 CMC, vol.47, no.3, pp.179-201, 2015

Research and Improvement on the Accuracy of Discontinuous Smoothed Particle Hydrodynamics (DSPH) Method

Rui Yan[1], **Fei Xu**[1,2]

Abstract: Discontinuous smoothed particle hydrodynamics (DSPH) method based on traditional SPH method, which can be used to simulate discontinuous physics problems near interface or boundary. Previous works showed that DSPH method has a good application prospect [Xu et al, 2013], but further verification and improvement are demanded. In this paper, we investigate the accuracy of DSPH method by some numerical models. Moreover, to improve the accuracy of DSPH method, first order and second order multidimensional RDSPH methods are proposed by following the idea of restoring particle consistency in SPH (RSPH) method which has shown good results in the improvement of particle consistency and accuracy for non-uniform particles. This restoring particle consistency in DSPH (RDSPH) method has the advantages from both RSPH method and DSPH method. In addition, the accuracy of RDSPH methods near the interface, boundary and in non-uniform interior region are tested in one-dimensional and two-dimensional spaces.

Keywords: Smoothed particle hydrodynamics; Interface; Boundary; Accuracy; Kernel consistency.

1 Introduction

Smoothed particle hydrodynamics (SPH) is a meshfree Lagrangian particle method. It was proposed by Gingold and Monaghan [Gingold and Monaghan (1977)] and Lucy [Lucy (1977)] for astrophysical problems. Comparing with finite element method (FEM), SPH method is easier to perform because the characteristic that particles can avoid remeshing frequently in large deformation problems. It has been used in fluid and solid mechanics, such as dam breaks, bodies impacting flu-

[1] Department of Aeronautical Structural Engineering, Northwestern Polytechnical University, 127 Youyi Xilu, Xi'an,Shaanxi, P. R. China
[2] Corresponding author. Tel.: +86 29 8849 3705, Email: xufei@nwpu.edu.cn

ids, turbulence, hypervelocity impact, etc. [Monaghan (2012); Liu and Liu (2010); Johnson, Stryk, and Beissel (1996); Jiang, Tang, and Ren (2014)].

SPH is a promising method but still suffers from some inherent numerical problems that motivate different modifications and corrections. In traditional SPH method, particles from different materials or objects influence each other and introduce errors when integrate Navier-Stokes equations from different objects. For example, the case of steel bullet impacting aluminum plate, particles from bullet influence particles from plate, which results in unacceptable errors near interface and has a negative effect on whole simulation further. To make it suitable for such kinds of problems, special algorithms deal with interface are required. The main methods used to contact algorithm and the major steps include (1) identifying the boundary, (2) detecting the contact and (3) adding repulsive contact forces [Liu and Liu (2010)]. Antoci, Gallati, and Sibilla (2007) enforced the interface condition between fluid and solid, which obtained by an approximate SPH evaluation of a surface integral of fluid pressure. Amini et al. (2011) used intermediate particles to impose the interaction between solid and fluid particles. Other methods can be found in [Vignjevic and Campbell (1999); Parshikov and Medin (2002); Mehra and Chaturvedi (2006)]. Different from the methods above, Liu, Liu, and Lam (2003) proposed discontinuous SPH (DSPH) method by dividing the support domain into smaller continuous domains, which is quite straightforward and simple. The advantage of DSPH method has been demonstrated in numerical models and simulations of shock waves. Xu, Zhao, Yan, and Furukawa (2013) proposed a technique to determinate the key particles near interface and extended DSPH method from one-dimension to multidimension. The simulation of metal penetration shows the advantage in solving discontinuous problem and the application prospect of DSPH method. Density, velocity and energy are calculated by SPH formulations when Navier-Stokes equations are used to control the general dynamic problems [Liu and Liu (2010)]. In our further work, we tested DSPH method in the calculation of those characters by simple impact model and found that DSPH method gives better results near interface [Yan, Xu, and Zhang (2013)].

