# Multiscale Nonlinear Thermo-Mechanical Coupling Analysis of Composite Structures with Quasi-Periodic Properties 

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#### Abstract

This paper reports a multiscale analysis method to predict the thermomechanical coupling performance of composite structures with quasi-periodic properties. In these material structures, the configurations are periodic, and the material coefficients are quasi-periodic, i.e., they depend not only on the microscale information but also on the macro location. Also, a mutual interaction between displacement and temperature fields is considered in the problem, which is our particular interest in this study. The multiscale asymptotic expansions of the temperature and displacement fields are constructed and associated error estimation in nearly pointwise sense is presented. Then, a finite element-difference algorithm based on the multiscale analysis method is brought forward in detail. Finally, some numerical examples are given. And the numerical results show that the multiscale method presented in this paper is effective and reliable to study the nonlinear thermo-mechanical coupling problem of composite structures with quasiperiodic properties.


Keywords: Thermo-mechanical coupling problem, quasi-periodic properties, multiscale asymptotic analysis, multiscale finite element-difference algorithm.

## 1 Introduction

Periodic composite material structures are widely used in the engineering practice due to their various advantageous physical and mechanical properties. Generally, both the material coefficients and geometric configurations of periodic composites are microscopically periodic. However, influenced by preparation technology, hot and humid environment, fatigue, damage and other factors, the coefficients reflecting properties of periodic composite material structures are no longer whole-periodic, but local-periodic, i.e., quasi-periodic. In other words, the material coefficients can depend not only on the

[^0]microscale information but also on the macro location. The functionally gradient material structure is a representative structure with quasi-periodic properties [Yin, Paulino, Buttlar, and Sun (2007); Shim, Yang, Liu, and Lee (2005); Zhang, Ni, Liu (2014)]. With the appearance of complex and extreme service environments, many composite structures work under transient thermo-mechanical circumstances. And the fully coupled analysis will lead to more accurate results. Therefore, it is necessary to study the transient thermomechanical coupling responses of quasi-periodic composite structures.
Up to now, some works have been performed on thermo-mechanical problems of composite structures. Feng et al. [Feng and Cui (2004)] proposed the multiscale asymptotic expansion for the problem under the conditions of coupled thermoelasticity for the structure of periodic composite materials. In [Zhang, Zhang, Bi, and Schrefler (2007); Yu and Tang (2007)], the authors investigated the thermo-mechanical problem of periodic composites by a multiscale asymptotic homogenization approach and a variational asymptotic micromechanics model, respectively. Terada et al. [Terada, Kurumatani, Ushida, and Kikuchi (2010)] considered the scale effect and derived the formal expansions for thermo-mechanical problem with periodically oscillatory coefficients. Goupee et al. [Goupee and Vel (2010)] presented multiscale thermoelastic analysis of random heterogeneous materials. Khan et al. [Khan, Barello, Muliana, and Lévesque (2011)] studied the coupled heat conduction and thermal stress analyses in particulate composites by introducing two micromechanical modeling approaches. Temizer et al. [Temizer and Wriggers (2011)] reported a survey of the known mathematical results of the homogenization method and the multiscale approach for the linear thermoelasticity. Guan et al. [Guan, Yu, and Tian (2016)] presented a thermo-mechanical model for strength prediction of concrete materials. However, these studies were devoted to one-way thermo-mechanical coupling problems, namely, the thermal effects affect the mechanical filed but not vice versa. As for the two-way coupling problems, Parnell [Parnell (2006)] has given the homogenized procedure for the transient thermo-mechanical problems with different periodic configurations. After that, Yang et al. [Yang, Cui, Wu, Wang, and Wan (2015)] investigated the transient thermo-mechanical coupling problems of periodic composites by second-order two-scale method. For quasi-periodic composites, Bensoussan et al. [Bensoussan, Lions, and Papanicolaou (1978)] presented the homogenization theory and Cao et al. [Cao and Cui (1999)] has given the first-order approximation and several basic estimations of the mechanical problems. After that, Su et al. [Su, Cui, Zhan, and Dong (2010)] present the multi-scale analysis of boundary value problems for second-order elliptic type equation for the quasi-periodic composites. Dong et al. [Dong, Nie, Cui (2017)] perform a second-order two-scale analysis and introduce a numerical algorithm for the damped wave equations of quasi-periodic composite materials. To our knowledge, we have not seen the study of the transient thermo-mechanical coupling problems of quasi-periodic composites in the existing literature.
The two-way coupling problem is strongly coupled by the hyperbolic and parabolic equations with nonlinear coefficients, and it is impossible to find the analytical solutions. As for numerical solutions, due to the quasi-periodic properties and oscillating rapidly in microscopic cells of material coefficients, in order to effectively capture the local fluctuation behaviors of temperature and displacement fields and their derivatives, the mesh size must be very small while employing the traditional numerical methods, which will lead to a prohibitive amount of computation time. Therefore, it is necessary to
develop highly efficient numerical methods for predicting the nonlinear thermo-mechanical coupling performance of composite material structures with quasi-periodic properties.
The homogenization method is developed to give the overall behavior of the composite by incorporating the fluctuations due to the heterogeneities of composites. However, numerous numerical results [Feng and Cui (2004); Bensoussan, Lions, and Papanicolaou (1978); Dong, Nie, Cui (2017)] have shown that the numerical accuracy of the standard homogenization method may not be satisfactory. And then, based on homogenization methods [Bensoussan, Lions, and Papanicolaou (1978); Marchenko and Khruslov (2008)], various multi-scale methods have been proposed [Efendiev and Hou (2009); Juanes (2005); E, Engquist, Li, Ren, and Vanden-Eijnden (2007)]. However, they only considered the first-order asymptotic expansions, which are not enough to describe the local fluctuation in many physical and mechanical problems. Hence, it is necessary to seek the more effective methods. This is the motivation for higher-order multiscale asymptotic methods and associated numerical algorithms. In recent years, Cui et al. [Cui and Yu (2006); Yang, Cui, Nie (2012); Zhang, Nie, Wu (2014)] introduced the secondorder multiscale analysis method to predict different physical and mechanical behaviors of composites. By the second-order correctors, the microscopic fluctuation behaviors inside the composite materials can be captured more accurately [Yang, Cui, Wu, Wang, and Wan (2015); Cao and Cui (1999); Su, Cui, Zhan, and Dong (2010); Dong, Nie, Cui (2017)]. However, the previous multiscale asymptotic expansions and algorithms cannot be directly employed to the thermo-mechanical problems due to the nonlinearity and twoway coupling. The aim of this paper is to establish a novel high-order multiscale method with less effort and computational cost to give a better approximation of the temperature and displacement fields in the transient thermo-mechanical coupling problems.
The remainder of this paper is outlined as follows. The formulation of the multiscale asymptotic expansions for the transient thermo-mechanical coupling problems of composite structures with quasi-periodic properties and associated error estimation in nearly pointwise sense are presented in section 2 . In section 3, a finite element-difference algorithm based on the multiscale method is given in details. Some numerical results are shown to verify the validity of the multiscale algorithms in section 4 . Finally, the conclusions are summarized in Section 5.
For convenience, the vector or matrix functions are denoted by bold letters like $\mathbf{u}, \mathbf{v}$, $\mathbf{w} . .$. , and the Einstein summation convention on repeated indices is used in this paper. Besides, we do not give the definitions of the associated Sobolev spaces in this paper, and we refer the reader to the book [Leoni (2009)].

## 2 Multiscale asymptotic expansion

Suppose that a structure made of the composite materials with small periodic configuration is denoted by $\Omega=\bigcup Y_{i}^{\varepsilon}$, as shown in Fig. 1, where $Y_{i}^{\varepsilon}$ is a basic microscopic cell, and all of $Y_{i}^{\varepsilon}$ are of the same configuration with size $\varepsilon$. Consider the transient thermo-mechanical coupling problem with mixed initial-boundary conditions for the composite structures $\Omega$ with quasi-periodic properties as follows

$$
\left\{\begin{array}{l}
b\left(x, \frac{x}{\varepsilon}\right) \frac{\partial \theta^{\varepsilon}(x, t)}{\partial t}-\frac{\partial}{\partial x_{i}}\left(a_{i j}\left(x, \frac{x}{\varepsilon}\right) \frac{\partial \theta^{\varepsilon}(x, t)}{\partial x_{j}}\right) \\
\quad+T_{0} \beta_{i j}\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^{2} u_{i}^{\varepsilon}(x, t)}{\partial x_{j} \partial t}=h(x, t) \\
\rho\left(x, \frac{x}{\varepsilon}\right) \frac{\partial^{2} u_{i}^{\varepsilon}(x, t)}{\partial t^{2}}-\frac{\partial}{\partial x_{j}}\left(C_{i j k l}\left(x, \frac{x}{\varepsilon}\right) \frac{\partial u_{k}^{\varepsilon}(x, t)}{\partial x_{l}}\right)  \tag{1}\\
\\
\quad+\frac{\partial}{\partial x_{j}}\left(\beta_{i j}\left(x, \frac{x}{\varepsilon}\right) \theta^{\varepsilon}(x, t)\right)=f_{i}(x, t) \\
\begin{array}{ll}
\theta^{\varepsilon}(x, t)=0 ; \mathbf{u}^{\varepsilon}(x, t)=\mathbf{0} & \text { in } \Omega \times(0, T) \\
\theta^{\varepsilon}(x, 0)=\theta^{0}(x) ; \mathbf{u}^{\varepsilon}(x, 0)=\mathbf{u}_{0}(x) ; \frac{\partial \mathbf{u}^{\varepsilon}}{\partial t}(x, 0)=\mathbf{u}_{1}(x) & \text { in } \Omega
\end{array}
\end{array}\right.
$$

where $i, j, k, l=1,2, \cdots, n ; \theta^{\varepsilon}(x, t)=\mathcal{T}(x, t)-T_{0}$ denotes the increment of temperature, $\mathcal{T}(x, t)$ the absolute temperature and $T_{0}$ the reference temperature; $T$ denotes the upper bound of time; $\mathbf{u}^{\varepsilon}(x, t)$ denotes the displacement vector; $a_{i j}(x, x / \varepsilon), \beta_{i j}(x, x / \varepsilon)$, $\rho(x, x / \varepsilon)$ and $C_{i j k l}(x, x / \varepsilon)$ are the thermal conductivity, thermal modulus, mass density and elastic tensor, $b(x, x / \varepsilon)=\rho(x, x / \varepsilon) c(x, x / \varepsilon)$ and $c(x, x / \varepsilon)$ is the specific heat; $h(x, t)$ and $\mathbf{f}(x, t)=\left(f_{1}(x, t), \cdots, f_{n}(x, t)\right)$ 'are the internal heat source and the body force, where 'denotes the matrix transposition; $\theta^{0}(x), \mathbf{u}_{0}(x)$ and $\mathbf{u}_{1}(x)$ are the initial increment of temperature, displacement and velocity respectively, which are chosen as $\theta^{0}(x)=0$ and $\mathbf{u}_{0}(x)=0$ in this paper. Let $y=x / \varepsilon$ be the local coordinates of unit cell $Y=[0,1]^{n}$, which contains the microscopic information, and then

$$
\begin{gathered}
b(x, x / \varepsilon)=b(x, y), a_{i j}(x, x / \varepsilon)=a_{i j}(x, y), \beta_{i j}(x, x / \varepsilon)=\beta_{i j}(x, y), \\
\rho(x, x / \varepsilon)=\rho(x, y), C_{i j k l}(x, x / \varepsilon)=C_{i j k l}(x, y)
\end{gathered}
$$

