 CMC, vol.53, no.4, pp.307-341, 2017

Generalized Rayleigh Wave Dispersion in a Covered Half-space Made of Viscoelastic Materials

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Abstract: Dispersion of the generalized Rayleigh waves propagating in a covered half-space made of viscoelastic materials is investigated by utilizing the exact equations of the theory of linear viscoelasticity. The dispersion equation is obtained for an arbitrary type of hereditary operator of the materials of the constituents and a solution algorithm is developed for obtaining numerical results on the dispersion of the waves under consideration. Dispersion curves are presented for certain attenuation cases and the influence of the viscosity of the materials is studied through three rheological parameters of the viscoelastic materials which characterize the characteristic creep time, long-term values and the mechanical behaviour of the viscoelastic material around the initial state of the deformation. Numerical results are presented and discussed for the case where the viscoelasticity of the materials is described through fractional-exponential operators by Rabotnov. As the result of this discussion, in particular, how the rheological parameters influence the dispersion of the generalized Rayleigh waves propagating in the covered half-space under consideration is established.

Keywords: Generalized Rayleigh wave, viscoelastic material, rheological parameters, dispersion, fractional-exponential operator.

1 Introduction

Surface waves propagating in viscoelastic layered media are of a particular importance for numerous scientific and engineering applications, from material science to biological science and from vibration reduction of different structural or mechanical elements to earthquake engineering and geophysical explorations. Several mathematical models have been used by many authors to study the dispersion and the attenuation behaviour of guided waves in such viscoelastic media. However, in most cases either they have described the viscoelasticity of the materials through some simple models such as the classical Kelvin-Voigt spring-dashpot models [Chiriţă, Ciarletta and Tibullo (2014); Quintanilla, Fan, Lowe et al. (2015); Mazzotti, Marzani, Bartoli et al. (2012); Manconi

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and Sorokin (2013)] or they have used complex elasticity modulus instead of the real one in the stress-strain relations of the viscoelastic materials [Vishwakarma and Gupta (2012); Barshinger and Rose (2004); Addy and Chakraborty (2005); Garg (2007); Jiangong (2011)]. Consequently, in general, such a simple viscoelastic models and the numerical results obtained within these models cannot illustrate the real character of the influence of the rheological parameters of the viscoelastic materials on the corresponding wave dispersion and attenuation. Recent efforts to use more realistic models for the wave propagation and attenuation problems in viscoelastic media were made by Meral, Royston and Magin (2009, 2011) by utilizing fractional order Voigt model to investigate Lamb wave propagation. In this way, a new rheological parameter which is the order of the fractional derivatives is introduced into the model and through this parameter the results are agreed more accurately with experiments as compared with conventional models.

Here we review in detail investigations related to the propagation of the Rayleigh waves in viscoelastic media which is close to the topic of the current paper and begin this review with the paper by Carcione (1992) in which the inelastic characteristics of the Rayleigh waves from the standpoint of balance energy is investigated. He calculated the quality factors as a function of the frequency and depth and showed that the viscoelastic properties calculated from energy considerations are consistent with those obtained from the Rayleigh secular equation. Based on the Cauchy residue theorem of complex analysis Lai and Rix (2002) presented a technique which permits simultaneous determination of the Rayleigh dispersion and attenuation curves for linear viscoelastic media with arbitrary values of material damping ratio. Fan (2004) obtained the analytical solution of the Rayleigh wave propagation phenomena considering the nonlinear damping mechanism of seismic waves by applying the perturbation method. Pasternak (2008) analysed the Rayleigh wave propagation problem in the elastic half-space and viscoelastic layer interface using in the Fourier-Laplace space using the Biot viscoelastic solid model. Sharma, Sharma and Sharma (2009) derived the complex secular equations for Rayleigh wave propagation in closed and isolated mathematical conditions and studied the thermos-elastic interaction in an infinite Kelvin-Voigt type viscoelastic, thermally conducting solid bordered with viscous liquid half-spaces/layers of varying temperature. Zhang, Luo and Xia et al. (2011) studied the dispersion of Rayleigh waves in viscoelastic media by applying pseudo spectral modelling method to obtain high accuracy for Rayleigh wave modelling in viscoelastic media. In pseudo spectral method the spatial derivatives in the vertical and horizontal directions are calculated using Chebyshev and Fourier difference operators, respectively. Chiriţă, Ciarletta and Tibullo (2014) studied the propagation of surface waves over an exponentially graded half-space of isotropic Kelvin-Voigt viscoelastic material by means of wave solutions with spatial and temporal finite energy. They showed that when there is just one wave solution it is found to be retrograde at the free surface, while when there is more than one viscoelastic surface wave, one is retrograde and the others are direct at the free surface.

We also note investigations carried out in the papers by Sharma (2005), Sharma and Othman (2007), Kumar and Parter (2009), Sharma and Kumar (2009), Abd-Alla, Aftab Khan and Abo-Dahab (2017) and others listed therein in which the Rayleigh-Lamb waves dispersion in the plate made of viscoelastic or thermo-viscoelastic materials are investigated. However, it should be noted that in these works the viscoelasticity of the plate material is described by the Voigt model. Moreover, it should be noted that the investigations carried out in these works are also have important significance in the methodological sense, i.e. in these works the functional iteration numerical technique is developed for determination of the complex roots of the secular equation. Detailed consideration of this method and its advantage and disadvantages are discussed in the paper by Sharma (2011). We will again turn below to this method in the text of the paper during the discussions the solution algorithm of the secular equation.

As follows from the foregoing discussion and works reviewed above, the investigations on the dispersion of the guided waves in the half-space or covering half-spaces made from viscoelastic materials mainly were carried out by employing simple Kelvin-Voigt classical models or by using frequency dependent complex modulus of viscoelastic materials which is obtain from the experiments. These simple methods were not actually connected with the more complicated and real behaviour of viscoelastic materials and they do not illustrate the influence of the rheological parameters of the viscoelastic materials on this dispersion. These considerations led the authors to study the generalized Rayleigh waves dispersion and attenuation for a system consisting of a viscoelastic covering layer and a viscoelastic half-space utilizing more realistic mathematical viscoelastic model using Rabotnov (1980) fractional exponential operator which are already used in the papers by Akbarov and Kepceler (2015), Akbarov, Kocal and Kepceler (2016a, 2016b) and Kocal and Akbarov (2017) under investigations of the axisymmetric torsional and longitudinal waves respectively in the layered hollow cylinders made of viscoelastic materials. Moreover, in the paper by Akbarov (2014) this model is employed to study of the axisymmetric time-harmonic Lamb's problem for a system consisting of s viscoelastic covering layer and viscoelastic half-space. Note that these results in these papers are also detailed in the monograph by Akbarov (2015).

Moreover, this study, actually extends the authors previous works Negin, Akbarov and Erguven (2014), Negin (2015) and Akbarov and Negin (2017) on propagation of the generalized Rayleigh waves in an initially stressed elastic covered half-space to viscoelastic cases, where the constitutive relations for the covering layer and the half-space materials are described through the fractional exponential operator by Rabotnov (1980). The investigations are carried out within the framework of the piecewise homogeneous body model by utilizing exact equations of motion of the linear theory of viscoelasticity and it is assumed that perfect contact conditions take place on the interface surface between the layer and the half-space. The theoretical results obtained in this paper can be utilized in many relevant practical problems of wave propagation in viscoelastic layered media which play roles in areas like engineering, earthquake and geophysical sciences etc. Some numerical calculations, discussions and conclusions will be discussed in their proper places.

2 Formulation of the problem

The system consists of a layer with thickness h which covers a half-space as shown in Figure 1. The layer and the half-space occupy the regions $\{-\infty < x_1 < +\infty,\ \ 0 \le x_2 \le h,\ -\infty < x_3 < +\infty\}$ and $\{-\infty < x_1 < +\infty,\ \ -\infty \le x_2 \le 0,\ -\infty < x_3 < +\infty\}$, respectively. We assume that the materials of the constituents are isotropic, homogeneous and hereditary-viscoelastic. Positions of the points are determined in the Cartesian system of coordinates $Ox_1x_2x_3$ and a plane-strain state in Ox_1x_2 plane is considered.

Below the values related to the layer and half-space are denoted by upper indices (1) and (2), respectively.

The governing equations of motion and mechanical relations for the case under consideration are as follows:

The equations of motion:

$$\frac{\partial \sigma_{11}^{(m)}}{\partial x_1} + \frac{\partial \sigma_{12}^{(m)}}{\partial x_2} = \rho^{(m)} \frac{\partial^2 u_1^{(m)}}{\partial t^2},$$

$$\frac{\partial \sigma_{12}^{(m)}}{\partial x_1} + \frac{\partial \sigma_{22}^{(m)}}{\partial x_2} = \rho^{(m)} \frac{\partial^2 u_2^{(m)}}{\partial t^2}, \qquad m = 1,2. \tag{1}$$

Constitutive and geometrical relations:

$$\sigma_{11}^{(m)} = \lambda^{(m)*}\theta^{(m)} + 2\mu^{(m)*}\varepsilon_{11}^{(m)},$$

$$\sigma_{22}^{(m)} = \lambda^{(m)*}\theta^{(m)} + 2\mu^{(m)*}\varepsilon_{22}^{(m)},$$

$$\sigma_{12}^{(m)} = 2\mu^{(m)*}\varepsilon_{12}^{(m)},$$

$$\theta^{(m)} = \sum_i \varepsilon_{ii}^{(m)}, \quad \varepsilon_{ij}^{(m)} = \frac{1}{2}\left(\frac{\partial u_i^{(m)}}{\partial x_j} + \frac{\partial u_j^{(m)}}{\partial x_i}\right), \quad i, j = 1,2 \tag{2}$$

where, $\lambda^{(m)*}$, $\mu^{(m)*}$ are the following type viscoelastic operators:

$$\lambda^{(m)*}\varphi(t) = \lambda_0^{(m)}\varphi(t) + \int_0^t \lambda_1^{(m)}(t-\tau)\varphi(\tau)d\tau,$$

$$\mu^{(m)*}\varphi(t) = \mu_0^{(m)}\varphi(t) + \int_0^t \mu_1^{(m)}(t-\tau)\varphi(\tau)d\tau. \tag{3}$$

In Eq. (3) $\lambda_0^{(m)}$, $\mu_0^{(m)}$ are the instantaneous values of Lame's constants and $\lambda_1^{(m)}$, $\mu_1^{(m)}$ are the corresponding kernel functions for describing the hereditary properties of the materials of the constituents.

We assume that the following boundary conditions on the free face plane and contact conditions on the interface of the covering layer and half-space satisfy:

Boundary conditions:

$$\sigma_{12}^{(1)}\Big|_{x_2=h} = 0, \qquad \sigma_{22}^{(1)}\Big|_{x_2=h} = 0. \tag{4}$$

Contact conditions:

$$u_1^{(1)}\big|_{x_2=0}=u_1^{(2)}\big|_{x_2=0},\quad u_2^{(1)}\big|_{x_2=0}=u_2^{(2)}\big|_{x_2=0},$$
$$\sigma_{12}^{(1)}\big|_{x_2=0}=\sigma_{12}^{(2)}\big|_{x_2=0},\quad \sigma_{22}^{(1)}\big|_{x_2=0}=\sigma_{22}^{(2)}\big|_{x_2=0}\ . \tag{5}$$

We also assume that the following decay conditions are satisfied:

$$\sigma_{ij}^{(2)}\Big|_{x_2\to-\infty}\to 0,\qquad u_i^{(2)}\Big|_{x_2\to-\infty}\to 0. \tag{6}$$

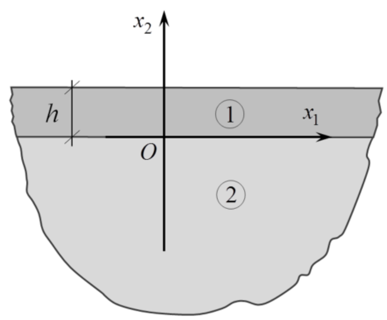


Figure 1: Geometry of the covered half-space.

This completes the formulation of the problem under consideration the novelty of which is the mathematical modelling of the near-surface (or generalized Rayleigh) wave propagation in the system consisting of the covering layer and half-space made of viscoelastic materials with arbitrary hereditary properties.

3 Method of Solution

According to the problem nature we can represent the sought values in the following form:

$$u_i^{(m)}=v_i^{(m)}\left(x_2\right)e^{i\left(kx_1-\omega t\right)},\ \ \varepsilon_{ij}^{(m)}=\gamma_{ij}^{(m)}\left(x_2\right)e^{i\left(kx_1-\omega t\right)},$$
$$\theta^{(m)}=\vartheta^{(m)}\left(x_2\right)e^{i\left(kx_1-\omega t\right)},\ \ \gamma_{ij}^{(m)}=\frac{1}{2}\left(\frac{\partial v_i^{(m)}}{\partial x_j}+\frac{\partial v_j^{(m)}}{\partial x_i}\right),\quad i,j=1,2. \tag{7}$$

where k is the wavenumber and ω is the circular frequency.