Numerical inaccuracy is one of the major drawbacks in SPH method, which is even worse in irregular distributed particles and boundary particles [Liu and Liu (2010)]. The same challenge is confronted by DSPH method. Therefore, we seek the improvement of its numerical accuracy in this paper. Chaniotis, Poulikakos, and Koumoutsakos (2001) researched the stability of viscous flows and proposed remeshing particle locations periodically. Liu and Chang (2010) found that the instability resulted from the particle inconsistency through Poiseuille flow, which also originated from the non-uniform particles. For the improvement of numerical accuracy, the primary strategy is to obtain higher order approximation and better consistency of the kernel function. The approach using renormalization to improve the accuracy of the simulations has been investigated and discussed, which prevented the tensile instability [Oger, Doring, Alessandrini, and Ferrant (2007)]. Belytschko, Kongrauz, Dolbow, and Gerlach (1998) examined several methods that restore various levels of completeness by satisfying reproducing conditions on the approximation or the derivatives of the approximation. Based on Taylor series expansion, Chen and Beraun [Chen, Beraun, and Carney (1999); Chen and Beraun (2000)] suggested a corrective smoothed particle method (CSPM), which enhances the ability of SPH method near boundary. Korzilius [Korzilius, Schilders, and Anthonissen (2013)] improved the accuracy of CSPM significantly by a normalization factor. Based on Taylor series expansion, Zhang and Batra (2004) got Modified SPH (MSPH) method, which has been successfully applied on wave propagation in elastic bar and transient heat conduction in a plate. Liu [Liu and Liu (2006)] proposed restoring particle consistency in SPH, as RSPH in this paper. It has better particle consistency for both boundary and non-uniformly interior particles. Song et al. [Song, Zhang, Liang, and Li (2009)] got 1D RDSPH method by combining RSPH method and one-dimensional (1D) DSPH method, which keeps advantages from both RSPH method and DSPH method. But this method is restricted in one-dimension and has limited validation on numerical performance.

In this paper, we introduce DSPH method and analyze its idea that deals with discontinuous problems first. Then, first order and second order multidimensional RDSPH methods are proposed to improve the accuracy of DSPH method. Finally, several numerical examples are described to validate and demonstrate the ability of RDSPH methods, via comparative studies.

2 Basic theory of SPH method

2.1 Principle of SPH method

The principle of SPH method is the integral expression of a function $f(x)$ as follows

$$f(x)=\int_{-\infty}^{\infty} f(x')\delta\left(x-x'\right)dx' \tag{1}$$

where δ is Dirac function. This representation is mathematically correct. For numerical analysis, $f(x)$ should be approximated by a finite integral form in the domain of Ω:

$$\langle f(x)\rangle=\int_{\Omega} f(x')W(x-x',h)dx' \tag{2}$$

where $\langle f(x)\rangle$ represents the discrete approximation of $f(x)$, $W(x-x',h)$ is called the kernel function (or weight function or smoothing function), h is the smoothing length, which controls the size of support domain (or influence domain, or smoothing domain) Ω. This discrete integral representation converges when the kernel function satisfies certain conditions such as normalization, Dirac function property and local support. In accordance with the discrete integration, the function and its partial derivatives can also be expressed in a discrete manner. The function and its partial derivative in the discrete form for particle simulation are described as

$$f(x_i) = \sum_{j=1}^{N} \frac{m_j}{\rho_j} f(x_j) \cdot W_{ij} \tag{3}$$

$$\frac{\partial f(x_i)}{\partial x} = \sum_{j=1}^{N} \frac{m_j}{\rho_j} (f(x_j) - f(x_i)) \frac{\partial W_{ij}}{\partial x_i} \tag{4}$$

Here x_i, x_j, m_j, ρ_j and N are position vectors of particles i and j, mass of particle j, density of particle j and the total particle number in support domain of particle i respectively.