Obviously, they are 1-periodic functions in $y$, respectively.
At first, we make following assumptions
(S1) $\rho(x, y), b(x, y), a_{i j}(x, y), C_{i j k l}(x, y)$ and $\beta_{i j}(x, y)$ are bounded measurable functions and smooth with respect to $x$, and
$0<\lambda_{1}<\rho(x, y), T_{0}, a_{i j}(x, y), C_{i j k l}(x, y), \beta_{i j}(x, y), b(x, y)<\lambda_{2}$
where are $\lambda_{1}$ and $\lambda_{2}$ are two positive constant independent of $\varepsilon$.
(S2) $a_{i j}(x, y), C_{i j k l}(x, y)$ and $\beta_{i j}(x, y)$ are symmetric, and there exist two positive constant $\tau_{1}$ and $\tau_{1}$ independent of $\varepsilon$ such that
$a_{i j}=a_{j i}, \tau_{1} \gamma_{i} \gamma_{i} \leq a_{i j}(x, y) \gamma_{i} \gamma_{j} \leq \tau_{2} \gamma_{j} \gamma_{j}$
$C_{i j k l}=C_{j k l}=C_{k l i j}, \tau_{1} \eta_{i j} \eta_{i j} \leq C_{i j k l}(x, y) \eta_{i j} \eta_{k l} \leq \tau_{2} \eta_{k l} \eta_{k l}$
$\beta_{i j}=\beta_{j i}, \tau_{1} \gamma_{i} \gamma_{i} \leq \beta_{i j}(x, y) \gamma_{i} \gamma_{j} \leq \tau_{2} \gamma_{j} \gamma_{j}$
where $\left\{\eta_{i j}\right\}$ is an arbitrary symmetric matrix and $\left\{\gamma_{i}\right\}$ is an arbitrary vector with real elements;


Figure 1: Macroscopic structure and microscopic unit cell
Now we derive the multiscale computation formulas for the transient thermo-mechanical coupling problem of quasi-periodic composite material structures. For convenience, we represent problem (1) as the operator equations. Let
$\mathbf{U}^{\varepsilon}(x, t)=\left(\theta^{\varepsilon}(x, t), \mathbf{u}^{\varepsilon}(x, t)^{\prime}, \mathbf{u}_{t}^{\varepsilon}(x, t)^{\prime}\right)^{\prime}$
and problem (1) can be rewritten as
$\left\{\begin{array}{l}\mathbf{U}_{t}^{\varepsilon}(x, t)+\mathcal{L}^{\varepsilon} \mathbf{U}^{\varepsilon}(x, t)=F^{\varepsilon}(x, t) \text { in } \Omega \times(0, T) \\ \mathbf{U}^{\varepsilon}(x, 0)=\mathbf{U}^{0}(x) \quad \text { in } \Omega \\ \mathbf{U}^{\varepsilon}(x, t)=\mathbf{0} \quad \text { on } \quad \partial \Omega \times(0, T)\end{array}\right.$
where $\mathcal{L}^{\varepsilon}$ is $2 n+1$ order matrix operator defined as

$$
\mathcal{L}^{\varepsilon}=\left(\begin{array}{ccc}
\frac{A^{\varepsilon}}{b(x, y)} & \boldsymbol{O}_{1 \times n} & \frac{T_{0}}{b(x, y)} \boldsymbol{B}^{\varepsilon}  \tag{3}\\
\boldsymbol{O}_{n \times 1} & \boldsymbol{O}_{n \times n} & -\boldsymbol{I}_{n \times n} \\
\frac{\boldsymbol{D}^{\varepsilon}}{\rho(x, y)} & \frac{\boldsymbol{C}^{\varepsilon}}{\rho(x, y)} & \boldsymbol{O}_{n \times n}
\end{array}\right)
$$

and
$\mathbf{F}^{\varepsilon}(x, t)=\left(\frac{h(x, t)}{b(x, y)}, \boldsymbol{O}_{1 \times n}, \frac{\mathbf{f}(x, t)}{\rho(x, y)}\right)^{\prime}, \quad \mathbf{U}^{0}(x)=\left(0, \boldsymbol{O}_{1 \times n}, \boldsymbol{u}_{1}(x)\right)^{\prime}$
where $\boldsymbol{O}_{m \times n}$ is $m \times n$ zero matrix operator, $\boldsymbol{I}_{n \times n}$ is $n$ order identity matrix operator, and $A^{\varepsilon}, \boldsymbol{B}^{\varepsilon}, \boldsymbol{C}^{\varepsilon}, \boldsymbol{D}^{\varepsilon}$ are expressed as
$A^{\varepsilon}=-\frac{\partial}{\partial x_{i}}\left(a_{i j}(x, y) \frac{\partial}{\partial x_{j}}\right)$
$\boldsymbol{B}^{\varepsilon}=\left(\beta_{1 j}(x, y) \frac{\partial}{\partial x_{j}}, \cdots, \beta_{n j}(x, y) \frac{\partial}{\partial x_{j}}\right)$
$\boldsymbol{C}^{\varepsilon}=\left(C_{i k}^{\varepsilon}\right)_{n \times n}, C_{i k}^{\varepsilon}=-\frac{\partial}{\partial x_{j}}\left(C_{i j k l}(x, y) \frac{\partial w_{k}}{\partial x_{l}}\right)$
$\boldsymbol{D}^{\varepsilon}=\left(D_{1}^{\varepsilon}, \cdots, D_{n}^{\varepsilon}\right)^{\prime}, D_{i}^{\varepsilon}=\frac{\partial}{\partial x_{j}}\left(\beta_{i j}(x, y) w\right)$
in which $\square$ denotes the functions.
Enlightened by the work in [Yang, Cui, Wu, Wang, and Wan (2015)], $\theta^{\varepsilon}(x, t)$ and $\mathbf{u}^{\varepsilon}(x, t)$ can be expanded into following forms
$\theta^{\varepsilon}(x, t)=\theta_{0}(x, y, t)+\varepsilon \theta_{1}(x, y, t)+\varepsilon^{2} \theta_{2}(x, y, t)+O\left(\varepsilon^{3}\right)$
$\mathbf{u}^{\varepsilon}(x, t)=\mathbf{u}^{0}(x, y, t)+\varepsilon \mathbf{u}^{1}(x, y, t)+\varepsilon^{2} \mathbf{u}^{2}(x, y, t)+O\left(\varepsilon^{3}\right)$
and $\mathbf{U}^{e}(x, t)$ can be expressed as
$\mathbf{U}^{\varepsilon}(x, t)=\mathbf{U}_{0}(x, y, t)+\varepsilon \mathbf{U}_{1}(x, y, t)+\varepsilon^{2} \mathbf{U}_{2}(x, y, t)+O\left(\varepsilon^{3}\right)$
where
$\mathbf{U}_{i}(x, y, t)=\left(\theta_{i}(x, y, t), \mathbf{u}^{i}(x, y, t)^{\prime}, \mathbf{u}_{t}^{i}(x, y, t)^{\prime}\right)^{\prime}, i=0,1,2$
Taking into account of the chain rule

$$
\begin{equation*}
\frac{\partial}{\partial x_{i}} \rightarrow \frac{\partial}{\partial x_{i}}+\varepsilon^{-1} \frac{\partial}{\partial y_{i}} \tag{9}
\end{equation*}
$$

operators $A^{\varepsilon}, \boldsymbol{B}^{\varepsilon}, \boldsymbol{C}^{\varepsilon}$ and $\boldsymbol{D}^{\varepsilon}$ can be expanded into

$$
\begin{array}{ll}
A^{\varepsilon}=\varepsilon^{-2} A_{0}+\varepsilon^{-1} A_{1}+A_{2}, & \boldsymbol{B}^{\varepsilon}=\varepsilon^{-1} \boldsymbol{B}_{0}+\boldsymbol{B}_{1}  \tag{10}\\
\boldsymbol{C}^{\varepsilon}=\varepsilon^{-2} \boldsymbol{C}_{0}+\varepsilon^{-1} \boldsymbol{C}_{1}+\boldsymbol{C}_{2}, & \boldsymbol{D}^{\varepsilon}=\varepsilon^{-1} \boldsymbol{D}_{0}+\boldsymbol{D}_{1}
\end{array}
$$

where

$$
\begin{align*}
& A_{0}=-\frac{\partial}{\partial y_{i}}\left(a_{i j}(x, y) \frac{\partial}{\partial y_{j}}\right) \\
& A_{1}=-\frac{\partial}{\partial y_{i}}\left(a_{i j}(x, y) \frac{\partial}{\partial x_{j}}\right)-\frac{\partial}{\partial x_{i}}\left(a_{i j}(x, y) \frac{\partial}{\partial y_{j}}\right) \\
& A_{2}=-\frac{\partial}{\partial x_{i}}\left(a_{i j}(x, y) \frac{\partial}{\partial x_{j}}\right) \\
& \boldsymbol{B}_{0}=\left(\beta_{1 j}(x, y) \frac{\partial}{\partial y_{j}}, \cdots, \beta_{n j}(x, y) \frac{\partial}{\partial y_{j}}\right) \\
& \boldsymbol{B}_{1}=\left(\beta_{1 j}(x, y) \frac{\partial}{\partial x_{j}}, \cdots, \beta_{n j}(x, y) \frac{\partial}{\partial x_{j}}\right) \\
& \boldsymbol{C}_{0}=\left(C_{i k 0}^{\varepsilon}\right)_{n \times n}, C_{i k 0}^{\varepsilon}=-\frac{\partial}{\partial y_{j}}\left(C_{i j k l}(x, y) \frac{\partial w_{k}}{\partial y_{l}}\right) \\
& \boldsymbol{C}_{1}=\left(C_{i k 1}^{\varepsilon}\right)_{n \times n}, C_{i k 1}^{\varepsilon}=-\frac{\partial}{\partial y_{j}}\left(C_{i j k l}(x, y) \frac{\partial w_{k}}{\partial x_{l}}\right)-\frac{\partial}{\partial x_{j}}\left(C_{i j k l}(x, y) \frac{\partial w_{k}}{\partial y_{l}}\right) \\
& \boldsymbol{C}_{2}=\left(C_{i k 2}^{\varepsilon}\right)_{n \times n}, C_{i k 2}^{\varepsilon}=-\frac{\partial}{\partial x_{j}}\left(C_{i j k l}(x, y) \frac{\partial w_{k}}{\partial x_{l}}\right) \\
& \boldsymbol{D}_{0}=\left(D_{10}^{\varepsilon}, \cdots, D_{n 0}^{\varepsilon}\right)^{\prime}, D_{i 0}^{\varepsilon}=\frac{\partial}{\partial y_{j}}\left(\beta_{i j}(x, y) w\right)  \tag{11}\\
& \boldsymbol{D}_{1}=\left(D_{11}^{\varepsilon}, \cdots, D_{n 1}^{\varepsilon}\right)^{\prime}, D_{i 1}^{\varepsilon}=\frac{\partial}{\partial x_{j}}\left(\beta_{i j}(x, y) w\right)
\end{align*}
$$