Note that the representation (7) is similar formally with the corresponding one related to the purely elastic case. However, in the present case, as it will be detailed below in section 4, the wavenumber $k\ (=k_1+ik_2)$ is selected as a complex one, the imaginary part $k_2>0$ of which characterizes the attenuation of the amplitudes of the stresses, strains and displacements in the constituents. Consequently, under investigations of the dispersion of the guided waves in the elements of constructions made of viscoelastic materials it is assumed that there is not any attenuation of the amplitudes with respect to time, and the attenuation of the amplitudes takes place only with respect to the coordinate x_1 on the coordinate axis which is directed along the wave propagation direction. This means that the magnitude of the amplitudes of the sought values at the point x_{12} is less than that at the point x_{11}, if $x_{12} > x_{11}$. Note that the same approach is also used in the papers Mazzotti, Marzani, Bartoli and Viola (2012), Barshinger and Rose (2004), Jiangong (2011), Sharma (2005), Sharma and Othman (2007), Kumar and Parter (2009), Sharma and Kumar (2009) and in many other investigations related to the study of the wave propagation in the viscoelastic materials. It is evident that the presentation of the sought values with the multiplying factor $\exp(i(kx_1 - \omega t))$ has unrealistic meaning only in the case where $x_1 \to -\infty$ (as $\operatorname{Im} k > 0$), nevertheless, all researchers have to put up with this contradiction. Consequently, the case where $x_1 > 0$ has a meaning without any doubt.

If free vibration of the elements of constructions made of viscoelastic materials is investigated, then all the sought quantities are presented with multiplying factor $\exp(i\omega t)$ where the circular frequency ω in this factor is assumed to be complex one (i.e. the natural frequencies are determined as complex frequencies). Under this free vibration the attenuation of the amplitudes of the sought values with respect to time takes place.

Thus, after the foregoing discussions we turn to consideration of the solution method and using the relation

$$\int_0^t f_1(t-\tau) f_2(\tau) d\tau \approx \int_{-\infty}^t f_1(t-\tau) f_2(\tau) d\tau, \tag{8}$$

in (2) and (3) and taking Eq. (7) and (8) into account we can write the following relation:

$$\begin{aligned} \sigma_{ii}^{(m)} &= \lambda_0^{(m)} \vartheta^{(m)}\left(x_2\right) e^{i\left(kx_1 - \omega t\right)} + 2\mu_0^{(m)} \gamma_{ii}^{(m)}\left(x_2\right) e^{i\left(kx_1 - \omega t\right)} \\ &+ \vartheta^{(m)}\left(x_2\right) e^{ikx_1} \int_{-\infty}^t \lambda_1^{(m)}(t-\tau) e^{-i\omega\tau} d\tau + 2\gamma_{ii}^{(m)}\left(x_2\right) e^{ikx_1} \int_{-\infty}^t \mu_1^{(m)}(t-\tau) e^{-i\omega\tau} d\tau. \end{aligned} \tag{9}$$

Employing the transformation $t - \tau = s$ we can make the following manipulations of the integrals which enter into (9),

$$\begin{aligned} \int_{-\infty}^t \lambda_1^{(m)}(t-\tau) e^{-i\omega\tau} d\tau &= -\int_{\infty}^0 \lambda_1^{(m)}(s) e^{-i\omega t} e^{i\omega s} ds \\ &= e^{-i\omega t} \int_0^\infty \lambda_1^{(m)}(s) e^{i\omega s} ds = e^{-i\omega t}\left(\lambda_{1c}^{(m)} + i\lambda_{1s}^{(m)}\right), \end{aligned} \tag{10}$$

where

$$\lambda_{1c}^{(m)} = \int_0^\infty \lambda_1^{(m)}(s) \cos\left(\omega s\right) ds, \qquad \lambda_{1s}^{(m)} = \int_0^\infty \lambda_1^{(m)}(s) \sin\left(\omega s\right) ds. \tag{11}$$

In a similar manner, we obtain

$$\int_{-\infty}^t \mu_1^{(m)}(t-\tau) e^{-i\omega\tau} d\tau = e^{-i\omega t}\left(\mu_{1c}^{(m)} + i\mu_{1s}^{(m)}\right), \tag{12}$$

where

$$\mu_{1c}^{(m)} = \int_0^\infty \mu_1^{(m)}(s) \cos\left(\omega s\right) ds, \qquad \mu_{1s}^{(m)} = \int_0^\infty \mu_1^{(m)}(s) \sin\left(\omega s\right) ds. \tag{13}$$

Taking the relations (10-13) into account we can write the following expressions for the stresses:

$$\sigma_{11}^{(m)} = \left[\Lambda^{(m)}(\omega)\vartheta^{(m)}(x_2) + 2M^{(m)}(\omega)\gamma_{11}^{(m)}(x_2)\right]e^{i(kx_1-\omega t)},$$

$$\sigma_{22}^{(m)} = \left[\Lambda^{(m)}(\omega)\vartheta^{(m)}(x_2) + 2M^{(m)}(\omega)\gamma_{22}^{(m)}(x_2)\right]e^{i(kx_1-\omega t)},$$

$$\sigma_{12}^{(m)} = 2M^{(m)}(\omega)\gamma_{12}^{(m)}(x_2)e^{i(kx_1-\omega t)}, \tag{14}$$

where

$$\Lambda^{(m)}(\omega) = \lambda_0^{(m)} + \lambda_{1c}^{(m)}(\omega) + i\lambda_{1s}^{(m)}(\omega),$$

$$M^{(m)}(\omega) = \mu_0^{(m)} + \mu_{1c}^{(m)}(\omega) + i\mu_{1s}^{(m)}(\omega). \tag{15}$$

Thus, we obtain the complex modulus $\Lambda^{(m)}(\omega)$, $M^{(m)}(\omega)$ instead of Lame constants in the relations (2) and (3), the real and imaginary parts of which are determined through the expressions (11), (13) and (15). This means that the complete system of field Eqs. (1), (2), (11), (12) and (15) for the viscoelastic system, can also be obtained from those written for the purely elastic system by replacing the elastic Lame constants $\lambda_0^{(m)}$ and $\mu_0^{(m)}$ with the complex constants $\Lambda^{(m)}(\omega)$ and $M^{(m)}(\omega)$, respectively. In other words, the foregoing mathematical calculations confirm the dynamic correspondence principle (see Fung (1965)) for the problem under consideration and the solution method used here agrees with this principle.

Thus, according to the foregoing procedures, the system of Eqs. (1-3) with boundary conditions (4) and (6), and contact condition (5), can be solved analytically by employing the so-called method of separation of variables. Also, it follows from the foregoing procedures that the presentation of the sought values through the variables x_1 and t is made through the known functions $\exp(ikx_1)$ and $\exp(-i\omega t)$, respectively. However, the presentation of the sought values through the variable x_2 is made through the unknown functions $v_1^{(m)}(x_2)$ and $v_2^{(m)}(x_2)$ ($m=1,2$) and substituting the expressions (7) and (14) into the equation of motion (1) we obtain the following equations for these unknown functions:

$$\frac{d^2v_1^{(m)}}{d(kx_2)^2} + b_{21}^{(m)}v_1^{(m)} + c_{21}^{(m)}\frac{dv_2^{(m)}}{d(kx_2)} = 0,$$

$$\frac{d^2v_2^{(m)}}{d(kx_2)^2} + b_{22}^{(m)}v_2^{(m)} + c_{22}^{(m)}\frac{dv_1^{(m)}}{d(kx_2)} = 0, \tag{16}$$

where

$$b_{21}^{(m)} = \frac{-\Lambda^{(m)} - 2M^{(m)} + \frac{\omega^2}{k^2}\rho^{(m)}}{M^{(m)}}, \qquad c_{21}^{(m)} = \frac{\left(\Lambda^{(m)} + M^{(m)}\right)i}{M^{(m)}}$$

$$b_{22}^{(m)} = \frac{-M^{(m)} + \frac{\omega^2}{k^2}\rho^{(m)}}{\Lambda^{(m)} + 2M^{(m)}}, \qquad c_{22}^{(m)} = \frac{\left(\Lambda^{(m)} + M^{(m)}\right)i}{\Lambda^{(m)} + 2M^{(m)}}. \tag{17}$$

It can be written from the second equation in (16) the following expressions

$$\frac{dv_1^{(m)}}{d(kx_2)} = -\frac{1}{c_{22}^{(m)}}\frac{d^2v_2^{(m)}}{d(kx_2)^2} - \frac{b_{22}^{(m)}}{c_{22}^{(m)}}v_2^{(m)}, \quad \frac{d^3v_1^{(m)}}{d(kx_2)^3} = -\frac{1}{c_{22}^{(m)}}\frac{d^4v_2^{(m)}}{d(kx_2)^4} - \frac{b_{22}^{(m)}}{c_{22}^{(m)}}\frac{d^2v_2^{(m)}}{d(kx_2)^2}, \tag{17$_1$}$$

and it can be written the following expression from the first equation in (16)

$$\frac{d^3v_1^{(m)}}{d(kx_2)^3} + b_{21}^{(m)}\frac{dv_1^{(m)}}{d(kx_2)} + c_{21}^{(m)}\frac{d^2v_2^{(m)}}{d(kx_2)^2} = 0. \tag{17$_2$}$$

Thus, substituting the expressions in (17_1) into the expression (17_2) we obtain:

$$\frac{d^4v_2^{(m)}}{d(kx_2)^4} + B_2^{(m)}\frac{d^2v_2^{(m)}}{d(kx_2)^2} + C_2^{(m)}v_2^{(m)} = 0,$$

$$B_2^{(m)} = b_{22}^{(m)} + b_{21}^{(m)} - c_{21}^{(m)}c_{22}^{(m)}, \quad C_2^{(m)} = b_{21}^{(m)}b_{22}^{(m)}. \tag{18}$$

The general solution of the Eq. (18) for the *m*-th layer can be written as follows:

$$v_2^{(1)}(x_2) = Z_1^{(1)}\exp\left(R_1^{(1)}kx_2\right) + Z_2^{(1)}\exp\left(-R_1^{(1)}kx_2\right) + Z_3^{(1)}\exp\left(R_2^{(1)}kx_2\right) + Z_4^{(1)}\exp\left(-R_2^{(1)}kx_2\right),$$

$$v_2^{(2)}(x_2) = Z_1^{(2)}\exp\left(R_1^{(2)}kx_2\right) + Z_3^{(2)}\exp\left(R_2^{(2)}kx_2\right).$$

$$\begin{aligned} v_1^{(1)}(x_2) = {} & \frac{Z_2^{(1)}R_1^{(1)}}{c_{22}^{(1)}}\exp\left(-R_1^{(1)}kx_2\right) - \frac{Z_1^{(1)}R_1^{(1)}}{c_{22}^{(1)}}\exp\left(R_1^{(1)}kx_2\right) - \frac{Z_3^{(1)}R_2^{(1)}}{c_{22}^{(1)}}\exp\left(R_2^{(1)}kx_2\right) \\ & + \frac{Z_4^{(1)}R_2^{(1)}}{c_{22}^{(1)}}\exp\left(-R_2^{(1)}kx_2\right) - \frac{Z_1^{(1)}b_{22}^{(1)}}{R_1^{(1)}c_{22}^{(1)}}\exp\left(R_1^{(1)}kx_2\right) + \frac{Z_2^{(1)}b_{22}^{(1)}}{R_1^{(1)}c_{22}^{(1)}}\exp\left(-R_1^{(1)}kx_2\right) \\ & - \frac{Z_3^{(1)}b_{22}^{(1)}}{R_2^{(1)}c_{22}^{(1)}}\exp\left(R_2^{(1)}kx_2\right) + \frac{Z_4^{(1)}b_{22}^{(1)}}{R_2^{(1)}c_{22}^{(1)}}\exp\left(-R_2^{(1)}kx_2\right) \end{aligned}$$

$$v_1^{(2)}(x_2) = -\frac{Z_1^{(2)}\left(R_1^{(2)}\right)^2 + Z_1^{(2)}b_{22}^{(2)}}{R_1^{(2)}c_{22}^{(2)}}\exp\left(R_1^{(2)}kx_2\right) - \frac{Z_3^{(2)}\left(R_2^{(2)}\right)^2 + Z_3^{(2)}b_{22}^{(2)}}{R_2^{(2)}c_{22}^{(2)}}\exp\left(R_2^{(2)}kx_2\right), \tag{19}$$

where

$$R_1^{(m)} = \sqrt{-\frac{B_2^{(m)}}{2} + \sqrt{\frac{\left(B_2^{(m)}\right)^2}{4} - C_2^{(m)}}}, \quad R_2^{(m)} = \sqrt{-\frac{B_2^{(m)}}{2} - \sqrt{\frac{\left(B_2^{(m)}\right)^2}{4} - C_2^{(m)}}}, \tag{20}$$

Finally, substituting the expressions in (20) into the Eq. (7) and (2) and employing well-known usual procedure we obtain the following dispersion equation from the boundary (4) and contact (5) conditions (4-6):

$$\det\left\|\alpha_{ij}\right\| = 0, \quad i;\, j = 1, 2, \ldots, 6. \tag{21}$$

The explicit expressions of the components of the matrix $\left(\alpha_{ij}\right)$ are given in Appendix A through the expressions (A1).