2.2 Principle of DSPH method

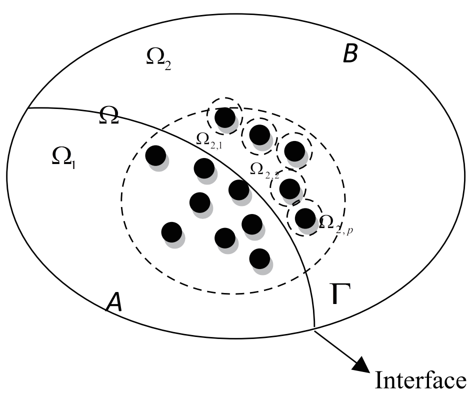


Figure 1: Discontinuous problems.

When two objects A and B are in contact, the integral domain Ω near their interface includes sub-domain Ω_1 from A and sub-domain Ω_2 from B. Sub-domains Ω_1 and Ω_2 are divided by the interface Γ as shown in Figure 1. DSPH method is proposed to deal this kind of problems. The most important work of multidimensional DSPH is to determine key particles, which is performed by using the following method [Xu, Zhao, Yan, and Furukawa (2013)]. The domain Ω_2 is divided into the smallest sub-domains within each of which only one particle is contained. Suppose that there are p particles $(x_{2,1}, x_{2,2}, \ldots, x_{2,p})$ in the domain Ω_2, accordingly the integral function of SPH can be written as

$$\int_{\Omega} f(x)W_i(x)dx = \int_{\Omega_1} f(x)W_i(x)dx + \int_{\Omega_{2,1}} f(x)W_i(x)dx + \int_{\Omega_{2,2}} f(x)W_i(x)dx + \cdots + \int_{\Omega_{2,p}} f(x)W_i(x)dx \quad (5)$$

where $W_i(x)$ is kernel function. The Taylor series expansions of the equation about particle x_i and $x_{2,1}, x_{2,2}, \ldots, x_{2,p}$ are as follows respectively

$$\int_{\Omega_1} f(x)W_i dx = f(x_i)\int_{\Omega_1} W_i dx + f_\alpha(x_i)\int_{\Omega_1}(x^\alpha - x_i^\alpha)W_i dx + \frac{f_{\alpha\beta}(x_i)}{2!}\int_{\Omega_1}(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)W_i dx + \ldots \quad (6a)$$

$$\int_{\Omega_{2,1}} f(x)W_i dx = f(x_{2,1})\int_{\Omega_{2,1}} W_i dx \quad (6b1)$$

$\ldots$

$$\int_{\Omega_{2,p}} f(x)W_i dx = f(x_{2,p})\int_{\Omega_{2,p}} W_i dx \quad (6bp)$$

Where α and β are the dimensional indexes that take values of 1 to 3. By neglecting the derivative terms in Equation (6) and replacing the terms in Equation (5), we obtain

$$\int_{\Omega_1} f(x)W_i dx = f(x_i)\int_{\Omega_1} W_i dx + f(x_{2,1})\int_{\Omega_{2,1}} W_i dx + f(x_{2,2})\int_{\Omega_{2,2}} W_i dx + \ldots + + f(x_{2,P})\int_{\Omega_{2,P}} W_i dx \quad (7)$$

and rearrangement Equation (7) yields the multidimensional discontinuous function as

$$f(x_i) = \frac{\int_{\Omega} f(x)W_i dx}{\int_{\Omega} W_i dx} - \left[\frac{[f(x_{2,1}) - f(x_i)]\int_{\Omega_{2,1}} W_i dx}{\int_{\Omega} W_i dx} + \ldots + \frac{[f(x_{2,P}) - f(x_i)]\int_{\Omega_{2,P}} W_i dx}{\int_{\Omega} W_i dx}\right] \quad (8)$$

In order to find the derivative of the multidimensional discontinuous function, the replacement of $W_i(x)$ by $W_{i,\beta}(x)$ in Equation (5) gives

$$\int_{\Omega_1} f(x)W_{i,\beta} dx = \int_{\Omega_1} f(x)W_{i,\beta} dx + \int_{\Omega_{2,1}} f(x)W_{i,\beta} dx$$

$$+\int_{\Omega_{2,2}} f(x)W_{i,\beta}dx+\ldots+\int_{\Omega_{2,P}} f(x)W_{i,\beta}dx \tag{9}$$