From above expansions (11), we can write $\mathcal{L}^{\varepsilon}$ as follows

$$
\begin{equation*}
\mathcal{L}^{\varepsilon}=\varepsilon^{-2} \mathcal{L}_{0}+\varepsilon^{-1} \mathcal{L}_{1}+\mathcal{L}_{2} \tag{12}
\end{equation*}
$$

where $\mathcal{L}_{i}, i=0,1,2$ are matrix operators defined as

$$
\mathcal{L}_{0}=\left(\begin{array}{ccc}
\frac{A_{0}}{b(x, y)} & \boldsymbol{O}_{1 \times n} & \boldsymbol{O}_{1 \times n}  \tag{13}\\
\boldsymbol{O}_{n \times 1} & \boldsymbol{O}_{n \times n} & -\boldsymbol{O}_{n \times n} \\
\boldsymbol{O}_{n \times 1} & \frac{\boldsymbol{C}_{0}}{\rho(x, y)} & \boldsymbol{O}_{n \times n}
\end{array}\right)
$$

$$
\begin{align*}
& \mathcal{L}_{1}=\left(\begin{array}{ccc}
\frac{A_{1}}{b(x, y)} & \boldsymbol{O}_{1 \times n} & \frac{T_{0}}{b(x, y)} \boldsymbol{B}_{0} \\
\boldsymbol{O}_{n \times 1} & \boldsymbol{O}_{n \times n} & -\boldsymbol{O}_{n \times n} \\
\frac{\boldsymbol{D}_{0}}{\rho(x, y)} & \frac{\boldsymbol{C}_{1}}{\rho(x, y)} & \boldsymbol{O}_{n \times n}
\end{array}\right)  \tag{14}\\
& \mathcal{L}_{2}=\left(\begin{array}{ccc}
\frac{A_{2}}{b(x, y)} & \boldsymbol{O}_{1 \times n} & \frac{T_{0}}{b(x, y)} \boldsymbol{B}_{1} \\
\boldsymbol{O}_{n \times 1} & \boldsymbol{O}_{n \times n} & -\boldsymbol{I}_{n \times n} \\
\frac{\boldsymbol{D}_{1}}{\rho(x, y)} & \frac{\boldsymbol{C}_{2}}{\rho(x, y)} & \boldsymbol{O}_{n \times n}
\end{array}\right) \tag{15}
\end{align*}
$$

Further, we can define

$$
\begin{equation*}
\mathcal{P}^{\varepsilon}=\frac{\partial}{\partial t}+\mathcal{L}^{\varepsilon}=\varepsilon^{-2} \mathcal{P}_{0}+\varepsilon^{-1} \mathcal{P}_{1}+\mathcal{P}_{2} \tag{16}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{P}_{0}=\mathcal{L}_{0}, \mathcal{P}_{1}=\mathcal{L}_{1}, \mathcal{P}_{2}=\frac{\partial}{\partial t}+\mathcal{L}_{2} \tag{17}
\end{equation*}
$$

Inserting (8) and (17) into (3) and equating the coefficients of the same powers $\varepsilon$, we have

$$
\begin{align*}
& O\left(\varepsilon^{-2}\right): \mathcal{P}_{0} \mathbf{U}_{0}=\mathbf{0}  \tag{18}\\
& O\left(\varepsilon^{-1}\right): \mathcal{P}_{0} \mathbf{U}_{1}+\mathcal{P}_{1} \mathbf{U}_{0}=\mathbf{0}  \tag{19}\\
& O\left(\varepsilon^{0}\right): \mathcal{P}_{0} \mathbf{U}_{2}+\mathcal{P}_{1} \mathbf{U}_{1}+\mathcal{P}_{2} \mathbf{U}_{0}=\boldsymbol{F}^{\varepsilon} \tag{20}
\end{align*}
$$

and we will study these equations successively and define the homogenized problems, homogenized coefficients and corresponding cell functions.

From (19) and considering (18), (14), (12) and (9), we have

$$
\begin{align*}
& -\frac{\partial}{\partial y_{i}}\left(a_{i j}(x, y) \frac{\partial \theta_{0}}{\partial y_{j}}\right)=0  \tag{21}\\
& -\frac{\partial}{\partial y_{j}}\left(C_{i j k l}(x, y) \frac{\partial u_{k}^{0}}{\partial y_{l}}\right)=0 \tag{22}
\end{align*}
$$

According to the theory of partial differential equations, we can acquire that $\theta_{0}$ and $\mathbf{u}^{0}$ are independent of the microscale $y$, namely
$\theta_{0}=\theta_{0}(x, t), \mathbf{u}^{0}=\mathbf{u}^{0}(x, t)$
Taking (24) into (20) and considering (18), (14), (15), (12) and (9), it can be obtained
$-\frac{\partial}{\partial y_{i}}\left(a_{i j}(x, y) \frac{\partial \theta_{1}}{\partial y_{j}}\right)=\frac{\partial a_{i j}}{\partial y_{i}} \frac{\partial \theta_{0}}{\partial x_{j}}$
$-\frac{\partial}{\partial y_{j}}\left(C_{i j k l}(x, y) \frac{\partial u_{k}^{1}}{\partial y_{l}}\right)=\frac{\partial C_{i j k l}}{\partial y_{j}} \frac{\partial u_{k}^{0}}{\partial x_{l}}-\frac{\partial \beta_{i j}}{\partial y_{j}} \theta_{0}$
$\theta_{1}$ and $\mathbf{u}^{1}$ can be defined as follows
$\theta_{1}(x, y, t)=H_{\alpha_{1}}(x, y) \frac{\partial \theta_{0}}{\partial x_{\alpha_{1}}}$
$\mathbf{u}^{1}(x, y, t)=\mathbf{N}_{\alpha_{1}}(x, y) \frac{\partial \mathbf{u}^{0}}{\partial x_{\alpha_{1}}}-\mathbf{M}(x, y) \theta_{0}$
where $\mathbf{N}_{\alpha_{1}}(x, y)$ are matrix-valued functions and $\mathbf{M}(x, y)$ are vector-valued functions
$\mathbf{N}_{\alpha_{1}}(x, y)=\left(N_{\alpha_{1} k m}(x, y)\right)_{n \times n}$
$\mathbf{N}_{\alpha_{1}}(x, y) \frac{\partial \mathbf{u}^{0}}{\partial x_{\alpha_{1}}}=\left(N_{\alpha_{1} k m}(x, y) \frac{\partial u_{m}^{0}(x)}{\partial x_{\alpha_{1}}}\right)_{1 \leq k \leq n}$
Taking (27) and (28) into (25) and (26), it can be obtained that $H_{\alpha_{1}}(x, y), \mathbf{N}_{\alpha_{1}}(x, y)$ and $\mathbf{M}(x, y)$ are the solutions of following cell problems
$\left\{\begin{array}{l}-\frac{\partial}{\partial y_{i}}\left(a_{i j}(x, y) \frac{\partial H_{\alpha_{1}}(x, y)}{\partial y_{j}}\right)=\frac{\partial a_{i \alpha_{1}}}{\partial y_{i}} \text { in } Y \\ \int_{Y} H_{\alpha_{1}}(x, y) d y=0\end{array}\right.$
$\left\{\begin{array}{l}-\frac{\partial}{\partial y_{j}}\left(C_{i j k l}(x, y) \frac{\partial N_{\alpha_{1} k m}(x, y)}{\partial y_{l}}\right)=\frac{\partial C_{i j m \alpha_{1}}}{\partial y_{j}} \text { in } Y \\ \int_{Y} N_{\alpha_{1} i m}(x, y) d y=0\end{array}\right.$
$\left\{\begin{array}{l}-\frac{\partial}{\partial y_{j}}\left(C_{i j k l}(x, y) \frac{\partial M_{k}(x, y)}{\partial y_{l}}\right)=\frac{\partial \beta_{i j}}{\partial y_{j}} \quad \text { in } \quad Y \\ \int_{Y} M_{i}(x, y) d y=0\end{array}\right.$
Existence and uniqueness of the cell problems (31) - (33) can be established based on
suppositions (S1) and (S2), Lax-Milgram lemma and Korn's Inequalities [Bensoussan, Lions, and Papanicolaou (1978)].
As for (21), using (18), (14), (15), (16) and (12), we get

$$
\begin{align*}
-\frac{\partial}{\partial y_{i}}\left(a_{i j}(x, y) \frac{\partial \theta_{2}}{\partial y_{j}}\right)= & \frac{\partial}{\partial y_{i}}\left(a_{i j} \frac{\partial \theta_{1}}{\partial x_{j}}\right)+\frac{\partial}{\partial x_{i}}\left(a_{i j} \frac{\partial \theta_{1}}{\partial y_{j}}\right)+\frac{\partial}{\partial x_{i}}\left(a_{i j} \frac{\partial \theta_{0}}{\partial x_{j}}\right)  \tag{30}\\
& -T_{0} \beta_{i j} \frac{\partial^{2} u_{i}^{1}}{\partial t \partial y_{j}}-b \frac{\partial \theta_{0}}{\partial t}+h \\
-\frac{\partial}{\partial y_{j}}\left(C_{i j k l}(x, y) \frac{\partial u_{k}^{2}}{\partial y_{l}}\right)= & \frac{\partial}{\partial y_{j}}\left(C_{i j k l} \frac{\partial u_{k}^{1}}{\partial x_{l}}\right)+\frac{\partial}{\partial x_{j}}\left(C_{i j k l} \frac{\partial u_{k}^{1}}{\partial y_{l}}\right)+\frac{\partial}{\partial x_{j}}\left(C_{i j k l} \frac{\partial u_{k}^{0}}{\partial x_{l}}\right)  \tag{31}\\
& -\frac{\partial}{\partial y_{j}}\left(\beta_{i j} \theta_{1}\right)-\frac{\partial}{\partial x_{j}}\left(\beta_{i j} \theta_{0}\right)-\rho \frac{\partial^{2} u_{i}^{0}}{\partial t^{2}}+f_{i}
\end{align*}
$$

Introducing (27) and (28) into (34) and (35), integrating over both sides of equations (34), (35) in $Y$ and respecting (31)-(33), following equations are obtained

$$
\begin{align*}
& b^{0}(x) \frac{\partial \theta_{0}(x, t)}{\partial t}-\frac{\partial}{\partial x_{i}}\left(a_{i j}^{0}(x) \frac{\partial \theta_{0}(x, t)}{\partial x_{j}}\right)+T_{0} d_{i j}^{0}(x) \frac{\partial^{2} u_{i}^{0}(x, t)}{\partial x_{j} \partial t}=h(x, t)  \tag{32}\\
& \rho^{0}(x) \frac{\partial^{2} u_{i}^{0}(x, t)}{\partial t^{2}}-\frac{\partial}{\partial x_{j}}\left(C_{i j k l}^{0}(x) \frac{\partial u_{k}^{0}(x, t)}{\partial x_{l}}\right)+\frac{\partial}{\partial x_{j}}\left(\beta_{i j}^{0}(x) \theta_{0}(x, t)\right)=f_{i}(x, t) \tag{33}
\end{align*}
$$

where $a_{i j}^{0}(x), C_{i j k l}^{0}(x), \beta_{i j}^{0}(x), d_{i j}^{0}(x), b^{0}(x), \rho^{0}(x)$ are homogenized coefficients and they can be defined as follows

$$
\begin{align*}
& a_{i j}^{0}(x)=\frac{1}{|Y|} \int_{Y}\left(a_{i j}(x, y)+a_{i k}(x, y) \frac{\partial H_{j}(x, y)}{\partial y_{k}}\right) d y  \tag{34}\\
& C_{i j k l}^{0}(x)=\frac{1}{|Y|} \int_{Y}\left(C_{i j k l}(x, y)+C_{i j \alpha_{1} m}(x, y) \frac{\partial N_{l \alpha_{1} k}(x, y)}{\partial y_{m}}\right) d y  \tag{35}\\
& \beta_{i j}^{0}(x)=\frac{1}{|Y|} \int_{Y}\left(\beta_{i j}(x, y)+C_{i j k l}(x, y) \frac{\partial M_{k}(x, y)}{\partial y_{l}}\right) d y  \tag{36}\\
& d_{i j}^{0}(x)=\frac{1}{|Y|} \int_{Y}\left(\beta_{i j}(x, y)+\beta_{k l}(x, y) \frac{\partial N_{j k i}(x, y)}{\partial y_{l}}\right) d y  \tag{37}\\
& b^{0}(x)=\frac{1}{|Y|} \int_{Y}\left(b(x, y)-T_{0} \beta_{i j}(x, y) \frac{\partial M_{i}(x, y)}{\partial y_{j}}\right) d y \tag{38}
\end{align*}
$$