Thus, in the present section the analytical expression for the dispersion equation related to the near-surface (or generalized Rayleigh) wave propagation in the system consisting of the covering layer and half-space made of viscoelastic materials with arbitrary hereditary properties are obtained.

4 Numerical results and discussions

As we consider the time harmonic wave propagation in a viscoelastic material, it is necessary to assume that the wave number k is a complex one and can be presented as follows:

$$k = k_1 + ik_2 = k_1(1 + i\beta), \quad \beta = k_2/k_1, \tag{22}$$

where k_2 (or parameter β in (22)), i.e. the imaginary part of the wave number k, defines the attenuation of the wave amplitude under consideration and β is called the coefficient of the attenuation. It should be noted that this attenuation takes place in the wave propagation direction (i.e. in the Ox_1 axis direction) and can be called as horizontal attenuation.

We determine the phase velocity of the studied waves through the expression:

$$c = \frac{\omega}{k_1}, \tag{23}$$

We assume that the viscoelasticity of the materials of the constituents is described through the fractional exponential operator by Rabotnov (1980), i.e. we assume that

$$\lambda^{(m)*}\varphi(t) = \lambda_0^{(m)}\left[\varphi(t) - \frac{\beta_0^{(m)}}{(1+\nu_0^{(m)})} R_{\alpha^{(m)}}^{(m)*}\left(-\frac{3\beta_0^{(m)}}{2(1+\nu_0^{(m)})} - \beta_\infty^{(m)}\right)\varphi(t)\right],$$

$$\mu^{(m)*}\varphi(t) = \mu_0^{(m)}\left[\varphi(t) - \frac{3\beta_0^{(m)}}{2(1+\nu_0^{(m)})} R_{\alpha^{(m)}}^{(m)*}\left(-\frac{3\beta_0^{(m)}}{2(1+\nu_0^{(m)})} - \beta_\infty^{(m)}\right)\varphi(t)\right], \tag{24}$$

where

$$R_{\alpha^{(m)}}^{(m)*}\left(x^{(m)}\right)\varphi(t) = \int_0^t R_{\alpha^{(m)}}^{(m)}\left(x^{(m)}, t-\tau\right)\varphi(\tau)d\tau,$$

$$R_{\alpha^{(m)}}^{(m)}\left(x^{(m)},t\right)\varphi(t)=t^{-\alpha^{(m)}}\sum_{n=0}^{\infty}\frac{(x^{(m)})^{n}t^{n(1-\alpha^{(m)})}}{\Gamma((1+n)(1-\alpha^{(m)}))}. \tag{25}$$

In (25) $\Gamma(x)$ is the gamma function. Moreover, the constants $\alpha^{(m)}$, $\beta_0^{(m)}$ and $\beta_\infty^{(m)}$ in (24) and (25) are the rheological parameters of the *m*-th material. The mechanical meanings of these rheological parameters are more explained in the papers by Akbarov (2014) and Akbarov and Kepceler (2015).

Introducing the following dimensionless rheological parameters,

$$d^{(m)}=\frac{\beta_\infty^{(m)}}{\beta_0^{(m)}},\qquad Q^{(m)}=\frac{\beta_\infty^{(m)}}{R\left(\beta_0^{(m)}+\beta_\infty^{(m)}\right)^{\frac{1}{1-\alpha^{(m)}}}}, \tag{26}$$

the following expressions are obtained for the long-term values of the mechanical constants and for $\lambda_c^{(m)}$, $\lambda_s^{(m)}$, $\mu_c^{(m)}$ and $\mu_s^{(m)}$ which enter into the Eq. (15):

$$\lambda_\infty^{(m)}=\lim_{t\to\infty}(\lambda^{(m)*}\cdot 1)=\lambda_0^{(m)}\left[1+\frac{(1-2\nu_0^{(m)})}{2\nu_0^{(m)}(1+\nu_0^{(m)})}\frac{1}{(3/(2(1+\nu_0^{(m)}))+d^{(m)}}\right],$$

$$\mu_\infty^{(m)}=\lim_{t\to\infty}(\mu^{(m)*}\cdot 1)=\mu_0^{(m)}\left[1-\frac{3}{2(1+\nu_0^{(m)})}\frac{1}{(3/(2(1+\nu_0^{(m)}))+d^{(m)}}\right], \tag{27}$$

$$\lambda_c^{(m)}=\lambda_0^{(m)}\left[1-\frac{1}{(1+\nu_0^{(m)})}\left(d^{(m)}+\frac{3\beta_0^{(m)}}{2(1+\nu_0^{(m)})}\right)^{-1}R_{\alpha^{(m)}c}^{(m)}(-\beta_{01}^{(m)}-\beta_\infty^{(m)},\omega)\right],$$

$$\lambda_s^{(m)}=\lambda_0^{(m)}\frac{1}{(1+\nu_0^{(m)})}\left(d^{(m)}+\frac{3\beta_0^{(m)}}{2(1+\nu_0^{(m)})}\right)^{-1}R_{\alpha^{(m)}s}^{(m)}(-\beta_{01}^{(m)}-\beta_\infty^{(m)},\omega),$$

$$\mu_c^{(m)}=\mu_0^{(m)}\left[1-\frac{3}{2(1+\nu_0^{(m)})}\left(d^{(m)}+\frac{3\beta_0^{(m)}}{2(1+\nu_0^{(m)})}\right)^{-1}R_{\alpha^{(m)}c}^{(m)}(-\beta_{01}^{(m)}-\beta_\infty^{(m)},\omega)\right],$$

$$\mu_s^{(m)}=-\mu_0^{(m)}\frac{3}{2(1+\nu_0^{(m)})}\left(d^{(m)}+\frac{3\beta_0^{(m)}}{2(1+\nu_0^{(m)})}\right)^{-1}R_{\alpha^{(m)}s}^{(m)}(-\beta_{01}^{(m)}-\beta_\infty^{(m)},\omega), \tag{28}$$

where

$$R_{\alpha^{(m)}c}^{(m)}(-\beta_{01}^{(m)}-\beta_\infty^{(m)},\omega)=\frac{(\xi^{(m)})^2+\xi^{(m)}\sin\frac{\pi\alpha^{(m)}}{2}}{(\xi^{(m)})^2+2\xi^{(m)}\sin\frac{\pi\alpha^{(m)}}{2}+1},$$

$$R_{\alpha^{(m)}s}^{(m)}(-\beta_{01}^{(m)}-\beta_\infty^{(m)},\omega)=\frac{\xi^{(m)}\cos\frac{\pi\alpha^{(m)}}{2}}{(\xi^{(m)})^2+2\xi^{(m)}\sin\frac{\pi\alpha^{(m)}}{2}+1}, \tag{29}$$

$$\xi^{(m)} = \left(Q^{(m)}\Omega\right)^{\alpha^{(m)}-1}, \quad \Omega = k_1 h \frac{c}{c_2^{(1)}}. \tag{30}$$

The dimensionless rheological parameter $d^{(m)}$ in (26) characterizes the long-term values of the viscoelastic materials and the rheological parameter $Q^{(m)}$ characterizes the creep time of the viscoelastic materials, and finally the rheological parameter $\alpha^{(m)}$ characterizes the form of the creep (or relaxation) function for the *m*-th material and the case where $\alpha^{(m)} = 0$ corresponds to the 'standard viscoelastic body' model (or the model by Kelvin).

With respect to solution of the dispersion Eq. (21), as the values of the determinant obtained in (21) are complex, the dispersion equation can be reduced to the following one

$$\left| \det\left\|\alpha_{ij}\right\| \right| = 0, \tag{31}$$

where $\left| \det\left\|\alpha_{ij}\right\| \right|$ means the modulus of the complex number $\det\left\|\alpha_{ij}\right\|$. Consequently, for construction of the attenuation or dispersion curves it is necessary to solve numerically the Eq. (31) for the selected problem parameters. Therefore, we use the algorithm which is based on direct calculation of the values of the moduli of the dispersion determinant $\det\left\|\alpha_{ij}\right\|$ and determination of the sought roots from the criterion $\left| \det\left\|\alpha_{ij}\right\| \right| \le 10^{-9}$, and the values of the wave dispersion velocity are determined under fixed values of the problem parameters.

According to the physico-mechanical consideration, in finding the velocity c which is the root of the Eq. (31) for the selected $k_1 h$ and β it is assumed that this velocity is greater (less) than that obtained in the corresponding purely elastic case with the elastic constants calculated at $t = 0$ (at $t = \infty$). This ensures the existence of the roots of the Eq. (31).

Thus, we consider the numerical results obtained from the solution of the dispersion Eq. (31) by employing the algorithm discussed above. First, we analyse the case where the attenuation of the materials is low and according to Ewing, Jazdetzky and Press (1957) and Kolsky (1963), we assume that

$$\beta = \frac{1}{2}\frac{\mu_{1s}^{(1)}(\omega)}{\mu_0^{(1)} + \mu_{1c}^{(1)}(\omega)} \quad \text{or} \quad \beta = \frac{1}{2}\frac{\mu_{1s}^{(2)}(\omega)}{\mu_0^{(2)} + \mu_{1c}^{(2)}(\omega)}. \tag{32}$$

It should be noted that the attenuation determined by the relation (32) relates to the dispersive attenuation case. At the same time, the non-dispersive attenuation case under which the selected values for $k_2 h$ (or β) in (22) do not depend on the wave frequency ω, also is considered in the present investigations.

Note that the solution technique of the secular equation described above has also been used in the paper by Barshinger and Rose (2004). In general, as noted in the paper by Sharma (2011), there is no general method for finding the complex roots of the transcendental secular equations. Theoretically, it is known that the functional iteration method detailed by Sharma (2011) can be applied for determination of the complex roots of an analytical function. In this method the analytical function is represented in the finite power series form and the obtained algebraic equation is solved through the iteration method. Namely this method is developed for solving the secular equations with complex roots and employed in the papers Sharma (2005), Sharma and Othman (2007), Kumar and Parter (2009), Sharma and Kumar (2009) and others listed therein. For instance, in the paper Sharma (2005) the Rayleigh-Lamb wave's dispersion in the viscoelastic plate is studied and the corresponding secular equation is solved by the use of the aforementioned functional iteration method. The key step in the application of this method is the successful selection of the initial guess. However, we have not found in the papers Sharma (2005) and Sharma and Othman (2007) what initial guess is taken under employing the iteration procedure. Nevertheless, it can be predicted that the sought complex root must be near to the certain complex root of the secular equation obtained in a special limit cases in which it is possible to obtain an analytical expression for the complex root. Note that such limit cases and determination of the exact complex roots in these cases take place in the investigations carried out in Sharma (2005), Sharma and Othman (2007), Kumar and Parter (2009) and Sharma and Kumar (2009). We think that namely this and similar type complex roots can be taken as initial guess for employing the functional iteration method.

In the cases where there is no the aforementioned situation, the selection of the initial guess for the complex root is difficult and there is no any rule for selection the mentioned initial guesses. This statement is the disadvantage of the functional iteration method. At the same time, this method allows to find the real and imaginary parts of the complex roots simultaneously. This is the advantage of the functional iteration method. However, the solution method described above and applied in the present and earlier works Akbarov and Kepceler (2015), Akbarov, Kocal and Kepceler (2016) the values of the attenuation coefficient are given a priori for finding the wave propagation velocity, or as in the paper Barshinger and Rose (2004) the values of the wave propagation velocity are selected a priori for finding the attenuation coefficient. Of course, this is disadvantages of the used method. However, this method does not require the selection of any initial guesses which is advantage of method.

Note that, as noted above, in the case under consideration we prefer to use the solution algorithm described above and used also in the papers [Barshinger and Rose (2004), Akbarov and Kepceler (2015) and Akbarov, Kocal and Kepceler (2016), Kocal and Akbarov (2017)]. This is because, employing the functional iteration method for the solution of the secular equation obtained in the present paper requires special consideration and development of this method which has not been done up to now.

Thus, we turn to the analysis of numerical results which are obtained for such cases where the conditions $\mathrm{Re}\{R_1^{(2)}k\}>0$ and $\mathrm{Re}\{R_2^{(2)}k\}>0$ are satisfied simultaneously where the $R_1^{(2)}$ and $R_2^{(2)}$ are determined through the expression in (20) and k is a complex wavenumber presented as in (22). Namely, satisfaction of these conditions provides the existence of the near-surface (or generalized Rayleigh) waves in the bi-material viscoelastic system under consideration. According to the mentioned conditions, numerical investigations are made within the scope of the assumptions $\nu_0^{(1)} = \nu_0^{(2)} = 0.3$, $\rho^{(1)} = \rho^{(2)}$ and $c_2^{(2)} / c_2^{(1)} = \sqrt{\mu_0^{(2)} / \mu_0^{(1)}}$ in the cases where $\mu_0^{(2)} / \mu_0^{(1)} = 2$ and $\mu_0^{(2)} / \mu_0^{(1)} = 9$ for which it can be provided the satisfaction of the conditions $\mathrm{Re}\{R_1^{(2)}k\} > 0$ and $\mathrm{Re}\{R_2^{(2)}k\} > 0$ throughout all the calculation procedures.