Substituting Equation (6) into Equation (9) and neglecting the terms higher than two orders, the derivative of the multidimensional discontinuous function can be obtained as

$$\begin{aligned} f_\alpha(x_i) =& \frac{\int_\Omega [f(x)-f(x_i)]W_{i,\beta}dx}{\int_\Omega (x^\alpha - x_i^\alpha)W_{i,\beta}dx} \\ &- \left[\frac{[f(x_{2,1})-f(x_i)]\int_{\Omega_{2,1}} W_{i,\beta}dx}{\int_\Omega (x^\alpha - x_i^\alpha)W_{i,\beta}dx} + \ldots + \frac{[f(x_{2,P})-f(x_i)]\int_{\Omega_{2,P}} W_{i,\beta}dx}{\int_\Omega (x^\alpha - x_i^\alpha)W_{i,\beta}dx}\right] \\ &- \left[\frac{\int_{\Omega_{2,1}} \left[(x^\alpha - x_{2,1}^\alpha)f_\alpha(x_{2,1}) - (x^\alpha - x_i^\alpha)f_\alpha(x_i)\right]W_{i,\beta}dx}{\int_\Omega (x^\alpha - x_i^\alpha)W_{i,\beta}dx} + \ldots\right. \\ &\left. + \frac{\int_{\Omega_{2,p}} \left[(x^\alpha - x_{2,p}^\alpha)f_\alpha(x_{2,p}) - (x^\alpha - x_i^\alpha)f_\alpha(x_i)\right]W_{i,\beta}dx}{\int_\Omega (x^\alpha - x_i^\alpha)W_{i,\beta}dx}\right] \end{aligned} \tag{10}$$

Where $f_i = f(x_i)$, $f_{i,\alpha} = f_\alpha(x_i) = (\partial f/\partial x^\alpha)_i$. Based on the theory of SPH method combining the items in Ω_2, the multidimensional discontinuous function (8) and its derivative (10) can be written in discrete form respectively,

$$f_i = \underbrace{\frac{\sum_{j\in\Omega}(\frac{m_j}{\rho_j})f_jW_{ij}}{\sum_{j\in\Omega}(\frac{m_j}{\rho_j})W_{ij}}}_{\text{(I)}} - \underbrace{\frac{\sum_{j\in\Omega_2}(f_j-f_i)(\frac{m_j}{\rho_j})W_{ij}}{\sum_{j\in\Omega}(\frac{m_j}{\rho_j})W_{ij}}}_{\text{(II)}} \tag{11}$$

$$\begin{aligned} f_{i,\alpha} = & \frac{\sum_{j\in\Omega}(\frac{m_j}{\rho_j})(f(x)-f(x_i))W_{ij,\beta}}{\sum_{j\in\Omega}(x^\alpha - x_i^\alpha)(\frac{m_j}{\rho_j})W_{ij,\beta}} - \left[\frac{\sum_{j\in\Omega_2}(\frac{m_j}{\rho_j})(f(x)-f(x_i))W_{ij,\beta}}{\sum_{j\in\Omega}(x^\alpha - x_i^\alpha)(\frac{m_j}{\rho_j})W_{ij,\beta}}\right. \\ & \left. - \frac{f_\alpha(x_i)\sum_{j\in\Omega_2}(\frac{m_j}{\rho_j})(x^\alpha - x_i^\alpha)W_{ij,\beta}}{\sum_{j\in\Omega}(x^\alpha - x_i^\alpha)(\frac{m_j}{\rho_j})W_{ij,\beta}}\right] \end{aligned} \tag{12}$$

In these two formulations of the multidimensional DSPH method, Equations (11) and (12), the leftmost terms are identical to the CSPM while the remaining terms are supplementary for considering discontinuity [Chen and Beraun (2000)].

3 The accuracy of DSPH method

We test the accuracy of DSPH method in this section, including the function and its derivate.