$\rho^{0}(x)=\frac{1}{|Y|} \int_{Y}(\rho(x, y)) d y$
According to (32) - (33) and definitions (40) - (41), it is easy to prove that $d_{i j}^{0}(x)$ is equivalent to $\beta_{i j}^{0}(x)$. And according to supposition (S2) and [Bensoussan, Lions, and Papanicolaou (1978)], it follows that $a_{i j}^{0}(x), C_{i j k l}^{0}(x), \beta_{i j}^{0}(x)$ and $d_{i j}^{0}(x)$ are symmetrical and positive definite. Thus, the homogenized problem associated with the original problem (1) can be defined as follows

$$
\left\{\begin{align*}
& b^{0}(x) \frac{\partial \theta_{0}(x, t)}{\partial t}- \frac{\partial}{\partial x_{i}}\left(a_{i j}^{0}(x) \frac{\partial \theta_{0}(x, t)}{\partial x_{j}}\right) \\
&+T_{0} d_{i j}^{0}(x) \frac{\partial^{2} u_{i}^{0}(x, t)}{\partial x_{j} \partial t}=h(x, t) \text { in } \Omega \times(0, T), \\
& \rho^{0}(x) \frac{\partial^{2} u_{i}^{0}(x, t)}{\partial t^{2}}- \frac{\partial}{\partial x_{j}}\left(C_{i j k l}^{0}(x) \frac{\partial u_{k}^{0}(x, t)}{\partial x_{l}}\right)  \tag{40}\\
&+\frac{\partial}{\partial x_{j}}\left(\beta_{i j}^{0}(x) \theta_{0}(x, t)\right)=f_{i}(x, t) \text { in } \Omega \times(0, T) \\
& \theta_{0}(x, t)=0 ; \mathbf{u}^{0}(x, t)=\mathbf{0} \quad \text { on } \quad \partial \Omega \times(0, T) \\
& \theta_{0}(x, 0)=0 ; \mathbf{u}^{0}(x, 0)=\mathbf{0} ; \frac{\partial \mathbf{u}^{0}}{\partial t}(x, 0)=\mathbf{u}_{1}(x) \text { in } \Omega
\end{align*}\right.
$$

Further, we resolve to obtain the expressions of the second-order terms $\theta_{2}$ and $\mathbf{u}^{2}$. Introducing the expressions (27), (28) of $\theta_{1}, \mathbf{u}^{1}$ and the homogenized equations (44) into (34) and (35), following equations are obtained

$$
\begin{align*}
-\frac{\partial}{\partial y_{i}}\left(a_{i j} \frac{\partial \theta_{2}}{\partial y_{j}}\right)= & \left(a_{\alpha_{2} j} \frac{\partial H_{\alpha_{1}}}{\partial y_{j}}+\frac{\partial}{\partial y_{i}}\left(a_{i \alpha_{2}} H_{\alpha_{1}}\right)+a_{\alpha_{1} \alpha_{2}}-a_{\alpha_{1} \alpha_{2}}^{0}\right) \frac{\partial^{2} \theta_{0}}{\partial x_{\alpha_{2}} \partial x_{\alpha_{1}}} \\
& +\left(\frac{\partial}{\partial x_{i}}\left(a_{i j} \frac{\partial H_{\alpha_{1}}}{\partial y_{j}}\right)+\frac{\partial}{\partial y_{i}}\left(a_{i j} \frac{\partial H_{\alpha_{1}}}{\partial x_{j}}\right)+\frac{\partial a_{i \alpha_{1}}}{\partial x_{i}}-\frac{\partial a_{i \alpha_{2}}^{0}}{\partial x_{i}}\right) \frac{\partial \theta_{0}}{\partial x_{\alpha_{1}}}  \tag{41}\\
& -T_{0}\left(\beta_{i j} \frac{\partial N_{\alpha_{2} i \alpha_{1}}}{\partial y_{j}}+\beta_{\alpha_{1} \alpha_{2}}-d_{\alpha_{1} \alpha_{2}}^{0}\right) \frac{\partial^{2} u_{\alpha_{1}}^{0}}{\partial t \partial x_{\alpha_{2}}} \\
& -\left(b-T_{0} \beta_{i j} \frac{\partial M_{i}}{\partial y_{j}}-b^{0}\right) \frac{\partial \theta_{0}}{\partial t}
\end{align*}
$$

$$
\begin{align*}
-\frac{\partial}{\partial y_{j}}\left(C_{i j k l} \frac{\partial u_{k}^{2}}{\partial y_{l}}\right)= & \left(C_{i \alpha_{2} k l} \frac{\partial N_{\alpha_{l} k m}}{\partial y_{l}}+\frac{\partial}{\partial y_{j}}\left(C_{i j \alpha_{2}} N_{\alpha_{l} k m}\right)+C_{i \alpha_{2} m \alpha_{1}}-C_{i \alpha_{2} m \alpha_{1}}^{0}\right) \frac{\partial^{2} u_{m}^{0}}{\partial x_{\alpha_{1}} \partial x_{\alpha_{2}}} \\
& +\left(\frac{\partial}{\partial x_{j}}\left(C_{i j k l} \frac{\partial N_{\alpha_{l} k m}}{\partial y_{l}}\right)+\frac{\partial}{\partial y_{j}}\left(C_{i j k l} \frac{\partial N_{\alpha_{k} k m}}{\partial x_{l}}\right)+\frac{\partial C_{i j m \alpha_{1}}}{\partial x_{j}}-\frac{\partial C_{i j m \alpha_{1}}^{0}}{\partial x_{j}}\right) \frac{\partial u_{m}^{0}}{\partial x_{\alpha_{1}}} \\
& -\left(\frac{\partial}{\partial y_{j}}\left(C_{i j k l} \frac{\partial M_{k}}{\partial x_{l}}\right)+\frac{\partial}{\partial x_{j}}\left(C_{i j k l} \frac{\partial M_{k}}{\partial y_{l}}\right)+\frac{\partial \beta_{i j}}{\partial x_{j}}-\frac{\partial \beta_{i j}^{0}}{\partial x_{j}}\right) \theta_{0}  \tag{42}\\
& -\left(C_{i \alpha_{l} k l} \frac{\partial M_{k}}{\partial y_{l}}+\frac{\partial}{\partial y_{j}}\left(C_{i j k \alpha_{1}} M_{k}\right)+\frac{\partial}{\partial y_{j}}\left(\beta_{i j} H_{\alpha_{1}}\right)+\beta_{i \alpha_{1}}-\beta_{i \alpha_{1}}^{0}\right) \frac{\partial \theta_{0}}{\partial x_{\alpha_{1}}} \\
& -\left(\rho-\rho^{0}\right) \frac{\partial^{2} u_{i}^{0}}{\partial t^{2}}
\end{align*}
$$

Then $\theta_{2}$ and $\mathbf{u}^{2}$ can be defined as follows

$$
\begin{align*}
\theta_{2}(x, y, t)= & H_{\alpha_{1} \alpha_{2}}(x, y) \frac{\partial^{2} \theta_{0}}{\partial x_{\alpha_{1}} \partial x_{\alpha_{2}}}(x, t)+F_{\alpha_{1}}(x, y) \frac{\partial \theta_{0}}{\partial x_{\alpha_{1}}}(x, t)  \tag{43}\\
& -G_{\alpha_{1} \alpha_{2}}(x, y) \frac{\partial^{2} u_{\alpha_{1}}^{0}}{\partial t \partial x_{\alpha_{2}}}(x, t)-R(x, y) \frac{\partial \theta_{0}}{\partial t}(x, t) \\
\mathbf{u}^{2}(x, y, t)= & \mathbf{N}_{\alpha_{1} \alpha_{2}}(x, y) \frac{\partial^{2} \mathbf{u}^{0}}{\partial x_{\alpha_{1}} \partial x_{\alpha_{2}}}(x, t)+\mathbf{P}_{\alpha_{1}}(x, y) \frac{\partial \mathbf{u}^{0}}{\partial x_{\alpha_{1}}}(x, t)  \tag{44}\\
& -\mathbf{Q}(x, y) \theta_{0}(x, t)-\mathbf{M}_{\alpha_{1}}(x, y) \frac{\partial \theta_{0}}{\partial x_{\alpha_{1}}}(x, t)-\mathbf{S}(x, y) \frac{\partial^{2} \mathbf{u}^{0}}{\partial t^{2}}(x, t)
\end{align*}
$$

where $H_{\alpha_{1} \alpha_{2}}(x, y), F_{\alpha_{1}}(x, y), G_{\alpha_{1} \alpha_{2}}(x, y)$ and $R(x, y)$ are scalar functions; $\mathbf{N}_{\alpha_{1} \alpha_{2}}(x, y)$, $\mathbf{P}_{\alpha_{1}}(x, y)$ and $\mathbf{S}(x, y)$ are matrix-valued functions; $\mathbf{Q}(x, y)$ and $\mathbf{M}_{\alpha_{1}}(x, y)$ are vectorvalued functions. They are all 1-periodic functions in $Y$ and solutions of following auxiliary cell problems

$$
\left\{\begin{array}{l}
-\frac{\partial}{\partial y_{i}}\left(a_{i j}(x, y) \frac{\partial H_{\alpha_{1} \alpha_{2}}(x, y)}{\partial y_{j}}\right)  \tag{45}\\
=a_{\alpha_{2} k} \frac{\partial H_{\alpha_{1}}}{\partial y_{k}}+\frac{\partial}{\partial y_{k}}\left(a_{k \alpha_{2}} H_{\alpha_{1}}\right)+a_{\alpha_{1} \alpha_{2}}-a_{\alpha_{1} \alpha_{2}}^{0} \quad \text { in } \quad Y \\
\int_{Y} H_{\alpha_{1} \alpha_{2}}(x, y) d y=0
\end{array}\right.
$$