First, we analyse the results obtained in the case where the viscoelasticity properties of the covering layer are the half-space are the same, i.e. the case where $Q^{(1)} = Q^{(2)} (= Q)$, $d^{(1)} = d^{(2)} (= d)$, $\alpha^{(1)} = \alpha^{(2)} (= \alpha)$; and denote it as the V.V. case. Moreover, unless otherwise specified, the results discussed below are obtained within the scope of the attenuation relation (32).

Consider the graphs given in Figure 2 and 3 which are constructed in the cases where $\mu_0^{(2)} / \mu_0^{(1)} = 2$ and $\mu_0^{(2)} / \mu_0^{(1)} = 9$, respectively under $\alpha = 0.5$. The graphs in Figure 2(a) and 3(a) illustrate the influence of the parameter Q on the dispersion curves under a fixed value of the parameter d (i.e. under $d = 10$) and the graphs in Figure 2(b) and 3(b) illustrate the influence of the parameter *d* on the dispersion curves under a fixed value of the parameter Q (i.e. under $Q = 50$). According to the discussions made in the paper by Akbarov, Kocal and Kepceler (2016), it can be predicted that the wave propagation velocity obtained for the all selected values of the parameter Q under a fixed value of the parameter d must have the same limit velocity as $k_1 h \to 0$ and this limit velocity coincides with that obtained for the corresponding purely elastic case with long-term values of the elastic constants determined with expressions in (27). Consequently, according to the expressions in (27), it can be concluded that these limit values of the wave propagation velocity must depend on the rheological parameter *d* and must not depend on the rheological parameters Q and α. Note that this conclusion is confirmed with the results illustrated in Figure 2 and 3 and with corresponding ones which will be discussed below. Moreover, these results show that the dispersion curves obtained under fixed values of the parameter d are limited with the corresponding dispersion curves obtained for the purely elastic cases under instantaneous values of the elastic constants (upper limits), i.e. under $t = 0$, and under long-term values of the elastic constants (lower limits), i.e. under $t = \infty$.

Now we note the following statement. According to the definition of the phase velocity $c = \omega / k_1$ of the wave propagation, the finite limit value of this velocity as $k_1 h \to \infty$ can be obtained only in the cases where $\omega \to \infty$. Moreover, according to the expressions (28), (29) and (30), it is obtained that $\left(\lambda_c^{(m)}; \lambda_s^{(m)}; \mu_c^{(m)}; \mu_s^{(m)}\right) \to 0$ as $\omega \to \infty$. This means that the limit values of the phase velocity of the wave propagation as $k_1 h \to \infty$ for fixed finite value of the layer thickness h must approach to the corresponding ones obtained in the purely elastic case with the instantaneous values of elastic constants at $t = 0$. This is because, the magnitude of the influence of the material viscosity on its vibration decreases with the vibration frequency ω and this influence disappears completely as

$\omega \to \infty$. This is well-known physico-mechanical statement which is confirmed again with the results given in Figure 2 and 3 and other corresponding ones given below.

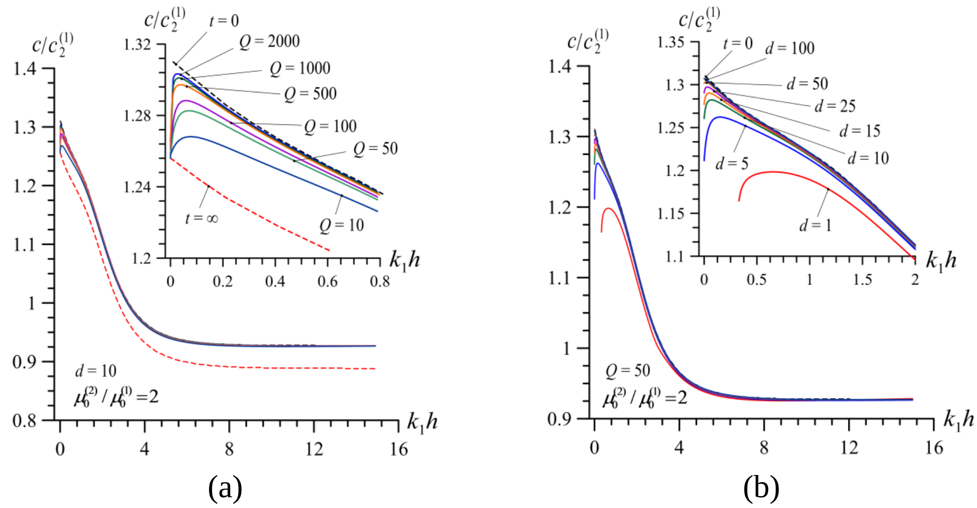


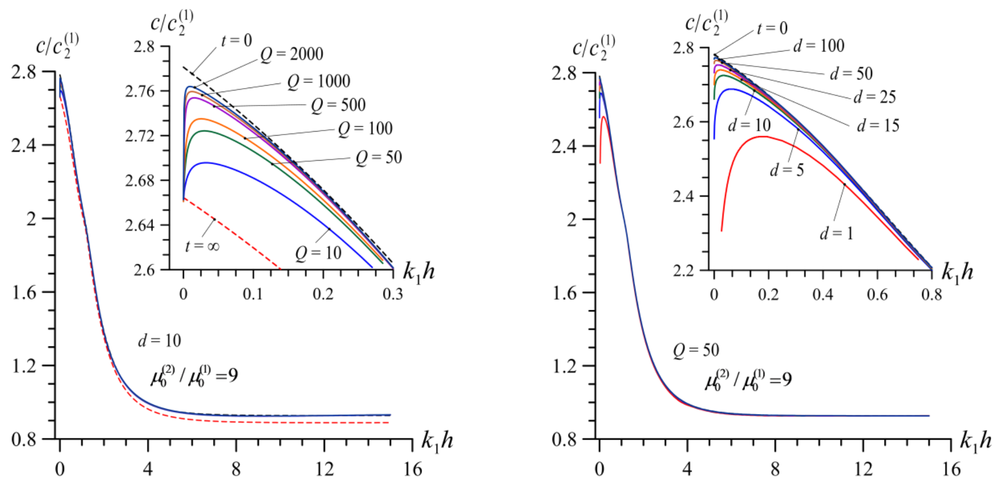
Figure 2: Dispersion curves in V.V. case $\mu_0^{(2)} / \mu_0^{(1)} = 2$ obtained for (a) various values of the parameter $Q\left(= Q^{(1)} = Q^{(2)}\right)$ under a fixed value of the parameter $d = 10$ (b) for various values of the parameter $d\left(= d^{(1)} = d^{(2)}\right)$ under a fixed value of the parameter $Q = 50$.

Note that the foregoing conformations of the obtained numerical with the related predictions and agreement of those with the known physico-mechanical considerations can be taken as validation and trustiness of the used calculation algorithm and PC programs which are composed by the authors and are realized in MATLAB. Unfortunately, we have not found any related numerical results in literature in order to compare these results with those.

We turn again to the consideration of the limit values of the wave propagation velocity as $k_1 h \to \infty$. Thus, in general, according to the foregoing discussions, and conclusions we can write the following relation

$$c \to \min(c_R^{(1)}; c_S) \text{ as } k_1 h \to \infty, \tag{33}$$

where $c_R^{(1)}$ is the Rayleigh wave propagation velocity in the covering layer material in the corresponding purely elastic case and c_S is the Stoneley wave propagation velocity for the selected pairs of materials for covering layer and half-space also in the purely elastic case. As in the cases under considerations, the Stoneley wave does not exist; therefore, the relation (33) can be replaced with the relation



(a) (b)

Figure 3: Dispersion curves in V.V. case $\mu_0^{(2)}/\mu_0^{(1)}=9$ obtained for (a) various values of the parameter $Q\left(=Q^{(1)}=Q^{(2)}\right)$ under a fixed value of the parameter $d=10$ (b) for various values of the parameter $d\left(=d^{(1)}=d^{(2)}\right)$ under a fixed value of the parameter $Q=50$.

$$c \to c_R^{(1)} \text{ as } k_1h \to \infty. \quad (34)$$

Now we analyse in detail the results given in Figure 2 and 3 from which, first of all, it follows that the viscoelasticity of the materials of the constituents causes a decrease in the wave propagation velocity. Moreover, these results show that the dispersion curves obtained for the viscoelastic case approach to the corresponding one obtained for the purely elastic case with instantaneous (long-term) values of the elastic constants at $t=0$ (at $t=\infty$) with increasing (decreasing) of the rheological parameters d and Q. It should be noted that the mentioned increase (decrease) has monotonic character and considerable effect in this increasing (decreasing) are observed in the cases where $k_1h \le 2.0$. According to the character of the dispersion curves obtained for the viscoelastic case and given in Figure 2, 3 and in other ones which will be illustrated below, it can be concluded that for each value of the rheological parameter Q and for each value of the rheological parameter d there exist the case where

$$\frac{d(c/c_2^{(2)})}{d(k_1h)}=0. \quad (35)$$

The wave propagation velocity and dimensionless wavenumber related to this case we denote by c_{cr} and $(k_1h)_{cr}$, respectively. Note that for the dispersion curves related to the purely elastic waves there is not the case where the relation (35) takes place. Consequently, the appearing of the cases where the relation (35) takes place is caused namely with the viscoelasticity of the materials of the constituents of the system under consideration.

We analyse the physico-mechanical meaning of the relation (35). In the wave propagation sense, the relation (35) means that in the case where this relation satisfies the phase velocity of the wave propagation becomes equal to the corresponding group velocity. According to this statement, in the cases where $0 < k_1 h < (k_1 h)_{cr}$ (in the cases where $(k_1 h)_{cr} < k_1 h < \infty$) the dispersion of the waves for viscoelastic case is anomalous (normal) dispersion. At the same time, the relation (35) means that the velocity $c = c_{cr}$ is root of the second order of the dispersion Eq. (21) (or (31)). According to the numerous investigations on the dynamics of the moving load acting on the 'covering layer + half-space' systems, for instance in the monograph by Akbarov (2015) and many others listed therein, the velocity which is second order root of the dispersion equation, i.e. under which the relation (35) takes place, is the critical velocity of the corresponding moving load. We recall that under this velocity of the moving load the resonance type accidents appear.

Table 1: The values of the $c / c_2^{(1)}$ obtained in the case indicated in Figure 2a under $k_1 h = 0.5$

Q	$c / c_2^{(1)}$
$Q = \infty (t = 0)$	1.2600
$Q = 2000$	1.2588
$Q = 1000$	1.2584
$Q = 500$	1.2576
$Q = 100$	1.2550
$Q = 50$	1.2527
$Q = 10$	1.2444
$Q = 0 (t = \infty)$	1.2116

Table 2: The values of the $c / c_2^{(1)}$ obtained in the case indicated in Figure 2b under $k_1 h = 0.5$

d	$c / c_2^{(1)}$
$d = \infty (t = 0)$	1.2600
$d = 100$	1.2590
$d = 50$	1.2582
$d = 25$	1.2574
$d = 15$	1.2558
$d = 10$	1.2534
$d = 5$	1.2456
$d = 1$	1.1960

Thus, it follows from the foregoing discussions that, the viscoelasticity of the materials of the constituents influences on the dispersion curves of the generalized Rayleigh waves not only in the quantitative sense but also in the qualitative sense. Moreover, it follows from the foregoing results that under investigations of the dynamics of the moving load acting on the systems which can be modelled as the 'covering layer+half-space' the appearing of the critical velocities as a result of viscoelasticity of materials of the constituents must be taken into consideration.

For a clear illustration of the amount of the influence of the materials' viscosity on the wave propagation velocity in the cases considered in Figure 2 and 3, the values of these velocity are presented in Tables 1, 2, 3 and 4 in the cases where $k_1 h = 0.5$ (Tables 1 and

2) and $k_1 h = 0.1$ (Tables 3 and 4). It follows from these data that in the quantitative sense the influence of the rheological parameter d on the wave propagation velocity is more significant than that of the rheological parameter Q.