3.1 The accuracy of function

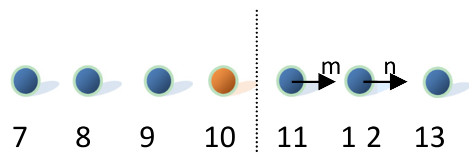


Figure 2: The distribution of particles.

The distribution of particles is shown in Figure 2 and all the distances between neighbor particles are equal to *1* initially. The numbers under the particles show their serial number and initial value (x). The positions of particles 11 and 12 are varied: the moving distance of particle 11 is m: $-0.5 \leq m < 0.5$ and the moving distance of particle 12 is n: $-0.5 \leq n < 1$. So the non-uniform particles can be illustrated by the varied m and n. The other particles keep their positions. The kernel function is cubic spline function and the detail can be found in reference [Liu and Liu (2003)]. The smoothing length h is chosen as 1.5. The accuracy and sensitivity on non-uniform particles of DSPH method are researched by a discontinuous function given by

$$f(x) = \begin{cases} 100 & 0.5 \leq x \leq 10.5 \\ -x^2 & 10.5 < x \leq 20.5 \end{cases} \tag{13}$$

All the errors from the calculation are 0 when different m and n are used (as shown in Figure 3), which illustrate that the value and distribution of particles 11 and 12 have no influence on the calculation of $f(10)$.

It is necessary to consider the real reason of the exact results from DSPH. Based on the method of dividing domains in section 1, we get Equation (14) through replace by Ω_1 and Ω_2 in the numerator of Equation (11) and rearranging it.

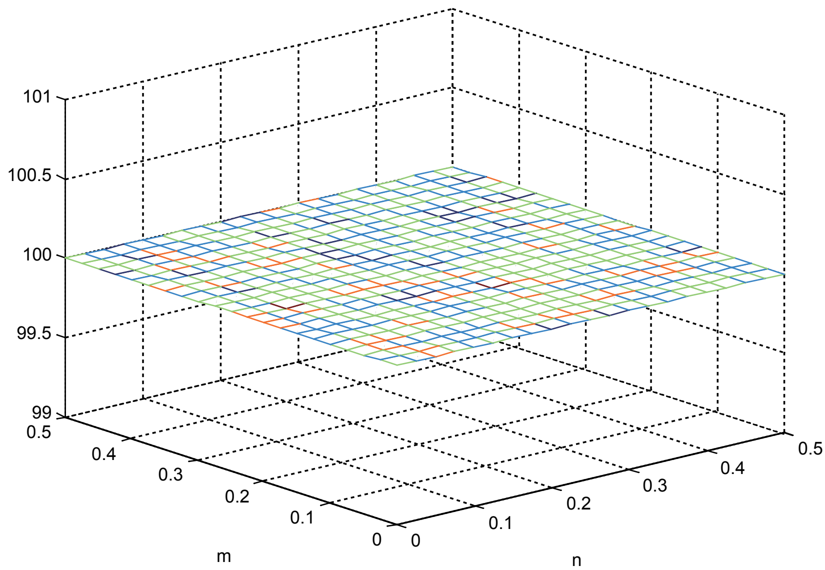


Figure 3: The results from DSPH method for $f(10)$.

$$f_i = \frac{\sum_{j\in\Omega_1} (\frac{m_j}{\rho_j}) f_i W_{ij} + \sum_{j\in\Omega_2} (\frac{m_j}{\rho_j}) f_i W_{ij}}{\sum_{j\in\Omega} (\frac{m_j}{\rho_j}) W_{ij}} \tag{14}$$

From Equation (14), we know that: the result of function for particle i has no relationship with the particles on the other side of the interface, and it is determined by the particles in the same side; in addition, the percentage of contribution from itself, which determined by the mass, density and distribution of particles in Ω_2, is stronger than SPH method. DSPH method is equal to CSPM when no particles in Ω_2. In physic problems, the result has high precision if the function in computational domain Ω_1 is a constant function.