$$
\begin{align*}
& -\frac{\partial}{\partial y_{i}}\left(a_{i j}(x, y) \frac{\partial F_{\alpha_{1}}(x, y)}{\partial y_{j}}\right) \\
& \left\{=\frac{\partial}{\partial x_{i}}\left(a_{i j} \frac{\partial H_{\alpha_{1}}}{\partial y_{j}}\right)+\frac{\partial}{\partial y_{i}}\left(a_{i j} \frac{\partial H_{\alpha_{1}}}{\partial x_{j}}\right)+\frac{\partial a_{i \alpha_{1}}}{\partial x_{i}}-\frac{\partial a_{i \alpha_{1}}^{0}}{\partial x_{i}} \text { in } Y\right.  \tag{46}\\
& \int_{Y} F_{\alpha_{1}}(x, y) d y=0 \\
& \left\{\begin{array}{l}
-\frac{\partial}{\partial y_{i}}\left(a_{i j}(x, y) \frac{\partial G_{\alpha_{1} \alpha_{2}}(x, y)}{\partial y_{j}}\right)=T_{0}\left(\beta_{i j} \frac{\partial N_{\alpha_{2} i \alpha_{1}}}{\partial y_{j}}+\beta_{\alpha_{1} \alpha_{2}}-d_{\alpha_{1} \alpha_{2}}^{0}\right) \text { in } Y \\
\int_{Y} G_{\alpha_{1} \alpha_{2}}(x, y) d y=0
\end{array}\right.  \tag{47}\\
& \left\{-\frac{\partial}{\partial y_{i}}\left(a_{i j}(x, y) \frac{\partial R(x, y)}{\partial y_{j}}\right)=b-T_{0} \beta_{i j} \frac{\partial M_{i}}{\partial y_{j}}-b^{0} \text { in } Y\right.  \tag{48}\\
& \int_{Y} R(x, y) d y=0 \\
& \int-\frac{\partial}{\partial y_{j}}\left(C_{i j k l}(x, y) \frac{\partial N_{\alpha_{1} \alpha_{2} k m}(x, y)}{\partial y_{l}}\right) \\
& \left\{=C_{i \alpha_{2} k l} \frac{\partial N_{\alpha_{1} k m}}{\partial y_{l}}+\frac{\partial}{\partial y_{j}}\left(C_{i j k \alpha_{2}} N_{\alpha_{1} k m}\right)+C_{i \alpha_{2} m \alpha_{1}}-C_{i \alpha_{2} m \alpha_{1}}^{0} \quad \text { in } \quad Y\right.  \tag{49}\\
& \int_{Y} N_{\alpha_{1} \alpha_{2} i m}(x, y) d y=0 \\
& \int-\frac{\partial}{\partial y_{j}}\left(C_{i j k l}(x, y) \frac{\partial P_{\alpha_{k} k m}(x, y)}{\partial y_{l}}\right) \\
& \left\{=\frac{\partial}{\partial x_{j}}\left(C_{i j k l} \frac{\partial N_{\alpha_{k} k m}}{\partial y_{l}}\right)+\frac{\partial}{\partial y_{j}}\left(C_{i j k l} \frac{\partial N_{\alpha_{1} k m}}{\partial x_{l}}\right)+\frac{\partial C_{i j m \alpha_{1}}}{\partial x_{j}}-\frac{\partial C_{i j m \alpha_{1}}^{0}}{\partial x_{j}} \text { in } Y\right.  \tag{50}\\
& \int_{Y} P_{\alpha_{i} m}(x, y) d y=0 \\
& \int-\frac{\partial}{\partial y_{j}}\left(C_{i j k l}(x, y) \frac{\partial Q_{k}(x, y)}{\partial y_{l}}\right) \\
& \left\{=\frac{\partial}{\partial x_{j}}\left(C_{i j k l} \frac{\partial M_{k}}{\partial y_{l}}\right)+\frac{\partial}{\partial y_{j}}\left(C_{i j k l} \frac{\partial M_{k}}{\partial x_{l}}\right)+\frac{\partial \beta_{i j}}{\partial x_{j}}-\frac{\partial \beta_{i j}^{0}}{\partial x_{j}} \text { in } Y\right.  \tag{51}\\
& \int_{Y} Q_{i}(x, y) d y=0
\end{align*}
$$

$$
\begin{align*}
& \left\{\begin{array}{l}
-\frac{\partial}{\partial y_{j}}\left(C_{i j k l}(x, y) \frac{\partial M_{\alpha_{k} k}(x, y)}{\partial y_{l}}\right) \\
=C_{i \alpha_{k} k l} \frac{\partial M_{k}}{\partial y_{l}}+\frac{\partial}{\partial y_{j}}\left(C_{i j k \alpha_{1}} M_{k}\right)+\frac{\partial}{\partial y_{j}}\left(\beta_{i j} H_{\alpha_{1}}\right)+\beta_{i \alpha_{1}}-\beta_{i \alpha_{1}}^{0} \quad \text { in } Y \\
\int_{Y} M_{\alpha_{i}}(x, y) d y=0
\end{array}\right. \\
& \left\{\begin{array}{l}
-\frac{\partial}{\partial y_{j}}\left(C_{i j k l}(x, y) \frac{\partial S_{k m}(x, y)}{\partial y_{l}}\right)=\left(\rho-\rho^{0}\right) \delta_{i m} \text { in } Y \\
\int_{Y} S_{i m}(x, y) d y=0
\end{array}\right. \tag{52}
\end{align*}
$$

Existence and uniqueness of the cell problems (49)-(57) can be established based on suppositions (S1) and (S2), Lax-Milgram lemma and Korn's Inequalities [Bensoussan, Lions, and Papanicolaou (1978)].
In summary, the multiscale approximate solutions of problem (1) are defined as follows

$$
\begin{align*}
\theta^{1, \varepsilon}(x, t) & =\theta_{0}(x, t)+\varepsilon \theta_{1}(x, y, t)  \tag{54}\\
\mathbf{u}^{1, \varepsilon}(x, t) & =\mathbf{u}^{0}(x, t)+\varepsilon \mathbf{u}^{1}(x, y, t)  \tag{55}\\
\theta^{2, \varepsilon}(x, t) & =\theta_{0}(x, t)+\varepsilon \theta_{1}(x, y, t)+\varepsilon^{2} \theta_{2}(x, y, t)  \tag{56}\\
\mathbf{u}^{2, \varepsilon}(x, t) & =\mathbf{u}^{0}(x, t)+\varepsilon \mathbf{u}^{1}(x, y, t)+\varepsilon^{2} \mathbf{u}^{2}(x, y, t) \tag{57}
\end{align*}
$$

where $\theta_{0}(x, t), \mathbf{u}^{0}(x, t)$ are solutions of homogenized problem (44); $\theta_{1}(x, y, t)$, $\mathbf{u}^{1}(x, y, t)$ and $\theta_{2}(x, y, t), \mathbf{u}^{2}(x, y, t)$ are defined by (27), (28) and (47), (48), respectively. $\theta^{1, \varepsilon}(x, y, t)$ and $\mathbf{u}^{1, \varepsilon}(x, y, t)$ denote first-order multiscale approximate solutions, $\theta^{2, \varepsilon}(x, y, t)$ and $\mathbf{u}^{2, \varepsilon}(x, y, t)$ second-order multiscale approximate solutions. Set
$\mathbf{Z}^{k, \varepsilon}(x, t)=\mathbf{U}^{\varepsilon}(x, t)-\sum_{i=0}^{k} \varepsilon^{i} \mathbf{U}_{i}(x, y, t), \mathrm{k}=1,2$
where $\mathbf{U}_{i}(x, y, t)=\left(\theta_{i}(x, y, t), \mathbf{u}^{i}(x, y, t)^{\prime}, \mathbf{u}_{t}^{i}(x, y, t)^{\prime}\right)^{\prime}$. To compare $\theta^{k, \varepsilon}(x, t)$ and $\mathbf{u}^{k, \varepsilon}(x, t)$ $(s=1,2)$ with the original solutions $\theta^{\varepsilon}(x, t)$ and $\mathbf{u}^{\varepsilon}(x, t)$, taking $\mathbf{Z}^{1, \varepsilon}(x, t)$ into (3) and according to (8), (17), (19)-(21), we have

$$
\begin{align*}
\mathrm{P}^{\varepsilon} \mathrm{Z}^{1, \varepsilon} & =\mathrm{P}^{\varepsilon} \mathrm{U}^{\varepsilon}-\left(\varepsilon^{-2} \mathrm{P}_{-0}+\varepsilon^{-1} \mathrm{P}_{-1}+\mathrm{P}_{-2}\right)\left(\mathrm{U}_{0}+\varepsilon \mathrm{U}_{1}\right) \\
& =\mathrm{F}^{\varepsilon}-\varepsilon^{-2} \mathrm{P}_{-0} \mathrm{U}_{0}-\varepsilon^{-1}\left(\mathrm{P}_{-0} \mathrm{U}_{1}+\mathrm{P}_{-1} \mathrm{U}_{0}\right)-\left(\mathrm{P}_{-1} \mathrm{U}_{1}+\mathrm{P}_{-2} \mathrm{U}_{0}\right)-\varepsilon \mathrm{P}_{-2} \mathrm{U}_{1}  \tag{59}\\
& =\mathrm{F}^{\varepsilon}-\mathrm{P}_{-1} \mathrm{U}_{1}-\mathrm{P}_{-2} \mathrm{U}_{0}-\varepsilon \mathrm{P}_{-2} \mathrm{U}_{1}
\end{align*}
$$

Then taking $\mathbf{Z}^{2, \varepsilon}(x, t)$ into (3) and according to (8), (17), (19) - (21), we have

$$
\begin{align*}
\mathrm{P}^{\varepsilon} \mathrm{Z}^{2, \varepsilon}= & \mathrm{P}^{\varepsilon} \mathrm{U}^{\varepsilon}-\left(\varepsilon^{-2} \mathrm{P}_{-0}+\varepsilon^{-1} \mathrm{P}_{-1}+\mathrm{P}_{-2}\right)\left(\mathrm{U}_{0}+\varepsilon \mathrm{U}_{1}+\varepsilon^{2} \mathrm{U}_{2}\right) \\
= & \mathrm{F}^{\varepsilon}-\varepsilon^{-2} \mathrm{P}_{-0} \mathrm{U}_{0}-\varepsilon^{-1}\left(\mathrm{P}_{-0} \mathrm{U}_{1}+\mathrm{P}_{-1} \mathrm{U}_{0}\right)-\left(\mathrm{P}_{-0} \mathrm{U}_{2}+\mathrm{P}_{-1} \mathrm{U}_{1}+\mathrm{P}_{-2} \mathrm{U}_{0}\right)  \tag{60}\\
& -\varepsilon\left(\mathrm{P}_{-2} \mathrm{U}_{1}+\mathrm{P}_{-1} \mathrm{U}_{2}\right)-\varepsilon^{2} \mathrm{P}_{-2} \mathrm{U}_{2} \\
= & -\varepsilon\left(\mathrm{P}_{-2} \mathrm{U}_{1}+\mathrm{P}_{-1} \mathrm{U}_{2}-\varepsilon \mathrm{P}_{-2} \mathrm{U}_{2}\right)
\end{align*}
$$

Remark 2.1 According to above detailed mathematical, it can be noted from (63) that the residual between the first-order multiscale approximate solutions and the solutions of original problem (1) is of order $O(1)$ that does not equal to 0 . In the practical engineering computation, it cannot be omitted for a constant $\varepsilon$, so engineers conclude that the firstorder multiscale approximate solutions cannot be acceptted and the microscale fluctuation of the temperature and displacement are far from being captured. This is the reason why it is necessary to seek the higher order expansions. It can be concluded from (64) that the second-order multiscale solutions are equivalent to the solutions of original problem (1) with order $O(\varepsilon)$ in nearly pointwise sense. Moreover, the numerical results presented in Section 4 clearly show that it is important to include the second-order corrector terms.
Summing up, one obtains following results
Theorem 2.1 The temperature and displacement fields for the transient thermomechanical coupling problem (1) of quasi-periodic composite materials have the multiscale asymptotic expansions as follows

$$
\begin{align*}
\theta^{\varepsilon}(x, t)= & \theta_{0}(x, t)+\varepsilon H_{\alpha_{1}}(x, y) \frac{\partial \theta_{0}(x, t)}{\partial x_{\alpha_{1}}}+\varepsilon^{2} H_{\alpha_{1} \alpha_{2}}(x, y) \frac{\partial^{2} \theta_{0}(x, t)}{\partial x_{\alpha_{1}} \partial x_{\alpha_{2}}} \\
& +\varepsilon^{2} F_{\alpha_{1}}(x, y) \frac{\partial \theta_{0}(x, t)}{\partial x_{\alpha_{1}}}-\varepsilon^{2} G_{\alpha_{1} \alpha_{2}}(x, y) \frac{\partial^{2} u_{\alpha_{1}}^{0}(x, t)}{\partial x_{\alpha_{2}} \partial t}  \tag{61}\\
& -\varepsilon^{2} R(x, y) \frac{\partial \theta_{0}(x, t)}{\partial t}+\varepsilon^{3} Z_{1}(\varepsilon, x, y, t) \\
\mathbf{u}^{\varepsilon}(x, t)= & \mathbf{u}^{0}(x, t)+\varepsilon \mathbf{N}_{\alpha_{1}}(x, y) \frac{\partial \mathbf{u}^{0}(x, t)}{\partial x_{\alpha_{1}}}-\varepsilon \mathbf{M}(x, y) \theta_{0}(x, t) \\
& +\varepsilon^{2} \mathbf{N}_{\alpha_{1} \alpha_{2}}(x, y) \frac{\partial^{2} \mathbf{u}^{0}(x, t)}{\partial x_{\alpha_{1}} \partial x_{\alpha_{2}}}+\varepsilon^{2} \mathbf{P}_{\alpha_{1}}(x, y) \frac{\partial \mathbf{u}^{0}(x, t)}{\partial x_{\alpha_{1}}}  \tag{62}\\
& -\varepsilon^{2} \mathbf{Q}(x, y) \theta_{0}(x, t)-\varepsilon^{2} \mathbf{M}_{\alpha_{1}}(x, y) \frac{\partial \theta_{0}(x, t)}{\partial x_{\alpha_{1}}} \\
& -\varepsilon^{2} \mathbf{S}(x, y) \frac{\partial^{2} \mathbf{u}^{0}(x, t)}{\partial t^{2}}+\varepsilon^{3} \mathbf{Z}_{2}(\varepsilon, x, y, t)
\end{align*}
$$