Table 3: The values of the $c/c_2^{(1)}$ obtained in the case indicated in Figure 3a under $k_1 h = 0.1$

Q	$c/c_2^{(1)}$
$Q = \infty (t = 0)$	2.7299
Q = 2000	2.7272
Q = 1000	2.7254
Q = 500	2.7226
Q = 100	2.7131
Q = 50	2.7058
Q = 10	2.6821
$Q = 0 (t = \infty)$	2.6197

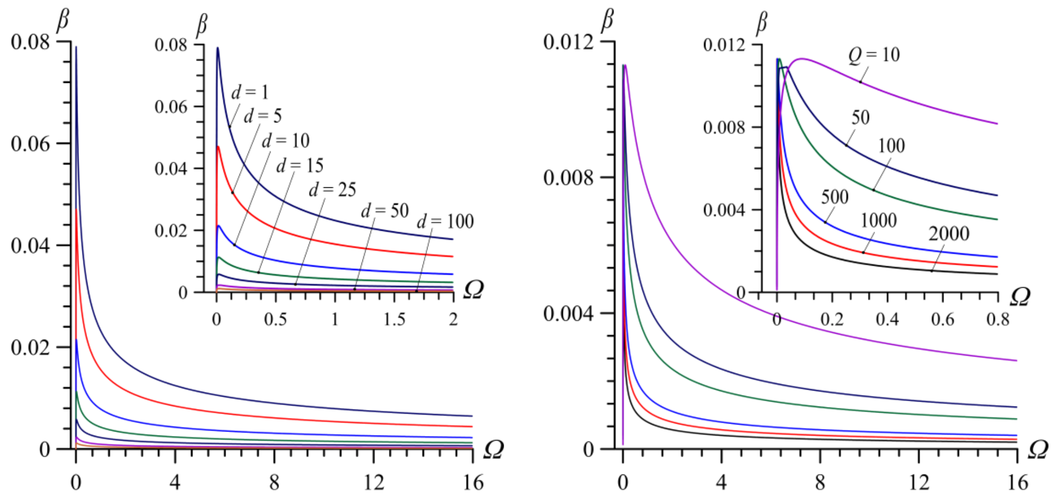
Table 4: The values of the $c/c_2^{(1)}$ obtained in the case indicated in Figure 3b under $k_1 h = 0.1$

d	$c/c_2^{(1)}$
$d = \infty (t = 0)$	2.7299
d = 100	2.7271
d = 50	2.7245
d = 25	2.7195
d = 15	2.7144
d = 10	2.7069
d = 5	2.6817
d = 1	2.5319

We recall that the all foregoing results on the dispersion curves and the results which will be discussed below are obtained within the scope of the attenuation determined by expression (32) and in the case where $Q^{(1)} = Q^{(2)} (= Q)$, $d^{(1)} = d^{(2)} (= d)$, $\alpha^{(1)} = \alpha^{(2)} (= \alpha)$ this expression can presented as

$$\beta = \frac{1}{2} \frac{\mu_{1s}^{(1)}(\omega)}{\mu_0^{(1)} + \mu_{1c}^{(1)}(\omega)} = \frac{1}{2} \frac{\mu_{1s}^{(2)}(\omega)}{\mu_0^{(2)} + \mu_{1c}^{(2)}(\omega)} . \tag{36}$$

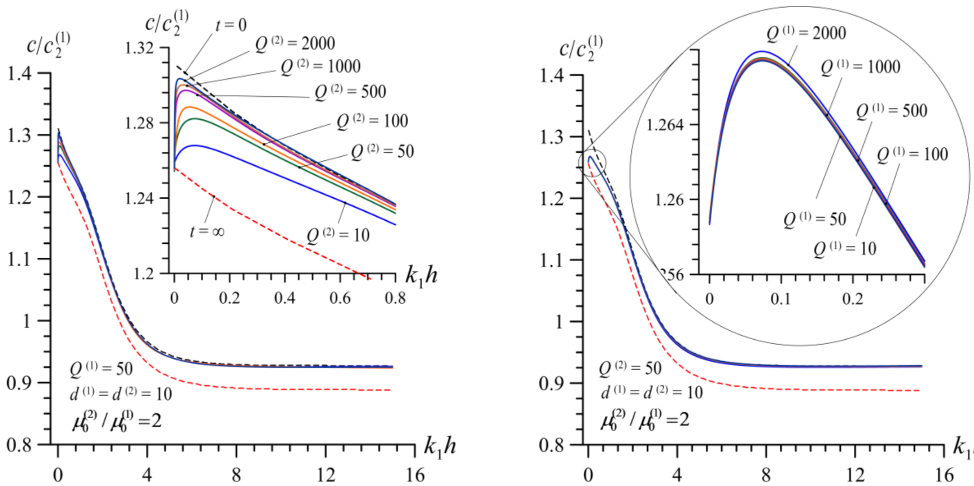
Consider graphs of the dependence of the attenuation coefficient β and dimensionless frequency $\Omega = \omega h / c_2^{(1)}$ which are given in Figure 4 for various values of the rheological parameter d under a fixed $Q(= 50)$ (Figure 4a) and for various values of the rheological parameter Q under a fixed $d(= 10)$ (Figure 4b) when $\alpha = 0.5$. Using these results and the results obtained for the dispersion curves, for instance, for the dispersion curves given in Figure 2 and 3, one can easily determine the value of the attenuation coefficient for each selected value of the wave propagation velocity. Note that this statement remains valid also for all the results which will be considered below.



(a) (b)

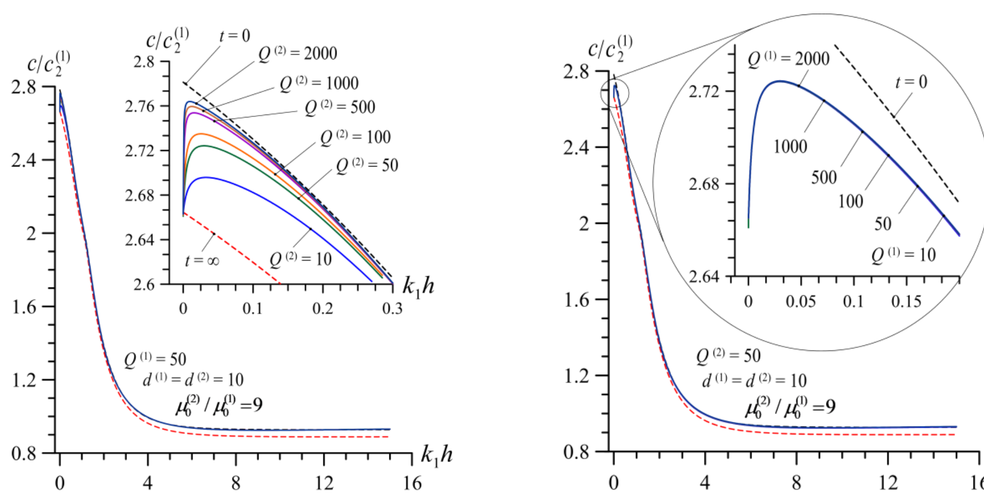
Figure 4: (a) The influence of parameter $d\left(=d^{(1)}=d^{(2)}\right)$ on β under $Q\left(=Q^{(1)}=Q^{(2)}\right)=50$ and (b) the influence of parameter $Q\left(=Q^{(1)}=Q^{(2)}\right)$ on β under $d\left(=d^{(1)}=d^{(2)}\right)=10$.

Now we attempt to answer a question what contribution is made the viscoelasticity of each constituent of the system under consideration on the dispersion curves and for this purpose we consider the graphs given in Figure 5-8 which are obtained in the case where $\alpha^{(1)}=\alpha^{(2)}=0.5$. Note that in these figures the graphs grouped by letter *a* (letter *b*) relate to the case where the rheological parameters related to the half-space (to the covering layer) material are changed and the rheological parameters related to the covering layer (to the half-space) are fixed. Moreover, note that the results given in Figure 5 and 7 (Figure 6 and 8) relate to the case where $\mu_0^{(2)}/\mu_0^{(1)}=2$ (to the case where $\mu_0^{(2)}/\mu_0^{(1)}=9$). The results given in Figure 5 and 6 (in Figure 7 and 8) illustrate the influence of the rheological parameters $Q^{(2)}$ and $Q^{(1)}$ (rheological parameters $d^{(2)}$ and $d^{(1)}$) on the dispersion curves. Thus, it follows from the graphs given in Figure 5-8 that the main contribution on the dispersion curves is made by the viscoelasticity of the half-space material.



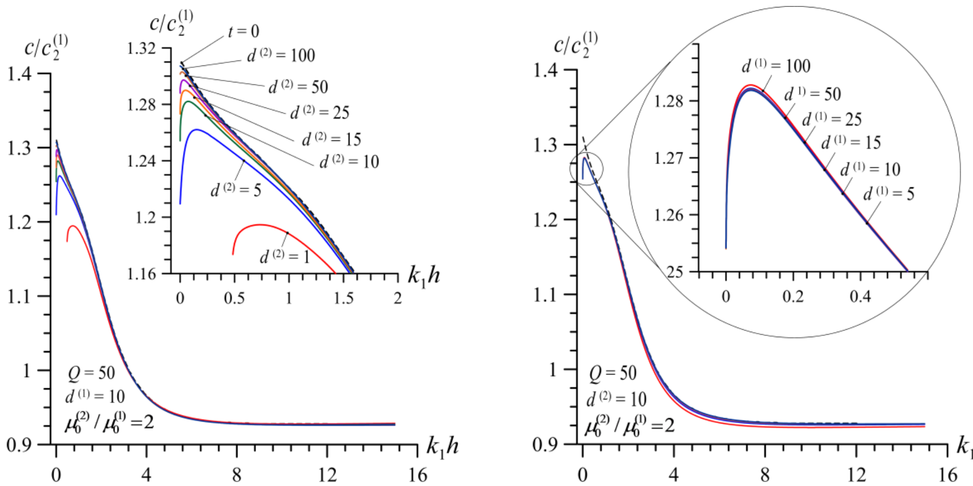
(a) (b)

Figure 5: Dispersion curves in V.V. case $\mu_0^{(2)} / \mu_0^{(1)} = 2$ (a) for different values of parameter $Q^{(2)}$ when $Q^{(1)} = 50$ and $d\left(= d^{(1)} = d^{(2)}\right) = 10$ (b) for different values of parameter $Q^{(1)}$ when $Q^{(2)} = 50$ and $d\left(= d^{(1)} = d^{(2)}\right) = 10$.



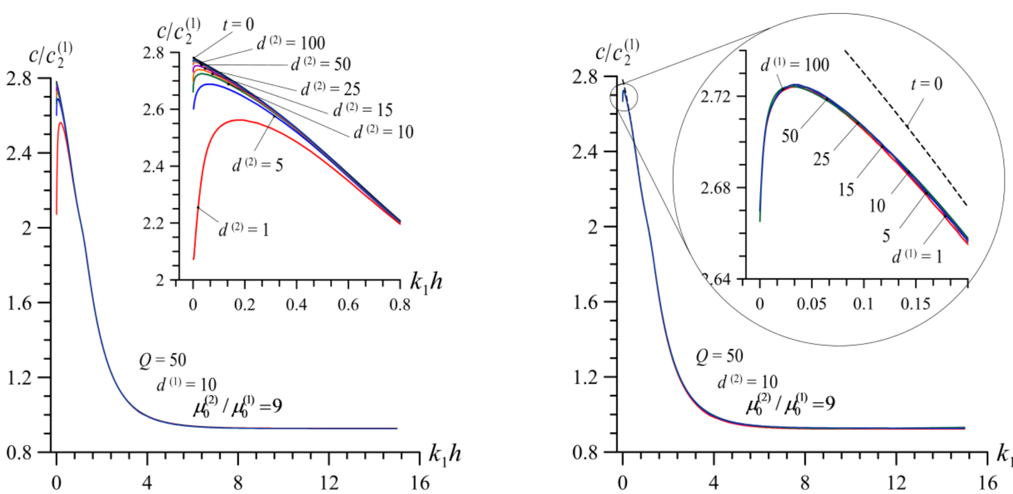
(a) (b)

Figure 6: Dispersion curves in V.V. case $\mu_0^{(2)} / \mu_0^{(1)} = 9$ (a) for different values of parameter $Q^{(2)}$ when $Q^{(1)} = 50$ and $d\left(= d^{(1)} = d^{(2)}\right) = 10$ (b) for different values of parameter $Q^{(1)}$ when $Q^{(2)} = 50$ and $d\left(= d^{(1)} = d^{(2)}\right) = 10$.



(a) (b)

Figure 7: Dispersion curves in V.V. case $\mu_0^{(2)} / \mu_0^{(1)} = 2$ (a) for different values of parameter $d^{(2)}$ when $d^{(1)} = 10$ and $Q\left(= Q^{(1)} = Q^{(2)}\right) = 50$ (b) for different values of parameter $d^{(1)}$ when $d^{(2)} = 10$ and $Q\left(= Q^{(1)} = Q^{(2)}\right) = 50$.



(a) (b)

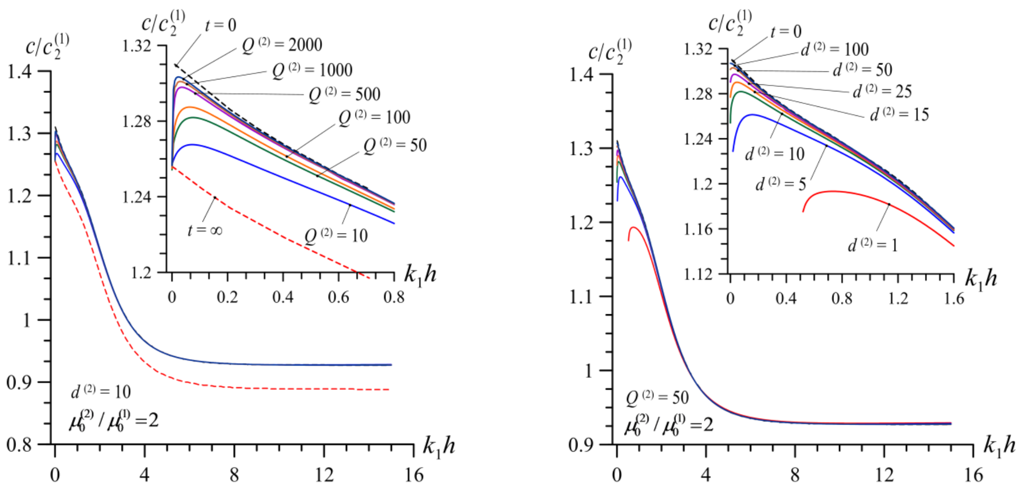
Figure 8: Dispersion curves in V.V. case $\mu_0^{(2)} / \mu_0^{(1)} = 9$ (a) for different values of parameter $d^{(2)}$ when $d^{(1)} = 10$ and $Q\left(= Q^{(1)} = Q^{(2)}\right) = 50$ (b) for different values of parameter $d^{(1)}$ when $d^{(2)} = 10$ and $Q\left(= Q^{(1)} = Q^{(2)}\right) = 50$.