3.2 The accuracy of function's derivate

In this section, the accuracy of function's derivate is researched. We still use the model shows in Figure 2 but different discontinuous function,

$$f(x) = \begin{cases} 10x & 0.5 \le x \le 10.5 \\ -x^2 & 10.5 < x \le 20.5 \end{cases} \tag{15}$$

Similar with the works in section 3.1, we calculate $f'(10)$ in different m and n. The result is accurate no matter where the particles move. Rewriting the Equation (12) to Equation (16),

$$f_{i,\alpha} = B_{i,\alpha\beta} \sum_{j\in\Omega_1} (\frac{m_j}{\rho_j})(f_j - f_i)W_{ij,\beta} + B_{i,\alpha\beta} \sum_{j\in\Omega_2} (\frac{m_j}{\rho_j})(F_j - F_i)W_{ij,\beta} \tag{16}$$

Where, $B_{i,\alpha\beta} = (\sum_{j\in\Omega} (\frac{m_j}{\rho_j})(x_j^\alpha - x_i^\alpha)W_{ij,\beta})^{-1}$, $F_i = x_i f_{i,\alpha} + f_i$ and $F_j = x_j f_{i,\alpha} + f_j$. Equation (16) shows the value of $f_{i,\alpha}$ has no relationship with the original function in Ω_2, but the particles' position, mass and density in Ω_2 are used. In physic problems, the result of function's derivate would be accurate when the function is similar to linear function in computational domain Ω_1.

According to the discussion above, DSPH method shows its advantage in the calculation of discontinuous constant function and linear function. But the models are much more complicate in physic problems, so higher order method is needed.

4 Principle of RDSPH method

The RSPH method in reference [Liu and Liu (2006)] provided a reasonable method to improve the accuracy of SPH method, which will be used in this section to improve the accuracy of DSPH method.

4.1 First order RDSPH method

Similar with DSPH method, expanding function $f(x)$ into Taylor series expansion about the point x_i and $x_{2,1}, x_{2,2}, \ldots, x_{2,p}$ in different domain and multiplying both sides of the function with conventional smoothing function $W_i(x)$ and its first order derivatives $W_{i,\beta}(x)$ respectively, neglecting the terms higher than one order derivatives, we have

$$\begin{aligned}\int_\Omega f(x)W_i(x)dx = &\int_{\Omega_1} f(x_i)W_i(x)dx + f_{i,\alpha}\int_{\Omega_1} (x^\alpha - x_i^\alpha)W_i(x)dx \\ &+ \int_{\Omega_{2,1}} f(x_{2,1})W_i(x)dx + \ldots + \int_{\Omega_{2,p}} f(x_{2,p})W_i(x)dx\end{aligned} \tag{17}$$

$$\begin{aligned}\int_\Omega f(x)W_{i,\beta}(x)dx = &f(x_i)\int_{\Omega_1} W_{i,\beta}(x)dx + f_{i,\alpha}\int_{\Omega_1} (x^\alpha - x_i^\alpha)W_{i,\beta}(x)dx \\ &+ \int_{\Omega_{2,1}} f(x_{2,1})W_{i,\beta}(x)dx + \ldots + \int_{\Omega_{2,p}} f(x_{2,p})W_{i,\beta}(x)dx\end{aligned} \tag{18}$$

Again α and β are the dimensional indexes repeated from 1 to 3. The approximations of f_i and $f_{i,\alpha}$ in continuous form (kernel approximation) are obtained by solving with respect to f_i and $f_{i,\alpha}$.

$$f_i = \frac{\begin{vmatrix} \int_{\Omega} f(x)W_i(x)dx - \left(f(x_{2,1}) \int_{\Omega_{2,1}} W_i(x)dx - f(x_{2,p}) \int_{\Omega_{2,p}} W_i(x)dx \right) & \int_{\Omega_1} (x^\alpha - x_i^\alpha) W_i(x)dx \\ \int_{\Omega} f(x)W_{i,\beta}(x)dx - \left(f(x_{2,1}) \int_{\Omega_{2,1}} W_{i,\beta}dx - f(x_{2,p}) \int_{\Omega_{2,p}} W_{i,\beta}dx \right) & \int_{\Omega_1} (x^\alpha - x_i^\alpha) W_{i,\beta}(x)dx \end{vmatrix}}{\begin{vmatrix} \int_{\Omega_1} W_i(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha) W_i(x)dx \\ \int_{\Omega_1} W_{i,\beta}(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha) W_{i,\beta}(x)dx \end{vmatrix}} \tag{19}$$

