4045
	21	3.5845	6.4015	9.2025	3.5845	3.5855	6.4015
1	3	4.3035	6.9155	9.5875	2.6495	5.6475	5.9935
	5	4.3015	6.9125	9.5815	2.6485	5.6455	5.9915
	7	3.5905	6.4125	9.2195	2.6485	5.6445	5.9905
	11	4.3005	6.9095	9.5755	2.6485	5.6435	5.9895
	15	4.3005	6.9085	9.5735	2.6485	5.6435	5.9895
	21	4.3005	6.9085	9.5715	2.6485	5.6435	5.9895
2	3	1.8505	5.2735	7.7725	2.0865	2.4305	3.5285
	5	1.8505	5.2725	7.7705	2.0175	3.0265	3.3995
	7	1.8505	5.2715	7.7685	2.0025	5.8135	7.6755
	11	1.8505	5.2705	7.7675	1.9925	5.7715	7.6755
	15	1.8505	5.2705	7.7675	1.9885	5.7415	7.6745
	21	1.8505	5.2705	7.7665	1.9855	5.7085	7.6745

3	3	2.8605	6.2485	8.7785	2.9165	6.6545	9.0025
	5	2.8595	6.2475	8.7755	2.9135	4.9765	8.9635
	7	2.8595	6.2465	8.7735	2.9125	4.9415	8.9295
	11	2.8595	6.2465	8.7725	2.9115	4.9245	8.8735
	15	2.8595	6.2455	8.7715	2.9115	4.9185	8.8295
	21	2.8595	6.2455	8.7715	2.9115	4.9145	8.7785
4	3	3.7705	7.1895	9.7785	3.7205	6.1015	9.9565
	5	3.7695	7.1875	9.7765	3.7205	6.0915	9.4585
	7	3.7695	7.1875	9.7755	3.7205	6.0885	8.7385
	11	3.7695	7.1865	9.7745	3.7205	6.0865	8.5255
	15	3.7695	7.1865	9.7735	3.7205	6.0865	8.4775
	21	3.7685	7.1865	9.7735	3.7205	6.0855	8.4485

Consider some fragments from these results obtained in the case where $b/a=0.2$ some of which are given in Tab. 2. Note that these results are obtained for various p/μ in the case where $q/\mu=0$ under which, according to the expressions in (1), the initial stresses $\sigma_{\theta\theta}^{(0)}$ and $\sigma_{\phi\phi}^{(0)}$ become compressional ones. According to the well-known mechanical considerations, as usual, the initial compression of the elements of constructions causes a decrease in the values of their natural frequencies. This trend is also observed in the considered case under comparison of the numerical result given in Tab. 2 and obtained for various p/μ. This statement proves again the validity and trustiness of the developed method and algorithm used for obtaining numerical results.

We also consider numerical results given in Tab. 3 which are obtained for various values of q/μ in the case where $p/\mu=0$. According to the expressions given in (1) for the initial stresses, in the case where $p/\mu=0$ the stresses $\sigma_{\theta\theta}^{(0)}$ and $\sigma_{\phi\phi}^{(0)}$ become tensional ones. It is known that, as usual, the initial stretching of the element of constructions causes an increase in the values of their natural frequencies. Note that this trend is also observed in the numerical results given in Tab. 3.

It follows from the analyses of the results given in Tab. 2 and Tab. 3 that the influence of the initial stresses caused by the external force p/μ is more significant than that caused by the external force q/μ.

Table 2: The influence of p/μ on the values of the natural frequencies $\Omega = \omega a\sqrt{\rho^{(1)} / \mu^{(1)}}$ obtained in the case where $b/a = 0.2$, $q/\mu = 0$

p/μ	n	Torsional vibration			Spheroidal vibration		
		Sequences of the roots					
		1	2	3	1	2	3
0	0	4.8375	8.6425	12.4285	4.8145	4.8375	8.6425
	1	5.7995	9.3105	12.9015	3.5825	7.6065	8.0775
	2	2.5005	7.1175	10.4795	2.5575	4.8895	8.0385
	3	3.8645	8.4385	11.8435	3.9175	6.5655	9.4575
	4	5.0945	9.7115	13.1975	5.0435	8.2045	11.0795
	5	6.2655	10.9505	14.5075	6.0835	9.7745	12.5425
0.001	0	4.8245	8.6195	12.3965	4.8025	4.8245	8.6195
	1	5.7845	9.2865	12.8685	3.5735	7.5865	8.0565
	2	2.4935	7.0995	10.4525	0.5405	2.5475	4.8755
	3	3.8545	8.4165	11.8125	1.0195	3.9065	6.5485
	4	5.0815	9.6855	13.1625	1.5085	5.0295	8.1825
	5	6.2495	10.9215	14.4695	2.0015	6.0675	9.7495
0.005	0	4.7745	8.5305	12.2675	4.7525	4.7745	8.5305
	1	5.7245	9.1905	12.7355	3.5365	7.5085	7.9735
	2	2.4675	7.0255	10.3435	1.1395	2.5055	4.8205
	3	3.8145	8.3285	11.6895	2.1275	3.8585	6.4755
	4	5.0285	9.5845	13.0265	3.1315	4.9755	8.0955
	5	6.1845	10.8075	14.3195	4.1415	6.0025	9.6475
0.01	0	4.7115	8.4185	12.1055	4.6915	4.7115	8.4185
	1	5.6495	9.0705	12.5685	3.4895	7.4105	7.8685
	2	2.4355	6.9325	10.2085	1.5785	2.4405	4.7505
	3	3.7645	8.2185	11.5355	2.9205	3.7895	6.3845
	4	4.9625	9.4585	12.8545	4.2775	4.9025	7.9875
	5	6.1025	10.6655	14.1305	5.6435	5.9205	9.5205
0.03	0	4.4615	7.9705	11.4605	4.4395	4.4585	7.9655
	1	5.3495	8.5895	11.9025	3.3025	7.0115	7.4465
	2	2.3055	6.5635	9.6655	1.4575	2.3135	4.4965
	3	3.5625	7.7805	10.9215	2.6985	3.5885	6.0425
	4	4.6965	8.9535	12.1705	3.9555	4.6415	7.5585