where $\theta_{0}(x, t)$ and $\mathbf{u}^{0}(x, t)$ are the solutions of the homogenized problem (44), $H_{\alpha_{1}}(x, y)$, $\mathbf{N}_{\alpha_{1}}(x, y)$ and $\mathbf{M}(x, y)$ are first-order auxiliary functions defined by (31)-(33),
$H_{\alpha_{1} \alpha_{2}}(x, y), F_{\alpha_{1}}(x, y), G_{\alpha_{1} \alpha_{2}}(x, y), R(x, y), \quad \mathbf{N}_{\alpha_{1} \alpha_{2}}(x, y), \mathbf{P}_{\alpha_{1}}(x, y), \mathbf{S}(x, y), \mathbf{Q}(x, y)$ and $\mathbf{M}_{\alpha_{1}}(x, y)$ are second-order auxiliary functions defined by (49)-(57), respectively, $Z_{1}(\varepsilon, x, y, t)$ and $\mathbf{Z}_{2}(\varepsilon, x, y, t)$ are the asymptotic expansion functions depending on the two-scale variables $x$ and $y$.

And then the temperature gradient, strains and stresses can be evaluated based on the chain rule (10) and the multiscale asymptotic expansions of temperature and displacement fields (65) and (66).

## 3 Numerical implementation of multiscale method

In this section, the multiscale algorithms based on the finite difference method in time direction and finite element method in spatial region for predicting the transient thermomechanical coupling behaviors of quasi-periodic composite materials is presented.
Note that all the cell problems (31) - (33) and (49) - (57) are associated with macroscopic coordinates $x$, which brings lots of complexities in numerical computation since we have to solve these cell problems at every point $x \in \Omega$. In practical applications, such as the damage analysis of composite materials, engineers often take the single cell as a unit to evaluate the damage degree of composite materials, which leads to a scale separation of material coefficients. The specific meaning of scale separation is written as follows
$a_{i j}(x, y)=\omega(x) \tilde{a}_{i j}(y)$
Thus, according to (31), the cell function $H_{\alpha_{1}}(x, y)$ becomes
$H_{\alpha_{l}}(x, y)=\tilde{H}_{a_{l}}(y)$
and it is a solution of following problem
$\left\{\begin{array}{l}-\frac{\partial}{\partial y_{i}}\left(\tilde{a}_{i j}(y) \frac{\partial \tilde{H}_{\alpha_{1}}(y)}{\partial y_{j}}\right)=\frac{\partial \tilde{a}_{i \alpha_{1}}}{\partial y_{i}} \text { in } Y \\ \int_{Y} \tilde{H}_{\alpha_{1}}(y) d y=0\end{array}\right.$
Besides, the homogenized coefficient can be rewritten as
$a_{i j}^{0}(x)=\omega(x) \tilde{a}_{i j}^{0}$
where
$\tilde{a}_{i j}^{0}=\frac{1}{|Y|} \int_{Y}\left(\tilde{a}_{i j}(y)+\tilde{a}_{i k}(y) \frac{\partial \tilde{H}_{j}(y)}{\partial y_{k}}\right) d y$
Other cell functions and the multiscale solutions also become simpler. Therefore, based on variable separations, all the cell functions are independent of $x$, which reduces the
complexity of computations effectively. And then we can give following multiscale algorithm based on the finite element and finite difference method.

### 3.1 Multiscale numerical formulations

### 3.1.1 Finite element computation of cell functions and homogenized parameters

It is easy to see that cell problems (31) - (33) and (49) - (57) are all elliptic equations and the finite element solutions of cell functions $H_{\alpha_{1}}^{h_{1}}$ can be obtained by solving the following FE virtual work equation on unit cell $Y$

$$
\begin{equation*}
\int_{Y} a_{i j} \frac{\partial H_{\alpha_{1}}^{h_{1}}}{\partial y_{j}} \frac{\partial v}{\partial y_{i}} d y=-\int_{Y} a_{i \alpha_{1}} \frac{\partial v}{\partial y_{i}} d y, \quad \forall v \in V^{h_{1}}(Y) \subseteq H_{0}^{1}(Y) \tag{68}
\end{equation*}
$$

And the finite element approximation of homogenized parameter $a_{i j}^{0}$ can be calculated as follows

$$
\begin{equation*}
a_{i j}^{0 h_{1}}=\frac{1}{|Y|} \int_{Y}\left(a_{i j}+a_{i k} \frac{\partial H_{\alpha_{1}}^{h_{1}}}{\partial y_{k}}\right) d y \tag{69}
\end{equation*}
$$

where $h_{1}$ is the mesh size and $V^{h_{1}}(Y)$ denotes the finite element space, similarly to (72) and (73), (32) - (33), (39) - (43) and (49) - (57) can be solved, successively.

### 3.1.2 Finite element-difference computation of homogenized equation

The homogenized equations (44) are dynamic problem coupled by hyperbolic and parabolic equations. Thus, the spatial region $\Omega$ is divided by using the finite element mesh first, and then the temporal domain $(0, T)$ is divided by using the finite difference. The semi-discrete scheme for solving homogenized equations is given as follows

$$
\begin{align*}
& \int_{\Omega} b^{0 h_{1}} \frac{\partial \theta_{0}^{h_{2}}}{\partial t} v^{h_{2}} d x+\int_{\Omega} T_{0} d_{i j}^{0 h_{1}} \frac{\partial^{2} u_{0 i}^{h_{2}}}{\partial t \partial x_{j}} v^{h_{2}} d x+\int_{\Omega} a_{i j}^{0 h_{1}} \frac{\partial \theta_{0}^{h_{2}}}{\partial x_{j}} \frac{\partial v^{h_{2}}}{\partial x_{i}} d x  \tag{70}\\
& =\int_{\Omega} h v^{h_{2}} d x, \quad \forall v^{h_{2}} \in V^{h_{2}}(\Omega) \\
& \int_{\Omega} \rho^{0 h_{1}} \frac{\partial^{2} u_{0 i}^{h_{2}}}{\partial t^{2}} v_{i}^{h_{2}} d x+\int_{\Omega}\left(C_{i j k l}^{0 h_{1}} \frac{\partial u_{0 k}^{h_{2}}}{\partial x_{l}}-\beta_{i j}^{0 h_{1}} \theta_{0}^{h_{2}}\right) \frac{\partial v_{i}^{h_{2}}}{\partial x_{j}} d x  \tag{71}\\
& =\int_{\Omega} f_{i} v_{i}^{h_{2}} d x, \quad \forall \mathbf{v}^{h_{2}} \in \mathbf{V}^{h_{2}}(\Omega)
\end{align*}
$$

where $V^{h_{2}}(\Omega)$ and $\mathbf{V}^{h_{2}}(\Omega)$ denotes the finite element spaces with mesh size $h_{2}$ on $\Omega$. Then for any fixed $t \in(0, T)$ the discrete variational equalities (74) and (75) are equivalent to the following coupled ordinary differential equation systems

$$
\left\{\begin{array}{l}
\tilde{M} \dot{\mathbf{T}}(t)+\tilde{K} \mathbf{T}(t)+\tilde{C} \dot{\mathbf{U}}(t)=\tilde{H}  \tag{72}\\
\breve{M} \ddot{\mathbf{U}}(t)+\tilde{K} \mathbf{U}(t)-\check{C} \mathbf{T}(t)=\breve{F} \\
\mathbf{U}(0)=\mathbf{U}^{0}, \dot{\mathbf{U}}(0)=\mathbf{U}^{1}, \mathbf{T}(0)=\tilde{\mathbf{T}}
\end{array}\right.
$$

where $\mathbf{T}(t)$ and $\mathbf{U}(t)$ are the vector of nodal temperature increment field and displacement field; $\mathbf{U}^{0}, \mathbf{U}^{1}$ and $\tilde{\mathbf{T}}$ are the initial displacement velocity and temperature increment field, respectively; and

$$
\begin{align*}
\tilde{M} & =\sum_{e} \tilde{M}^{e}, \tilde{\mathrm{C}}=\sum_{e} \tilde{\mathrm{C}}^{e}, \tilde{K}=\sum_{e} \tilde{K}^{e}, \tilde{H}=\sum_{e} \tilde{H}^{e} \\
\breve{M} & =\sum_{e} \breve{M}^{e}, \breve{K}=\sum_{e} \breve{K}^{e}, \breve{C}=\sum_{e} \breve{C}^{e}, \breve{F}=\sum_{e} \breve{F}^{e} \tag{73}
\end{align*}
$$

They are expressed as the following forms

$$
\left\{\begin{array}{l}
\tilde{M}^{e}=\int_{\Omega^{e}} \mathbf{b}^{0 h_{1}} \mathbf{N}_{T}^{\prime} \mathbf{N}_{T} d x, \quad \tilde{C}^{e}=\int_{\Omega^{e}} T_{0} \mathbf{N}_{T}^{\prime}\left(\boldsymbol{\beta}^{0 h_{1}}\right)^{\prime} \mathbf{B}_{U} d x \\
\tilde{K}^{e}=\int_{\Omega^{e}} \mathbf{B}_{T}^{\prime} \mathbf{a}^{0 h_{h}} \mathbf{B}_{T} d x, \quad \tilde{H}^{e}=\int_{\Omega^{e}} \mathbf{N}_{T}^{\prime} h d x  \tag{74}\\
\breve{M}^{e}=\int_{\Omega^{e}} \mathbf{o}^{0 h_{1}} \mathbf{N}_{U}^{\prime} \mathbf{N}_{U} d x, \quad \breve{K}^{e}=\int_{\Omega^{e}} \mathbf{B}_{U}^{\prime} \mathbf{C}^{0 h_{1}} \mathbf{B}_{U} d x \\
\breve{C}^{e}=\int_{\Omega^{e}} \mathbf{B}_{U}^{\prime} \boldsymbol{\beta}^{0 h_{1}} \mathbf{N}_{T} d x, \quad \breve{F}^{e}=\int_{\Omega^{e}} \mathbf{N}_{U}^{\prime} \mathbf{f} d x
\end{array}\right.
$$

where $\mathbf{N}_{T}$ and $\mathbf{N}_{U}$ are the shape function matrix; $\mathbf{B}_{T}$ and $\mathbf{B}_{U}$ are the matrix of symmetric gradient of $\mathbf{N}_{T}$ and $\mathbf{N}_{U}$, respectively; $\mathbf{a}^{0 h_{1}}$ is the element thermal conductivity matrix; $\boldsymbol{\beta}^{0 h_{1}}$ is the element thermal modulus matrix; $\mathbf{C}^{0 h_{1}}$ is the element stiffness matrix.