We recall that the foregoing results are obtained in the cases where both the materials of the covering layer and the half-space are viscoelastic, i.e. the V.V. case. Now we consider the results obtained in the case where the material of the covering layer is purely elastic, but the material of the half-space is viscoelastic and denote this case as the

E.V. case. Note that the all foregoing results for this case are obtained within the scope of the attenuation determined by second expression in (32). Figure 9 shows the graphs obtained for the case E.V. when $\mu_0^{(2)} / \mu_0^{(1)} = 2$ for different values of parameter $Q^{(2)}$ (Figure 9a) under fixed values of parameter $d^{(2)}(=10)$ and for different values of parameter $d^{(2)}$ (Figure 9b) under fixed values of parameter $Q^{(2)}(=50)$.

Comparison of the graphs given in Figure 9a with the corresponding V.V. ones given in Figure 5a shows that the influence of the rheological parameters of the half-space $Q^{(2)}$ on the dispersion curves obtained in the E.V. case under $\mu_0^{(2)} / \mu_0^{(1)} = 2$ is almost the same as in the V.V. case. However, comparison of the graphs given Figure 9b with the corresponding V.V. ones given in Figure 7a shows that the influence of the rheological parameters of the half-space $d^{(2)}$ on the dispersion curves obtained in the E.V. case is more significant than that obtained in the V.V. case. We do not consider here the numerical results obtained for $\mu_0^{(2)} / \mu_0^{(1)} = 9$ in the E.V. case because these results in the qualitative sense are the same.

Now we consider the results related to the effect of rheological parameters $\alpha^{(1)} = \alpha^{(2)} = \alpha$ on the wave dispersion curves in the case where $d^{(1)} = d^{(2)} = d$ and $Q^{(1)} = Q^{(2)} = Q$. Note that the influence of this rheological parameter on the wave dispersion is considered for the first time in the present paper and this effect has not been examined in the papers by Akbarov and Kepceler (2015) and in the paper by Akbarov, Kocal and Kepceler (2016). Thus, we consider graphs given in Figure 10-13 which illustrate the mentioned influence. Note that these graphs are constructed in the cases where $\mu_0^{(2)} / \mu_0^{(1)} = 2$ (Figure 11 and 12) and $\mu_0^{(2)} / \mu_0^{(1)} = 9$ (Figure 12 and 13). Moreover, note that the graphs given in Figure 10 and 12 (Figure 11 and 13) show the effect of the rheological parameter α on the dispersion curves under various values of the rheological parameter Q (of the rheological parameter d) for a fixed value of the rheological parameter $d(=10)$ (for a fixed value of the rheological parameter $Q(=10)$). The graphs given in Figure 10 and 12 (in Figure 11 and 13) and grouped by letters *a*, *b*, *c* and *d* correspond the cases where $Q = 10$, 50, 100 and 500 (the case where $d = 5$, 10, 25 and 50) respectively.



(a) (b)

Figure 9: Dispersion curves in E.V. case $\mu_0^{(2)} / \mu_0^{(1)} = 2$ (a) for different values of parameter $Q^{(2)}$ when $d^{(2)} = 10$ (b) for different values of parameter $d^{(2)}$ when $Q^{(2)} = 50$.

Thus, it follows from the results given in Figure 10-13 that in the all considered cases there exists such value of the dimensionless wavenumber $k_1 h$ (denote it by $(k_1 h)*$ at which the change in the values of the rheological parameter α does not influence on the values of the wave propagation velocity. However, in the cases where $k_1 h > (k_1 h)*$ (in the cases where $k_1 h < (k_1 h)*$) an increase in the values of the parameter α causes a decrease (an increase) in the wave propagation velocities. According to the aforementioned numerical results, it can be concluded that the $(k_1 h)*$ depends on the values of the rheological parameters Q and d and an increase in the values of these parameters causes to decrease of the $(k_1 h)*$. Moreover, it can be concluded the values of the $(k_1 h)*$ depends also on the ratio $\mu_0^{(2)} / \mu_0^{(1)}$ and an increase of this ratio causes a decrease in the values of the $(k_1 h)*$.

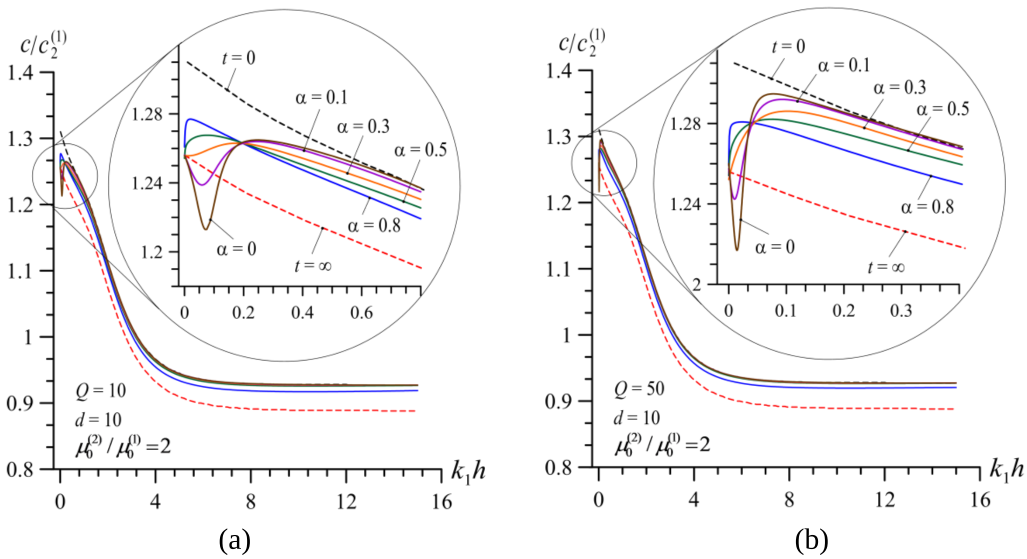


Figure 10: Dispersion curves in V.V. case for different values of parameter α and Q when $d\left(=d^{(1)}=d^{(2)}\right)=10$ and $\mu_0^{(2)} / \mu_0^{(1)} = 2$.

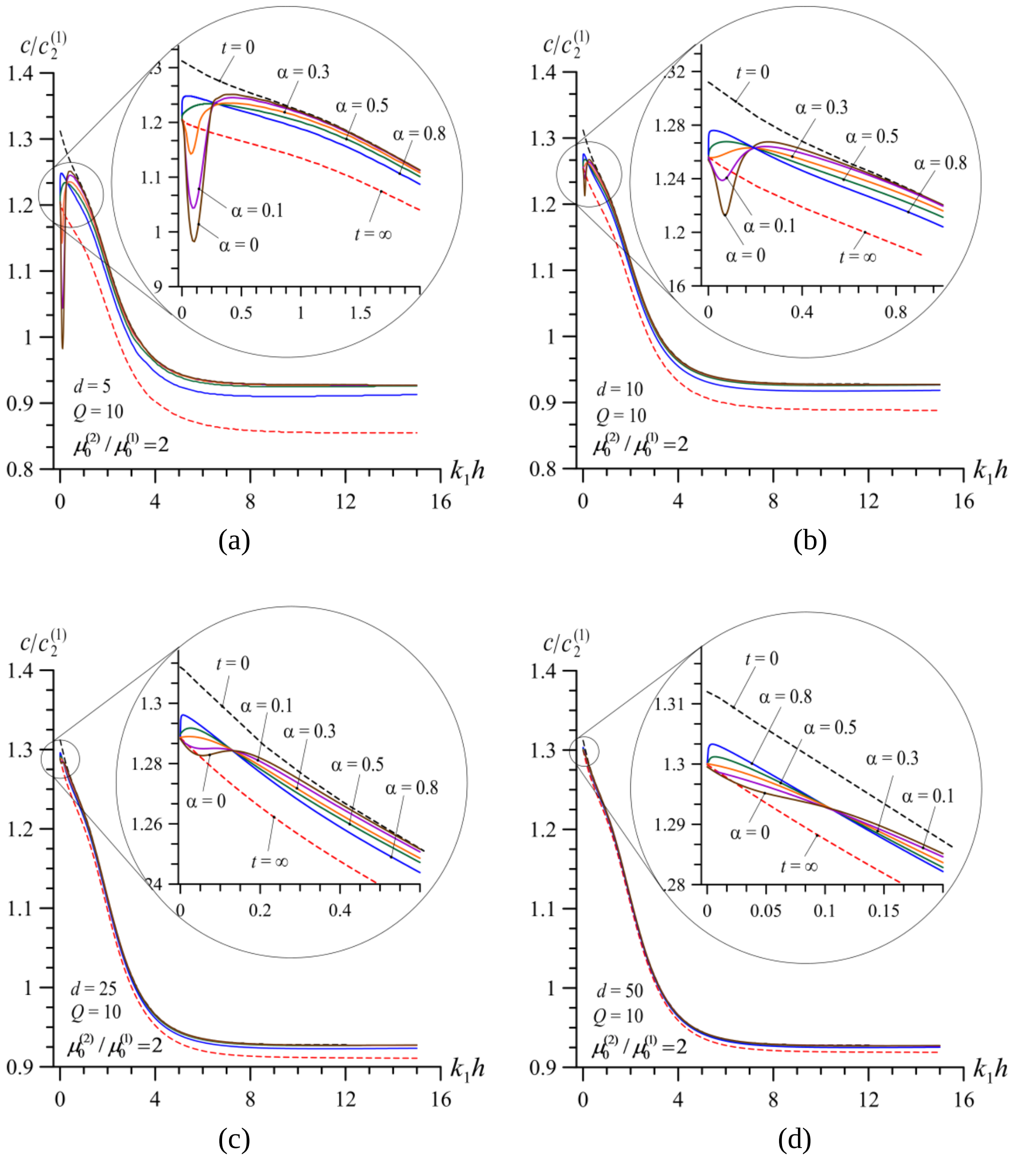


Figure 11: Dispersion curves in V.V. case for different values of parameter α and d when $Q\left(=Q^{(1)}=Q^{(2)}\right)=10$ and $\mu_0^{(2)} / \mu_0^{(1)}=2$.

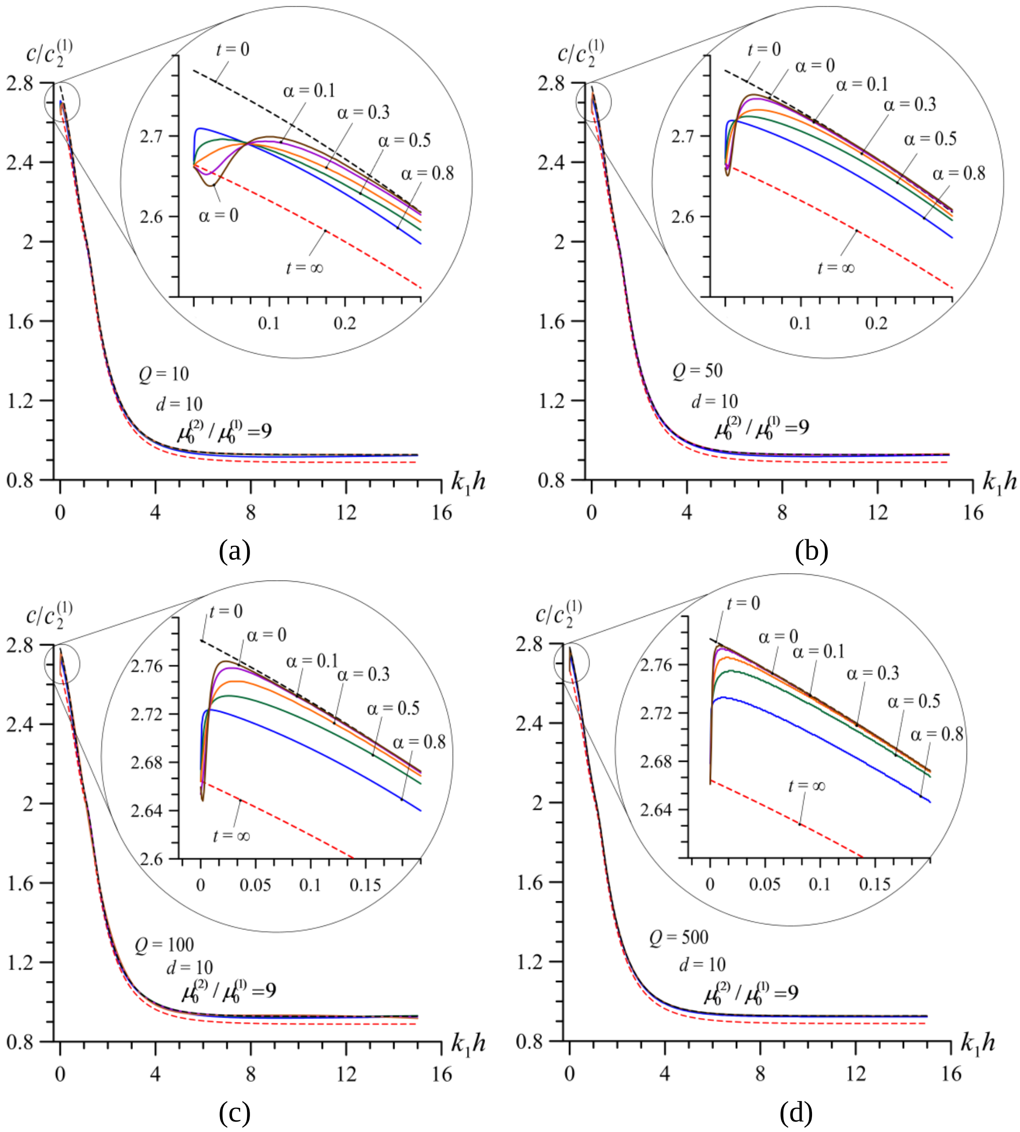


Figure 12: Dispersion curves in V.V. case for different values of parameter α and Q when $d\left(=d^{(1)}=d^{(2)}\right)=10$ and $\mu_0^{(2)}/\mu_0^{(1)}=9$.