$$f_{i,\alpha} = \frac{\begin{vmatrix} \int_{\Omega_1} W_i(x)dx & \int_{\Omega} f(x)W_i(x)dx - \left(f(x_{2,1}) \int_{\Omega_{2,1}} W_i dx - f(x_{2,p}) \int_{\Omega_{2,p}} W_i dx \right) \\ \int_{\Omega_1} W_{i,\beta}(x)dx & \int_{\Omega} f(x)W_{i,\beta}(x)dx - \left(f(x_{2,1}) \int_{\Omega_{2,1}} W_{i,\beta}dx - f(x_{2,p}) \int_{\Omega_{2,p}} W_{i,\beta}dx \right) \end{vmatrix}}{\begin{vmatrix} \int_{\Omega_1} W_i(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha) W_i(x)dx \\ \int_{\Omega_1} W_{i,\beta}(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha) W_{i,\beta}(x)dx \end{vmatrix}} \tag{20}$$

The multi-dimensional discontinuous function (19) and its derivative (20) can be written in discrete form respectively. The equations can be written in a simple way when we combine the items.

$$f_i = \frac{\begin{vmatrix} \sum_{j\in\Omega_1} f_j W_{ij} \frac{m_j}{\rho_j} & \sum_{j\in\Omega_1} (x_j^\alpha - x_i^\alpha) W_{ij} \frac{m_j}{\rho_j} \\ \sum_{j\in\Omega_1} f_j W_{ij,\beta} \frac{m_j}{\rho_j} & \sum_{j\in\Omega_1} (x_j^\alpha - x_i^\alpha) W_{ij,\beta} \frac{m_j}{\rho_j} \end{vmatrix}}{\begin{vmatrix} \sum_{j\in\Omega_1} W_{ij} \frac{m_j}{\rho_j} & \sum_{j\in\Omega_1} (x_j^\alpha - x_i^\alpha) W_{ij} \frac{m_j}{\rho_j} \\ \sum_{j\in\Omega_1} W_{ij,\beta} \frac{m_j}{\rho_j} & \sum_{j\in\Omega_1} (x_j^\alpha - x_i^\alpha) W_{ij,\beta} \frac{m_j}{\rho_j} \end{vmatrix}} \tag{21}$$

$$f_{i,\alpha} = \frac{\begin{vmatrix} \sum_{j\in\Omega_1} W_{ij} \frac{m_j}{\rho_j} & \sum_{j\in\Omega_1} f_j W_{ij} \frac{m_j}{\rho_j} \\ \sum_{j\in\Omega_1} W_{ij,\beta} \frac{m_j}{\rho_j} & \sum_{j\in\Omega_1} f_j W_{ij,\beta} \frac{m_j}{\rho_j} \end{vmatrix}}{\begin{vmatrix} \sum_{j\in\Omega_1} W_{ij} \frac{m_j}{\rho_j} & \sum_{j\in\Omega_1} (x_j^\alpha - x_i^\alpha) W_{ij} \frac{m_j}{\rho_j} \\ \sum_{j\in\Omega_1} W_{ij,\beta} \frac{m_j}{\rho_j} & \sum_{j\in\Omega_1} (x_j^\alpha - x_i^\alpha) W_{ij,\beta} \frac{m_j}{\rho_j} \end{vmatrix}} \tag{22}$$

First order RDSPH method would as same as CSPM when we calculate inner uniform particles, which is known by simplifying Equations (21) and (22).