	5	5.7765	10.0955	13.3775	5.2195	5.6035	9.0095
0.05	0	4.2105	7.5215	10.8145	4.1905	4.2065	7.5145
	1	5.0505	8.1095	11.2365	3.1145	6.6155	7.0255
	2	2.1755	6.1935	9.1235	1.9065	2.0505	4.2335
	3	3.3615	7.3415	10.3075	5.6905	8.1585	10.6265
	4	4.4315	8.4485	11.4855	4.3955	4.8405	7.1265
	5	5.4505	9.5265	12.6245	5.2875	6.4045	8.4975

Table 3: The influence of q/μ on the values of the natural frequencies $\Omega = \omega a \sqrt{\rho^{(1)} / \mu^{(1)}}$ obtained in the case where $b/a = 0.2$, $p/\mu = 0$

q/μ	n	Torsional vibration			Spheroidal vibration		
		Sequences of the roots					
		1	2	3	1	2	3
0	0	4.8375	8.6425	12.4285	4.8145	4.8375	8.6425
	1	5.7995	9.3105	12.9015	3.5825	7.6065	8.0775
	2	2.5005	7.1175	10.4795	2.5575	4.8895	8.0385
	3	3.8645	8.4385	11.8435	3.9175	6.5655	9.4575
	4	5.0945	9.7115	13.1975	5.0435	8.2045	11.0795
	5	6.2655	10.9505	14.5075	6.0835	9.7745	12.5425
0.01	0	4.8375	8.6435	12.4305	4.8145	4.8375	8.6435
	1	5.8005	9.3115	12.9035	3.5835	7.6075	8.0785
	2	2.5005	7.1185	10.4815	2.5635	4.8925	8.0435
	3	3.8655	8.4395	11.8445	3.9205	6.5685	9.4665
	4	5.0955	9.7125	13.1995	5.0445	8.2065	11.0855
	5	6.2665	10.9525	14.5095	6.0845	9.7765	12.5455
0.02	0	4.8385	8.6455	12.4325	4.8135	4.8385	8.6445
	1	5.8015	9.3125	12.9055	3.5835	7.6075	8.0785
	2	2.5005	7.1195	10.4825	2.5725	4.8965	8.0485
	3	3.8655	8.4405	11.8465	3.9235	6.5715	9.4805
	4	5.0965	9.7145	13.2015	5.0455	8.2075	11.0945
	5	6.2675	10.9545	14.5125	6.0855	9.7765	12.5485
0.03	0	4.8395	8.6465	12.4345	4.8135	4.8385	8.6445
	1	5.8025	9.3145	12.9075	3.5845	7.6075	8.0785
	2	2.5015	7.1205	10.4845	2.5785	4.8995	8.0535
	3	3.8665	8.4425	11.8485	3.9255	6.5745	9.4915

	4	5.0975	9.7155	13.2035	5.0465	8.2085	11.1005
	5	6.2695	10.9555	14.5145	6.0865	9.7775	12.5515
0.05	0	4.8375	8.6435	12.4305	4.8135	4.8375	8.6435
	1	5.8015	9.3115	12.9035	3.5845	7.6075	8.0795
	2	2.5015	7.1205	10.4825	2.5815	4.9015	8.0565
	3	3.8665	8.4415	11.8475	3.9275	6.5765	9.4975
	4	5.0975	9.7155	13.2025	5.0485	8.2095	11.1055
	5	6.2685	10.9555	14.5135	6.0895	9.7795	12.5545

With this, we restrict ourselves the consideration the numerical results and their analysis. Note that more detail analysis of these and other related numerical results would be considered in the other papers by the authors.

5 Conclusions

Thus, in the present paper the discrete-analytical method is developed and employed for solution of the dynamical problems related to the hollow sphere with non-homogeneous initial stresses. The case where the initial stresses are symmetric with respect to the sphere's center and depend only on the spherical radial coordinate is considered. The essence of the developed method is to divide the spherical layer into a certain number of corresponding spherical sublayers each of which the initial stresses are homogeneous and try to find an analytical solution for the field equations within each sublayers separately. It is assumed that on the interface surfaces between the sublayers the conditions on the continuity of the force and displacement vectors are satisfied and from which the unknown constants contained in the mentioned analytical solutions are determined.

The proposed method is examined for the solution of the natural frequencies of the hollow sphere with the aforementioned type initial stresses. Numerical results illustrating the convergence of these results with respect to the sublayers' number, as well as illustrating the influence of the initial stresses on the values of the natural frequencies are presented and discussed briefly. According to these results, it is established that, in general, the initial compression of the hollow sphere with the uniformly distributed normal forces acting on its external (inner) surface causes a decrease (an increase) in the values of the natural frequencies of the sphere.

Moreover, it is established that the character of the influence of the initial stresses on the values of the natural frequencies depends on the vibration harmonic and sequences of the roots determined from the frequency equations.

More detailed analysis of these and other related numerical results will be considered in the future works by the authors.

The method developed in the present paper can be employed also to solution of the corresponding dynamical problems for layered solid and hollow spheres with inhomogeneous initial stresses similar with that considered in the present paper.

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