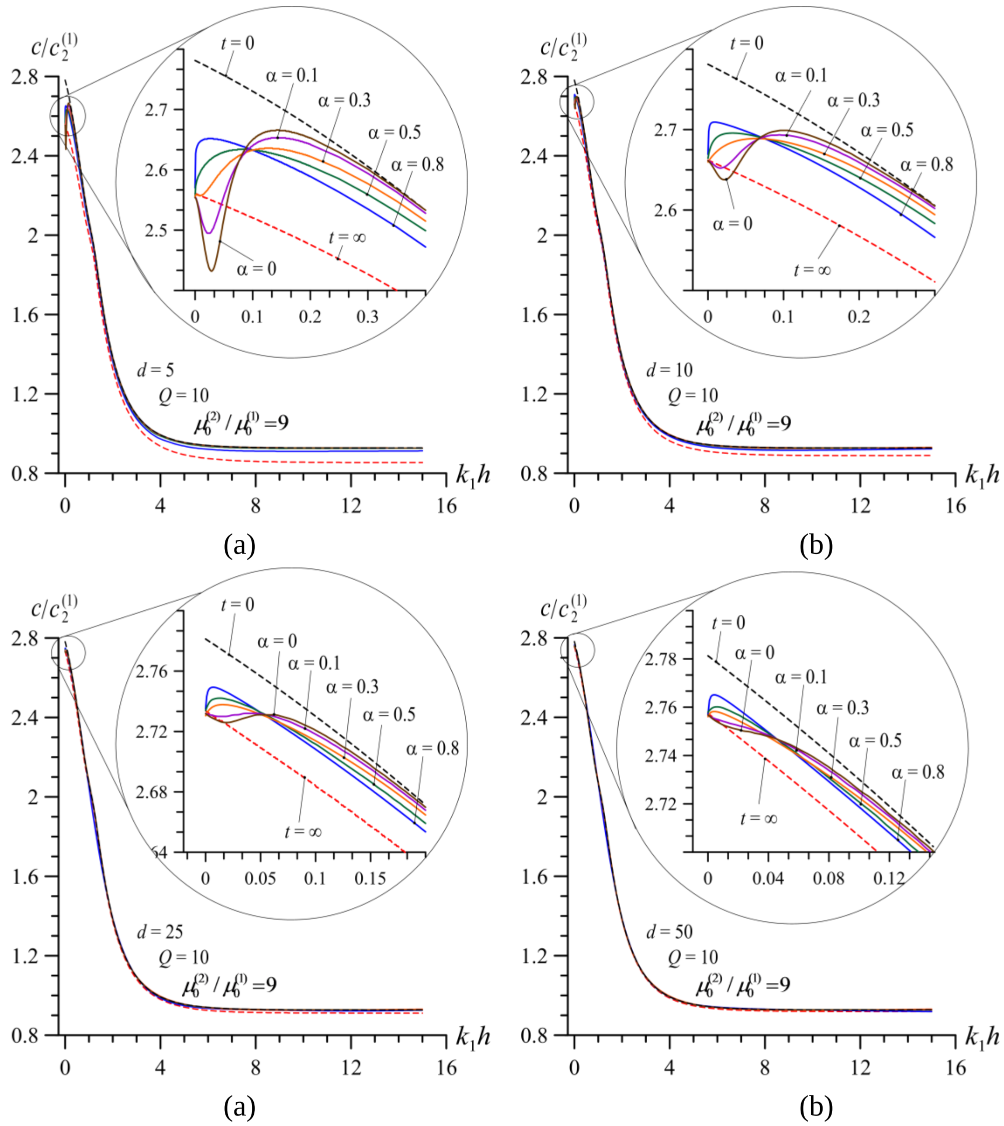


Figure 13: Dispersion curves in V.V. case for different values of parameter α and d when $Q\left(=Q^{(1)}=Q^{(2)}\right)=10$ and $\mu_0^{(2)}/\mu_0^{(1)}=9$.

Note that besides of all these it can be concluded that the change in the values of the rheological parameter α does not influence on the limit values of the wave propagation velocity as $k_1h \to 0$. However, in the near vicinity of this limit case, if to say more precisely in the region $0 < k_1h < (k_1h)^*$ the influence of the rheological parameter α on the dispersion curves is significant not only in the quantitative sense but also in the qualitative sense. So that under small values of the α, for instance under $\alpha = 0$, 0.1, the dispersion curves have well-defined minimum in the region $0 < k_1h < (k_1h)^*$ and at this minimum the relation (35) takes place. Moreover, in the near vicinity of this minimum the wave propagation velocity obtained for the viscoelastic cases become less than that obtained for the purely elastic case with long-term values of elastic constants at $t = \infty$. Consequently, in the region $0 < k_1 h < (k_1 h)^*$ a decrease in the values of the α causes to change the character of the dispersion curves. However, with increasing of α the aforementioned minimum disappears in the dispersion curves and wave propagation velocities are limited with the wave propagation velocities obtained for the purely elastic cases with instantaneous values of elastic constants at $t = 0$ (upper limit) and with long-term values of elastic constants at $t = \infty$ (lower limit).

Thus, the all foregoing results and discussions allow us to conclude that in investigations of the wave propagation in elements of constructions made of viscoelastic material the viscoelastic relations of which are described through the fractional exponential operators it is necessary to take into consideration the influence of the rheological parameter α on this propagation.

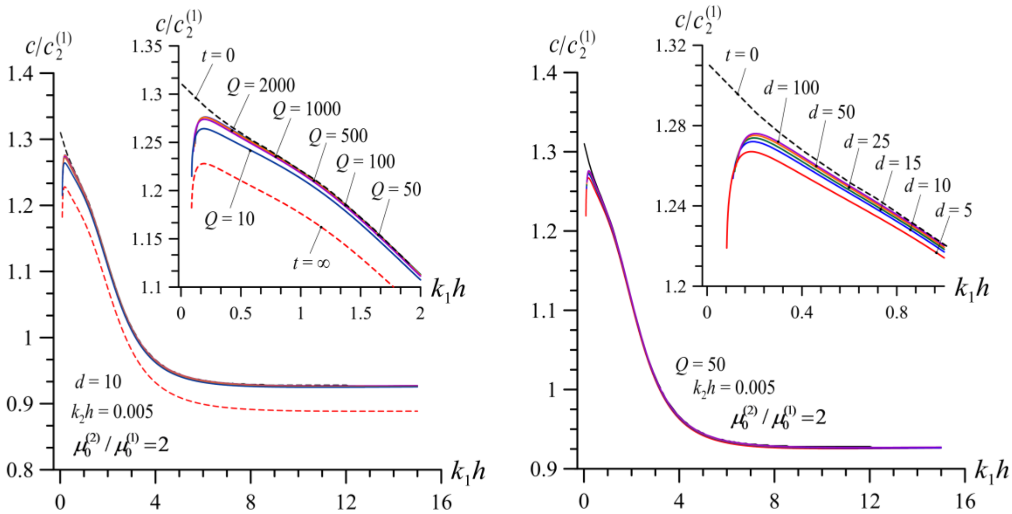
We recall that all the foregoing results are obtained for the dispersion attenuation case with the attenuation coefficient β determined through the expression (32), according to which $\beta \to 0$ as $k_1 h \to 0$. This statement can be proven as follows:

According to the expressions in (30) we can see that $\Omega \to 0$ and $\xi^{(m)} \to \infty$ as $k_1 h \to 0$. This is because $(\alpha^{(n)} - 1) < 0$ in the first expression in (30). Taking this limit value of the parameter $\xi^{(m)}$ into account we obtain from the expressions given in (29) that $R^{(m)}_{\alpha^{(m)}c} \to 1$ and $R^{(m)}_{\alpha^{(m)}s} \to 0$ as $k_1 h \to 0$. According to these limit values, it follows from the expressions in (28) that $\mu_s^{(m)} \to 0$ and also, it follows from the expressions in (32) (or from the expressions in (36)) that $\beta \to 0$ as $k_1 h \to 0$.

Thus, for estimation of the influence of this character of the dispersion attenuation on the wave dispersion curves we also consider a few results obtained for a non-dispersive attenuation case. Note that under this non-dispersive attenuation the value of the k_2 (or $k_2 h$) in (22) given a priori as a constant and then the dispersion curves are determined from the dispersion Eq. (31). Let us call the $k_2 h$ as an 'attenuation order'.

Thus, analyse the graphs given in Figure 14 which illustrate the dispersion curves obtained for the non-dispersive attenuation case under $k_2 h = 0.005$, $d^{(1)} = d^{(2)} = d$ and $Q^{(1)} = Q^{(2)} = Q$ for the case where $\mu_0^{(2)} / \mu_0^{(1)} = 2$. Note that these graphs are constructed for the various values of the parameter Q under fixed value of the $d\,(=10)$ (Figure 14a) and for the various values of the parameter d under fixed $Q\,(=50)$ (Figure 14b). It follows from these results that as a result of the non-dispersity of the attenuation of the waves propagated in the viscoelastic materials the 'cut off' values of the wavenumber $k_1 h$ appear. We denote this 'cut off' value through $(k_1 h)_{c.f.}$ and note that if we multiply

$(k_1 h)_{c.f.}$ with the corresponding wave propagation velocity c then we obtain the corresponding 'cut off' frequency $\omega_{c.f.} = \left(c(k_1 h)_{c.f.}\right)$.



(a) (b)

Figure 14: Dispersion curves obtained in the non-dispersive attenuation case under $k_2 h =$ 0.005 in the V.V. case (a) for various values of the parameter Q under a fixed value of the parameter $d(=10)$ and (b) for various values of the parameter d under a fixed value of the parameter $Q(=50)$ in the V.V. case under $\mu_0^{(2)} / \mu_0^{(1)} = 2$.

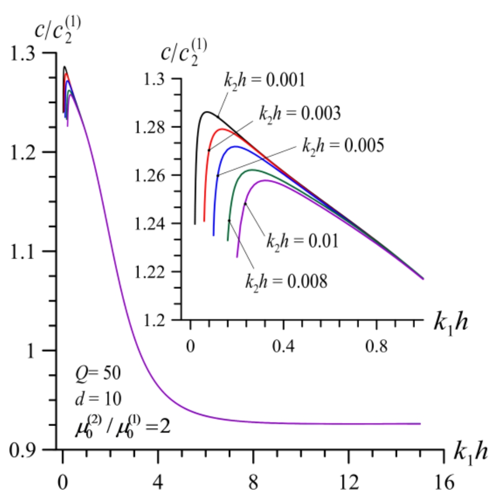


Figure 15: The influence of the 'attenuation order' $k_2 h$ on the cut off values of $k_1 h$, i.e. on the values of $(k_1 h)_{c.f.}$ in the case considered in Figure 14.

It follows from Figure 14 that the change in the rheological parameter Q does not influence on the values of the $(k_1h)_{c.f.}$, however an increase in the values of the rheological parameter d causes to increase the values of the $(k_1h)_{c.f.}$. Figure 15 shows the dispersion curves constructed for various values of the "attenuation order" k_2h in the case where $Q = 50$ and $d = 10$ from which follows that the values of the $(k_1h)_{c.f.}$ increase monotonically with the k_2h.

Thus, the results given in Figure 14 and 15 allow us to conclude that as a result of the dispersity of the attenuation of the waves under consideration the 'cut off' wavenumbers or frequencies disappear.