And then, the first equation of (76) is integrated in time using the backward difference scheme [Leoni (2009)], and the second one is solved using the Newmark difference scheme [Leoni (2009)]. Then the above coupled system (76) can be rewritten as follows
$\left\{\begin{array}{l}(\tilde{M}+\Delta t \tilde{K}) \mathbf{T}^{n+1}=\tilde{M} \mathbf{T}^{n}-\tilde{C}\left(\mathbf{U}^{n+1}-\mathbf{U}^{n}\right)+\Delta t \tilde{H}^{n+1} \\ \left(\breve{M}+\alpha \Delta t^{2} \breve{K}\right) \mathbf{U}^{n+1}-\alpha \Delta t^{2} \breve{C} \mathbf{T}^{n+1}=\breve{M} \mathbf{U}^{n}+\Delta t \breve{M}^{n} \\ +\alpha \Delta t^{2} \breve{M}\left(\frac{1}{2 \alpha}-1\right) \ddot{\mathbf{U}}^{n}+\alpha \Delta t^{2} \breve{F}^{n+1}\end{array}\right.$
where $\Delta t$ is the time step for one iteration and the temporal domain is divided by;
$t_{n}=(n-1) \Delta t,(n=1,2, \cdots, l), 0=t_{1}<t_{2}<\cdots<t_{l}=T$
Set
$\mathbf{U}^{n+1}=\mathbf{U}^{n}+\varpi\left(\mathbf{U}^{n}-\mathbf{U}^{n-1}\right)$
and $\varpi$ is a correction coefficient. Then the coupled system is decomposed into the following two sub-systems

$$
\begin{align*}
& \left\{\begin{array}{l}
(\tilde{M}+\Delta t \tilde{K}) \mathbf{T}^{n+1}=\tilde{M} \mathbf{T}^{n}-\tilde{C} \varpi\left(\mathbf{U}^{n}-\mathbf{U}^{n-1}\right)+\Delta t \tilde{H}^{n+1} \\
\mathbf{T}(0)=\tilde{\mathbf{T}}
\end{array}\right.  \tag{78}\\
& \left\{\begin{array}{l}
\left(\breve{M}+\alpha \Delta t^{2} \breve{K}\right) \mathbf{U}^{n+1}=\breve{M} \mathbf{U}^{n}+\Delta t \breve{M} \breve{\mathbf{U}}^{n} \\
+\alpha \Delta t^{2} \breve{M}\left(\frac{1}{2 \alpha}-1\right) \ddot{\mathbf{U}}^{n}+\alpha \Delta t^{2} \breve{F}^{n+1}+\alpha \Delta t^{2} \breve{C} \mathbf{T}^{n+1} \\
\mathbf{U}(0)=\mathbf{U}^{0}, \dot{\mathbf{U}}(0)=\mathbf{U}^{1}
\end{array}\right. \tag{79}
\end{align*}
$$

Finally, we can obtain the temperature and displacement fields at any time step through (82) and (83). It is well known that the Newmark scheme is unconditionally stable when $\delta \geq 0.5, \alpha \geq 0.25(0.5+\delta)^{2}$ and we choose $\delta=0.5, \alpha=0.25$ in this paper.

### 3.1.3 Multiscale numerical solutions

According to (58) - (61), the multiscale approximation solutions based on global structure $\Omega$ can be evaluated by

$$
\begin{align*}
\theta_{h_{1}, h_{2}}^{1, \varepsilon}(x, t)= & \theta_{0}^{h_{2}}(x, t)+\varepsilon H_{\alpha_{1}}^{h_{1}}(x, y) \frac{\partial \theta_{0}^{h_{2}}(x, t)}{\partial x_{\alpha_{1}}} \\
\theta_{h_{1}, h_{2}}^{2, \varepsilon}(x, t)= & \theta_{0}^{h_{2}}(x, t)+\varepsilon H_{\alpha_{1}}^{h_{1}}(x, y) \frac{\partial \theta_{0}^{h_{2}}(x, t)}{\partial x_{\alpha_{1}}}+\varepsilon^{2} H_{\alpha_{1} \alpha_{2}}^{h_{1}}(x, y) \frac{\partial^{2} \theta_{0}^{h_{2}}(x, t)}{\partial x_{\alpha_{1}} \partial x_{\alpha_{2}}} \\
& +\varepsilon^{2} F_{\alpha_{1}}^{h_{1}}(x, y) \frac{\partial \theta_{0}^{h_{2}}(x, t)}{\partial x_{\alpha_{1}}}-\varepsilon^{2} G_{\alpha_{1} \alpha_{2}}^{h_{1}}(x, y) \frac{\partial^{2} u_{\alpha_{1}}^{0 h_{2}}(x, t)}{\partial x_{\alpha_{2}} \partial t}-\varepsilon^{2} R^{h_{1}}(x, y) \frac{\partial \theta_{0}^{h_{2}}(x, t)}{\partial t} \\
\mathbf{u}_{h_{1}, h_{2}}^{1, \varepsilon}(x, t)= & \mathbf{u}^{0 h_{2}}(x, t)+\varepsilon \mathbf{N}_{\alpha_{1}}^{h_{1}}(x, y) \frac{\partial \mathbf{u}^{0 h_{2}}(x, t)}{\partial x_{\alpha_{1}}}-\varepsilon \mathbf{M}^{h_{1}}(x, y) \theta_{0}^{h_{2}}(x, t)  \tag{80}\\
\mathbf{u}_{h_{1}, h_{2}}^{2, \varepsilon}(x, t)= & \mathbf{u}^{0 h_{2}}(x, t)+\varepsilon \mathbf{N}_{\alpha_{1}}^{h_{1}}(x, y) \frac{\partial \mathbf{u}^{0 h_{2}}(x, t)}{\partial x_{\alpha_{1}}}-\varepsilon \mathbf{M}^{h_{1}}(x, y) \theta_{0}^{h_{2}}(x, t) \\
& +\varepsilon^{2} \mathbf{N}_{\alpha_{1} \alpha_{2}}^{h_{1}}(x, y) \frac{\partial^{2} \mathbf{u}^{0 h_{2}}(x, t)}{\partial x_{\alpha_{1}} \partial x_{\alpha_{2}}}+\varepsilon^{2} \mathbf{P}_{\alpha_{1}}^{h_{1}}(x, y) \frac{\partial \mathbf{u}^{0 h_{2}}(x, t)}{\partial x_{\alpha_{1}}} \\
& -\varepsilon^{2} \mathbf{Q}^{h_{1}}(x, y) \theta_{0}^{h_{2}}(x, t)-\varepsilon^{2} \mathbf{M}_{\alpha_{1}}^{h_{1}}(x, y) \frac{\partial \theta_{0}^{h_{2}}(x, t)}{\partial x_{\alpha_{1}}}-\varepsilon^{2} \mathbf{S}^{h_{1}}(x, y) \frac{\partial^{2} \mathbf{u}^{0 h_{2}}(x, t)}{\partial t^{2}}
\end{align*}
$$

### 3.2 Algorithm procedures for the multiscale method

The algorithm procedure for the multiscale method to predict the transient thermomechanical coupling performance is stated as follows

1) Determine the geometrical constructions of the macroscopic structure $\Omega$ and cell domain $Y$, and verify the material parameters of various constituents.
2) Solve the cell problems (31)-(33) to get the finite element solutions of $H_{\alpha_{1}}(x, y)$, $\mathbf{N}_{\alpha_{1}}(x, y)$ and $\mathbf{M}(x, y)$, respectively. Furthermore, the homogenized parameters $a_{i j}^{0}(x)$, $C_{i j k l}^{0}(x), \beta_{i j}^{0}(x), d_{i j}^{0}(x), b^{0}(x)$ and $\rho^{0}(x)$ are evaluated by formulas (38)-(43).
3) With the homogenized parameters obtained in previous step, compute the homogenized solutions $\theta_{0}(x, t)$ and $\mathbf{u}^{0}(x, t)$ by solving the homogenized problem (44).
4) Solve problems (49)-(57) by using the same FE meshes as in step 2 to get the FE solutions of $H_{\alpha_{1} \alpha_{2}}(x, y), F_{\alpha_{1}}(x, y), G_{\alpha_{1} \alpha_{2}}(x, y), R(x, y), \quad \mathbf{N}_{\alpha_{1} \alpha_{2}}(x, y), \mathbf{P}_{\alpha_{1}}(x, y), \mathbf{S}(x, y)$, $\mathbf{Q}(x, y)$ and $\mathbf{M}_{\alpha_{1}}(x, y)$, respectively.
5) Solve the derivatives of the homogenized solutions $\theta_{0}(x, t)$ and $\mathbf{u}^{0}(x, t)$ with respect to spatial and temporal variables. The derivatives with respect to spatial variable are evaluated by the average technique on relative elements [Thomas (2013)] and the derivatives with respect to temporal variable are evaluated using the difference schemes in step 3.
6) Compute the temperature and displacement fields using formulas (65) and (66), respectively.

## 4 Numerical examples

To illustrate to the effectiveness of the multiscale method for studying the transient thermo-mechanical coupling problem of quasi-periodic composite materials, some numerical results are given here. The macrostructure $\Omega$ and unit cell $Y=[0,1]^{2}$ are shown in the Fig. 2. We consider that $\varepsilon=1 / 8$. The cell is composed of two kinds of materials $Y_{1}$ and $Y_{2}$, each of which is homogeneous and isotropic. And the material properties are listed in Table 1. Besides, in problem (1), let $T_{0}=500, T=0.1$, $h(x, t)=10000, \mathbf{f}(x, t)=(0,-10), \theta^{0}(x)=0, \mathbf{u}_{0}(x)=0$ and $\mathbf{u}_{1}(x)=0$. The macrostructure $\Omega$ is clamped on its boundaries and the time step is chosen as $\Delta t=0.001$.


Figure 2: (a) Unit cell $Y=[0,1]^{2}$; (b) domain $\Omega$

Since it is difficult to find the exact solutions of above problem, we have to take $\theta^{\varepsilon}(x, t)$ and $\mathbf{u}^{\varepsilon}(x, t)$ to be their finite element (FE) solutions $\theta_{F E}^{\varepsilon}(x, t)$ and $\mathbf{u}_{F E}^{\varepsilon}(x, t)$ in the very fine mesh for comparison with different order approximate solutions. The triangulation partition is implemented, and the information of the FE meshes is listed in Table 2. Set
$e_{\theta}^{0}=\theta_{F E}^{\varepsilon}(x, t)-\theta_{F E}^{0}(x, t), e_{\theta}^{1}=\theta_{F E}^{\varepsilon}(x, t)-\theta_{F E}^{1, \varepsilon}(x, t), e_{\theta}^{2}=\theta_{F E}^{\varepsilon}(x, t)-\theta_{F E}^{2, \varepsilon}(x, t)$
$\mathbf{z}_{\mathbf{u}}^{0}=\mathbf{u}_{F E}^{\varepsilon}(x, t)-\mathbf{u}_{F E}^{0}(x, t), \mathbf{z}_{\mathbf{u}}^{1}=\mathbf{u}_{F E}^{\varepsilon}(x, t)-\mathbf{u}_{F E}^{1, \varepsilon}(x, t), \mathbf{z}_{\mathbf{u}}^{2}=\mathbf{u}_{F E}^{\varepsilon}(x, t)-\mathbf{u}_{F E}^{2, \varepsilon}(x, t)$
where $\theta_{F E}^{0}(x, t)$ and $\mathbf{u}_{F E}^{0}(x, t)$ denote the FE numerical solutions of the homogenized problem (44); $\theta_{F E}^{1, \varepsilon}(x, t)$ and $\mathbf{u}_{F E}^{1, \varepsilon}(x, t)$ denote the first-order multiscale numerical solutions, and $\theta_{F E}^{2, \varepsilon}(x, t)$ and $\mathbf{u}_{F E}^{2, \varepsilon}(x, t)$ denote the second-order multiscale numerical solutions based on (84). And the norm $\|\cdot\|_{L^{\infty}\left(0, T ; L^{2}(\Omega)\right)}$ is denoted by $\|\cdot\|_{L^{\infty}\left(L^{2}\right)}$ for simplicity.