The experimental evaluation application of the numerical results obtained in the present paper can be made in the following manner. First, it is necessary to model the viscoelasticity of the materials of the constituents of the system through the fractional exponential operators by Rabotnov and to determine the values of the corresponding rheological parameters which enter to these operators. After it can be used the traditional methods on the measurement of the near surface wave propagation velocities for determination of the influence of the rheological parameters of the materials to this velocity. The obtained numerical results allow to use those not only in the aforementioned for measurements procedures in the concrete selected cases but also these results allows to have theoretical knowledge on the character of this influence in the principal sense.

This completes the discussions of the numerical results.

5 Conclusions

Thus, in the present paper dispersion of the generalized Rayleigh waves in the 'covering layer+half-space' system made of viscoelastic materials is investigated. The investigations carried out within the scope of the piecewise homogeneous body model by utilizing the exact equations of motion of the theory of linear viscoelasticity in the plane strain state.

The main processing flow of investigations carried out in the present paper is as follows:

1) the analytical expressions of the sought values are determined for arbitrary kernel functions in the operators described the viscoelasticity of the materials by employing the method of separation of variables;

2) the corresponding dispersion equation is also obtained for arbitrary hereditary type viscoelastic operators;

3) for numerical investigations the viscoelasticity operators (the kernel functions in these operators) of the materials are specialized through the fractional exponential operators by Rabotnov (1980), according to which, dimensionless rheological parameters characterizing the characteristic creep time (denoted by Q), the long-term values of the elastic constants (denoted by d) and the form of the creep (or relaxation) function of the materials in the beginning region of deformations (denoted by α) are introduced and through these

parameters the viscoelasticity of the materials of the covered layer and the half-space on the dispersion curves is studied;

4) numerical investigations are made for the cases where the wave attenuation coefficients are determined through the expressions in (32);

5) These results are obtained for the cases under which the generalized Rayleigh waves exists, i.e. under which the conditions $\mathrm{Re}\{R_1^{(2)}k\} > 0$ and $\mathrm{Re}\{R_2^{(2)}k\} > 0$ satisfy simultaneously, where the $R_1^{(2)}$ and $R_2^{(2)}$ are determined through the expression in (20) and k is a complex wavenumber determined as in (22);

and finally

6) The numerical results are presented and discussed mainly for the attenuation dispersion case and at the same time a few numerical examples are also presented and discussed for the non-dispersive attenuation case.

Moreover, these results are obtained for the case where the viscoelasticity properties of the covering layer and the half-space materials are the same (denoted as V.V. case) and for the case where the material of the covering layer is purely elastic, but the material of the half-space is viscoelastic (denoted as E.V. case). According to these numerical results, the following main conclusions can be drawn:

- In both V.V. and E.V. cases in the considered attenuation dispersion case the viscoelasticity of the materials causes the generalized Rayleigh wave propagation velocity to decrease. The magnitude of this decrease increases with a decrease in the aforementioned dimensionless rheological parameters d and Q;
- The character of the influence of the parameter α on the wave velocities and on the dispersion corves depends on the values of the dimensionless wave number $k_1 h$ and on the values of the rheological parameters d and Q. There exist such value of the $k_1 h$ after (before) which an increase in the values of the α causes to decrease (to increase) the wave propagation velocity;
- The lower wavenumber limit values of the wave propagation velocity depends only on the rheological parameter d and coincide with that obtained for the corresponding purely elastic case with long-term values of elastic constants at $t = \infty$ and as a result of this statement the viscoelasticity of the materials of the constituents causes to change the character of the dispersion of the waves under consideration and to appear the cases where the relation (35) satisfies;
- The satisfaction of the relation (35) conforms the appearing of the critical velocities of the moving load acting on the system under consideration as a result of the viscoelasticity of the materials of the constituents of the system;
- In general, the dispersion curves obtained to viscoelastic cases are limited by the dispersion curves corresponding to the purely elastic case with instantaneous values of the elastic constants (upper limit) and by those obtained for the purely elastic case with long-term values of the elastic constants (lower limit), however this rule is violated in the relatively small values of the rheological parameter α under which the wave propagation velocity can become less than the mentioned lower limit;

- A significant effect of the viscoelasticity of the materials on the wave propagation velocity appears in the case where $k_1 h \le 2$;
- For all the considered cases, the high wavenumber limit values of the wave propagation velocity does not depend on the rheological parameters of the materials;
- The main influence on the dispersion curves is caused by the viscoelasticity of the half-space material and the influence of the viscoelasticity of the covering layer material on these curves is insignificant;
- In the nondispersive attenuation case, the cut off values of the $k_1 h$ arise and these values increase with the non-dispersive 'attenuation order' $k_2 h$.

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Appendix A

The expressions of the components α_{ij} in Eq. (21):

$$\alpha_{11} = -\frac{R_1^{(1)}}{c_{22}^{(1)}} - \frac{b_{22}^{(1)}}{R_1^{(1)} c_{22}^{(1)}}, \quad \alpha_{12} = \frac{R_1^{(1)}}{c_{22}^{(1)}} + \frac{b_{22}^{(1)}}{R_1^{(1)} c_{22}^{(1)}},$$

$$\alpha_{13} = -\frac{R_2^{(1)}}{c_{22}^{(1)}} - \frac{b_{22}^{(1)}}{R_2^{(1)} c_{22}^{(1)}}, \quad \alpha_{14} = \frac{R_2^{(1)}}{c_{22}^{(1)}} + \frac{b_{22}^{(1)}}{R_2^{(1)} c_{22}^{(1)}},$$

$$\alpha_{15} = \frac{\left(R_1^{(2)}\right)^2 + b_{22}^{(2)}}{R_1^{(2)} c_{22}^{(2)}}, \quad \alpha_{16} = \frac{\left(R_2^{(2)}\right)^2 + b_{22}^{(2)}}{R_2^{(2)} c_{22}^{(2)}},$$

$$\alpha_{21} = 1, \quad \alpha_{22} = 1, \quad \alpha_{23} = 1, \quad \alpha_{24} = 1, \quad \alpha_{25} = -1, \quad \alpha_{26} = -1,$$

$$\alpha_{31} = -\frac{b_{22}^{(1)}}{c_{22}^{(1)}} - \frac{\left(R_1^{(1)}\right)^2}{c_{22}^{(1)}} - 1, \quad \alpha_{32} = -\frac{b_{22}^{(1)}}{c_{22}^{(1)}} - \frac{\left(R_1^{(1)}\right)^2}{c_{22}^{(1)}} - 1,$$

$$\alpha_{33} = -\frac{b_{22}^{(1)}}{c_{22}^{(1)}} - \frac{\left(R_2^{(1)}\right)^2}{c_{22}^{(1)}} - 1, \quad \alpha_{34} = -\frac{b_{22}^{(1)}}{c_{22}^{(1)}} - \frac{\left(R_2^{(1)}\right)^2}{c_{22}^{(1)}} - 1,$$

$$\alpha_{35}=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\frac{M^{(2)}(\omega)}{M^{(1)}(\omega)}\left(\frac{\left(\left(R_1^{(2)}\right)^2+b_{22}^{(2)}\right)}{c_{22}^{(2)}}+1\right),\quad \alpha_{36}=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\frac{M^{(2)}(\omega)}{M^{(1)}(\omega)}\left(\frac{\left(\left(R_2^{(2)}\right)^2+b_{22}^{(2)}\right)}{c_{22}^{(2)}}+1\right),$$

$$\alpha_{41}=R_1^{(1)}\left(\frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega)+2M^{(1)}(\omega)\right)-\frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega)\left(\frac{R_1^{(1)}}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{R_1^{(1)}c_{22}^{(1)}}\right),$$

$$\alpha_{42}=-R_1^{(1)}\left(\frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega)+2M^{(1)}(\omega)\right)+\frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega)\left(\frac{R_1^{(1)}}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{R_1^{(1)}c_{22}^{(1)}}\right),$$

$$\alpha_{43}=R_2^{(1)}\left(\frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega)+2M^{(1)}(\omega)\right)-\frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega)\left(\frac{R_2^{(1)}}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{R_2^{(1)}c_{22}^{(1)}}\right),$$

$$\alpha_{44}=-R_2^{(1)}\left(\frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega)+2M^{(1)}(\omega)\right)+\frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega)\left(\frac{R_2^{(1)}}{c_{22}^{(1)}}+\frac{b_{22}^{(1)}}{R_2^{(1)}c_{22}^{(1)}}\right),$$

$$\alpha_{45}=\frac{\lambda_0^{(2)}}{\mu_0^{(1)}}\Lambda^{(2)}(\omega)\frac{\left(\left(R_1^{(2)}\right)^2+b_{22}^{(2)}\right)}{R_1^{(2)}c_{22}^{(2)}}-R_1^{(2)}\left(\frac{\lambda_0^{(2)}}{\mu_0^{(1)}}\Lambda^{(2)}(\omega)+2\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(2)}(\omega)\right),$$

$$\alpha_{46}=\frac{\lambda_0^{(2)}}{\mu_0^{(1)}}\Lambda^{(2)}(\omega)\frac{\left(\left(R_2^{(2)}\right)^2+b_{22}^{(2)}\right)}{R_2^{(2)}c_{22}^{(2)}}-R_2^{(2)}\left(\frac{\lambda_0^{(2)}}{\mu_0^{(1)}}\Lambda^{(2)}(\omega)+2\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(2)}(\omega)\right),$$

$$\alpha_{51}=-e^{R_1^{(1)}k_1h(1+i\beta)}-\frac{b_{22}^{(1)}e^{R_1^{(1)}k_1h(1+i\beta)}}{c_{22}^{(1)}}-\frac{\left(R_1^{(1)}\right)^2e^{R_1^{(1)}k_1h(1+i\beta)}}{c_{22}^{(1)}},$$

$$\alpha_{52}=-e^{-R_1^{(1)}k_1h(1+i\beta)}-\frac{b_{22}^{(1)}e^{-R_1^{(1)}k_1h(1+i\beta)}}{c_{22}^{(1)}}-\frac{\left(R_1^{(1)}\right)^2e^{-R_1^{(1)}k_1h(1+i\beta)}}{c_{22}^{(1)}},$$

$$\alpha_{53}=-e^{R_2^{(1)}k_1h(1+i\beta)}-\frac{b_{22}^{(1)}e^{R_2^{(1)}k_1h(1+i\beta)}}{c_{22}^{(1)}}-\frac{\left(R_2^{(1)}\right)^2e^{R_2^{(1)}k_1h(1+i\beta)}}{c_{22}^{(1)}},$$

$$\alpha_{52}=-e^{-R_2^{(1)}k_1h(1+i\beta)}-\frac{b_{22}^{(1)}e^{-R_2^{(1)}k_1h(1+i\beta)}}{c_{22}^{(1)}}-\frac{\left(R_2^{(1)}\right)^2e^{-R_2^{(1)}k_1h(1+i\beta)}}{c_{22}^{(1)}},$$

$$\alpha_{55}=0,\ \alpha_{56}=0,$$

$$\alpha_{61} = -\frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega)\left(\frac{R_1^{(1)} e^{R_1^{(1)} k_1 h(1+i\beta)}}{c_{22}^{(1)}} + \frac{b_{22}^{(1)} e^{R_1^{(1)} k_1 h(1+i\beta)}}{R_1^{(1)} c_{22}^{(1)}}\right)$$

$$+ R_1^{(1)} e^{R_1^{(1)} k_1 h(1+i\beta)}\left(\frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega) + 2M^{(1)}(\omega)\right),$$

$$\alpha_{62} = \frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega)\left(\frac{R_1^{(1)} e^{-R_1^{(1)} k_1 h(1+i\beta)}}{c_{22}^{(1)}} + \frac{b_{22}^{(1)} e^{-R_1^{(1)} k_1 h(1+i\beta)}}{R_1^{(1)} c_{22}^{(1)}}\right)$$

$$- R_1^{(1)} e^{-R_1^{(1)} k_1 h(1+i\beta)}\left(\frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega) + 2M^{(1)}(\omega)\right),$$

$$\alpha_{63} = -\frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega)\left(\frac{R_2^{(1)} e^{R_2^{(1)} k_1 h(1+i\beta)}}{c_{22}^{(1)}} + \frac{b_{22}^{(1)} e^{R_2^{(1)} k_1 h(1+i\beta)}}{R_2^{(1)} c_{22}^{(1)}}\right)$$

$$+ R_2^{(1)} e^{R_2^{(1)} k_1 h(1+i\beta)}\left(\frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega) + 2M^{(1)}(\omega)\right),$$

$$\alpha_{64} = \frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega)\left(\frac{R_2^{(1)} e^{-R_2^{(1)} k_1 h(1+i\beta)}}{c_{22}^{(1)}} + \frac{b_{22}^{(1)} e^{-R_2^{(1)} k_1 h(1+i\beta)}}{R_2^{(1)} c_{22}^{(1)}}\right)$$

$$- R_2^{(1)} e^{-R_2^{(1)} k_1 h(1+i\beta)}\left(\frac{\lambda_0^{(1)}}{\mu_0^{(1)}}\Lambda^{(1)}(\omega) + 2M^{(1)}(\omega)\right),$$

$$\alpha_{65} = 0,\ \alpha_{66} = 0. \qquad \text{(A1)}$$