Table 1: Material properties

|  | Material $1\left(Y_{1}\right)$ | Material 2 $\left(Y_{2}\right)$ |
| :--- | :--- | :--- |
| $\tilde{a}_{i j}$ | $1 \delta_{i j}$ | $0.01 \delta_{i j}$ |
| $\tilde{C}_{i j k l}$ | $6.25 \times 10^{5} \delta_{i j} \delta_{k l}$ | $7.14 \times 10^{5} \delta_{i j} \delta_{k l}$ |
|  | $+1.25 \times 10^{6}\left(\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right)$ | $+5.36 \times 10^{5}\left(\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right)$ |
| $\tilde{\beta}_{i j}$ | $500 \delta_{i j}$ | $10 \delta_{i j}$ |
| $\tilde{b}$ | 10 | 0.1 |
| $\tilde{\rho}$ | 10 | 1 |

For convenience, we introduce the following notation
$|v|_{H^{1}(\Omega)}=\left(\int_{\Omega}(\nabla v)^{2} d x\right)^{1 / 2}$
And we consider two cases
Case 1: $\omega(x)=5+\sin \left(4 \pi x_{1}\right)+\sin \left(4 \pi x_{2}\right)$
Case 2: $\omega(x)=10+x_{1}\left(1-x_{1}\right)+x_{2}\left(1-x_{2}\right)$
According to the previous multiscale algorithms, the corresponding homogenized coefficients can be computed as $\tilde{a}_{i j}^{0}=0.68 \delta_{i j}, \tilde{\beta}_{i j}^{0}=393.94 \delta_{i j}, \tilde{b}^{0}=15.22, \tilde{\rho}^{0}=8.38$ and $\tilde{C}_{i j k l}^{0}=6.77 \times 10^{5} \delta_{i j} \delta_{k l}+1.07 \times 10^{6}\left(\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right)$.

The relative numerical errors of the homogenization, first-order multiscale, and secondorder multiscale methods in $L^{\infty}\left(L^{2}\right)$ - and $L^{\infty}\left(H^{1}\right)$-norm for two cases are listed in Tables 3-6. Fig. 3 and Fig. 4 illustrate the numerical results for the temperature increment
gradient and strain distribution of Case 1 at time $t=0.1$, including the homogenization, first-order multiscale, second-order multiscale solutions and the FE solutions in very fine mesh. Fig. 5 and Fig. 6 display previous four numerical results for the temperature, displacement, temperature gradient and strain along the line of $x_{1}=x_{2}$ for Case 1 at time $t=0.05$, respectively. Fig. 7 clearly shows the evolution of the relative errors of different approximate solutions with time $t$ for Case 1 .

Table 2: Comparison of computational cost

|  | Multiscale FE computation |  | Classical FE computation <br> with refined meshes |
| :--- | :--- | :--- | :--- |
|  | Unit cell | Homogenized equation | 291456 |
| Elements number | 5566 | 20000 | 146449 |
| Nodes number | 2884 | 10201 | 1928.7 s |
| Running time | 16.2 s | 432.8 s |  |

Table 3: Comparison of computing results for temperature increment in $L^{2}$-norm

|  | $\frac{\left\\|e_{\theta}^{0}\right\\|_{L^{\infty}\left(L^{2}\right)}}{}$ | $\frac{\left\\|e_{\theta}^{1}\right\\|_{L^{\infty}\left(L^{2}\right)}}{}$ | $\frac{\left\\|e_{\theta}^{2}\right\\|_{L^{\infty}\left(L^{2}\right)}}{\left\\|\theta_{F E}^{\varepsilon}\right\\|_{L^{\infty^{\circ}}\left(L^{2}\right)}}$ |
| :--- | :--- | :--- | :--- |
|  | $\left\\|\theta_{F E}^{\varepsilon}\right\\|_{L^{\infty}\left(L^{2}\right)}$ | $\left\\|\theta_{F E}^{\varepsilon}\right\\|_{L^{\infty}\left(L^{2}\right)}$ |  |
| Case 1 | 0.726833 | 0.727374 | 0.009774 |
| Case 2 | 0.720257 | 0.720645 | 0.014425 |

Table 4: Comparison of computing results for temperature increment in $H^{1}$-norm

|  | $\left\\|e_{\theta}^{0}\right\\|_{L^{\circ}\left(H^{\prime}\right)}$ | $\left\\|e_{\theta}^{1}\right\\|_{L^{\circ}\left(H^{1}\right)}$ | $\left\\|e_{\theta}^{2}\right\\|_{L^{\circ}\left(H^{1}\right)}$ |
| :---: | :---: | :---: | :---: |
|  | $\widehat{\left\\|\theta_{F E}^{e}\right\\|_{L^{*}\left(H^{1}\right)}}$ | $\overline{\left\\|\theta_{F E}^{e}\right\\|_{L^{\infty}\left(H^{\text {a }}\right)}}$ | $\widehat{\left\\|\theta_{F E}^{e}\right\\|_{L^{ \pm}\left(H^{\prime}\right)}}$ |
| Case 1 | 0.997141 | 0.994945 | 0.125979 |
| Case 2 | 0.997349 | 0.995377 | 0.180538 |

Table 5: Comparison of computing results for displacement in $L^{2}$-norm

|  | $\frac{\left.\left\\|\mathbf{z}_{\mathbf{u}}^{0}\right\\|_{L^{\infty}} L^{2}\right)}{}$ |  | $\left\\|\mathbf{z}_{u}^{1}\right\\|_{L^{\infty}\left(L^{2}\right)}$ |  |
| :--- | :--- | :--- | :--- | :--- |
|  | $\left\\|\mathbf{u}_{F E}^{\varepsilon}\right\\|_{L^{\infty}\left(L^{2}\right)}$ | $\left\\|\mathbf{z}_{\mathbf{u}}^{2}\right\\|_{L^{\infty}\left(L^{2}\right)}$ |  |  |
|  | $\left\\|\mathbf{u}_{F E}^{\varepsilon}\right\\|_{L^{\infty}\left(L^{2}\right)}$ | $\left\\|\mathbf{u}_{F E}^{\varepsilon}\right\\|_{L^{\infty}\left(L^{2}\right)}$ |  |  |
| Case 1 | 0.143016 | 0.039820 | 0.036475 |  |
| Case 2 | 0.160566 | 0.042948 | 0.037055 |  |

Table 6: Comparison of computing results for displacement in $H^{1}$-norm

|  | $\frac{\left\\|\mathbf{z}_{\mathbf{u}}^{0}\right\\|_{L^{\infty}\left(H^{\prime}\right)}}{\left\\|\mathbf{u}_{F E}^{\varepsilon}\right\\|_{L^{\infty}\left(H^{1}\right)}}$ | $\frac{\left\\|\mathbf{z}_{\mathbf{u}}^{1}\right\\|_{L^{\infty}\left(H^{1}\right)}}{\left\\|\mathbf{u}_{F E}^{\varepsilon}\right\\|_{L^{\infty}\left(H^{1}\right)}}$ | $\frac{\left\\|\mathbf{z}_{\mathbf{u}}^{2}\right\\|_{L^{\infty}\left(H^{1}\right)}}{\left\\|\mathbf{u}_{F E}^{\varepsilon}\right\\|_{L^{\infty}\left(H^{\prime}\right)}}$ |
| :--- | :--- | :--- | :--- |
| Case 1 | 0.788751 | 0.206704 | 0.200866 |
| Case 2 | 0.804421 | 0.209827 | 0.202001 |

From Table 2, we can see that the mesh partition numbers of second-order multiscale approximate solutions are much less than that of refined FE solutions. Both the secondorder multiscale method and the direct FE numerical computations are performed on the same computer. And the approximate running times for the multiscale algorithm and classical FE computation with refined mesh are 471.2 seconds and 1928.7 seconds, respectively. We cannot easily solve the problem (1) directly by the classical numerical methods because it would require very fine meshes and the convergence of the FE method based on fine meshes for the nonlinear coupled problem is not very easy. Moreover, the proposed second-order multiscale method is suitable for the composite materials with a great number of cells, which can greatly save computer memory and CPU time without losing precision. And it is very important in engineering computations. From Fig.3-Fig. 6 and Tables 3-6, it can be found that the newly second-order multiscale approximate solutions are in good agreement with the FE solutions in a refined mesh. But the homogenized solutions and first-order multiscale solutions have less effect approaching the refined-mesh FE solutions. The homogenized solutions give the original problem an asymptotic behavior, which is not enough for $\varepsilon$ that is not so small. So, the correctors are necessary, and the results show that the second-order correctors give much better approximation of the displacement, strain, temperature increment and its gradient. Furthermore, numerical results also show that only second-order multiscale solutions can accurately capture the microscale oscillating information of the multiscale problem. Besides, the relative errors between different approximate solutions and FE solutions obtained on refined mesh are also exhibited in Figure 7. It is worth to note that the relative errors are not growing significantly as time increases. This indicates that the multiscale method is a very good method for treating a long-time problem in some cases. Consequently, all the results demonstrate that the multiscale method is effective and efficient to predict the transient thermo-mechanical coupling behaviors of quasi-periodic composite materials.


Figure 3: Temperature increment gradient at $t=0.1$ for Case 1: (a) $\frac{\partial \theta_{F E}^{\varepsilon}}{\partial x_{1}}$; (b) $\frac{\partial \theta_{F E}^{0}}{\partial x_{1}}$; (c) $\frac{\partial \theta_{F E}^{1, \varepsilon}}{\partial x_{1}} ;$ (d) $\frac{\partial \theta_{F F}^{2, \varepsilon}}{\partial x_{1}}$

(a)

(c)

(b)

(d)

Figure 4: Strain at $t=0.1$ for Case 1: (a) $\frac{\partial u_{1, F E}^{\varepsilon}}{\partial x_{1}}$; (b) $\frac{\partial u_{1, F E}^{0}}{\partial x_{1}}$; (c) $\frac{\partial u_{1, F E}^{1, \varepsilon}}{\partial x_{1}}$; (d) $\frac{\partial u_{1, F E}^{2, \varepsilon}}{\partial x_{1}}$


Figure 5: Comparison of different solutions on the line $x_{1}=x_{2}$ at $t=0.05$ for Case 1: (a) temperature increment; (b) displacement

(b)

Figure 6: Comparison of different solutions on the line $x_{1}=x_{2}$ at $t=0.05$ for Case 1: (a) temperature increment gradient; (b) strain


Figure 7: The evolution of relative errors with $t$ of Case 1 for (a) temperature increment gradient and (b) strain

## 5 Conclusion

In this paper, the multiscale analysis method and related numerical algorithms are presented to predict the transient thermo-mechanical coupling behaviors of quasi-periodic composite structures. The multiscale formulations for the nonlinear and coupling problem are obtained, including the local cell problems, effective thermal and mechanical parameters, homogenized equations and second-order multiscale asymptotic expansions of temperature and displacement fields. The error analysis is given to indicate that the second-order multiscale approximate solutions have a much better approximation to the solutions of the original problem. Numerical results demonstrate that the local steep variations of the temperature, displacement and their gradient can be captured more precisely by adding the second-order correctors. And it can be also concluded that the multiscale method is not only feasible, but also accurate and efficient to predict the transient thermo-mechanical coupling behaviors of quasi-periodic composite structures. The high quality of the results encourage the application of proposed multiscale model and related numerical technique to deal with thermo-mechanical analysis of heterogeneous medias with much more complicated multiscale structures. And it is very helpful to the design and optimization of the composite structures.

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