4.2 Second order RDSPH method

In some complicate problems, the higher precision is needed, so we provide second order method for reference. Neglecting the third and higher order derivatives and multiplying both sides of the function with a conventional smoothing function $W_i(x)$, its first order derivatives $W_{i,\chi}(x)$ and second order derivatives $W_{i,\chi\delta}(x)$ respectively, we obtain

$$\begin{aligned}\int_{\Omega} f(x)W_i(x)dx =& \int_{\Omega_1} f(x_i)W_i(x)dx + f_{i,\alpha}\int_{\Omega_1}(x^\alpha - x_i^\alpha)W_i(x)dx \\ &+ \frac{1}{2}f_{i,\alpha\beta}\int_{\Omega_1}(x^\alpha - x_i^\alpha)\left(x^\beta - x_i^\beta\right)W_i(x)dx \\ &+ \int_{\Omega_{2,1}} f(x_{2,1})W_i(x)dx + \ldots + \int_{\Omega_{2,p}} f(x_{2,p})W_i(x)dx\end{aligned} \tag{23}$$

$$\begin{aligned}\int_{\Omega} f(x)W_{i,\chi}(x)dx =& f(x_i)\int_{\Omega_1} W_{i,\chi}(x)dx + f_{i,\alpha}\int_{\Omega_1}(x^\alpha - x_i^\alpha)W_{i,\chi}(x)dx \\ &+ \frac{1}{2}f_{i,\alpha\beta}\int_{\Omega_1}(x^\alpha - x_i^\alpha)\left(x^\beta - x_i^\beta\right)W_{i,\chi}(x)dx \\ &+ \int_{\Omega_{2,1}} f(x_{2,1})W_{i,\chi}(x)dx + \ldots + \int_{\Omega_{2,p}} f(x_{2,p})W_{i,\chi}(x)dx\end{aligned} \tag{24}$$

$$\begin{aligned}\int_{\Omega} f(x)W_{i,\chi\delta}(x)dx =& f(x_i)\int_{\Omega_1} W_{i,\chi\delta}(x)dx + f_{i,\alpha}\int_{\Omega_1}(x^\alpha - x_i^\alpha)W_{i,\chi\delta}(x)dx \\ &+ \frac{1}{2}f_{i,\alpha\beta}\int_{\Omega_1}(x^\alpha - x_i^\alpha)\left(x^\beta - x_i^\beta\right)W_{i,\chi\delta}(x)dx \\ &+ \int_{\Omega_{2,1}} f(x_{2,1})W_{i,\chi\delta}(x)dx + \ldots + \int_{\Omega_{2,p}} f(x_{2,p})W_{i,\chi\delta}(x)dx\end{aligned} \tag{25}$$

Where α, β, χ and δ are the dimensional indexes repeated from 1 to 3. The approximations of f_i, $f_{i,\alpha}$ and $f_{i,\alpha\beta}$ in continuous form (kernel approximation) are obtained by solving with respect to f_i, $f_{i,\alpha}$ and $f_{i,\alpha\beta}$. Equations (26)–(28) show the second order RDSPH method in two-dimension, which are used in the numerical calculation in Section 4.

$$f_i = \frac{\begin{vmatrix} \int_{\Omega_1} f(x)W_i(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha)W_i(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2} W_i(x)dx \\ \int_{\Omega_1} f(x)W_{i,\chi}(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha)W_{i,\chi}(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2} W_{i,\chi}(x)dx \\ \int_{\Omega_1} f(x)W_{i,\chi\delta}(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha)W_{i,\chi\delta}(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2} W_{i,\chi\delta}(x)dx \end{vmatrix}}{\begin{vmatrix} \int_{\Omega_1} W_i(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha)W_i(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2} W_i(x)dx \\ \int_{\Omega_1} W_{i,\chi}(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha)W_{i,\chi}(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2} W_{i,\chi}(x)dx \\ \int_{\Omega_1} W_{i,\chi\delta}(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha)W_{i,\chi\delta}(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2} W_{i,\chi\delta}(x)dx \end{vmatrix}} \tag{26}$$

$$f_{i,x} = \frac{\begin{vmatrix} \int_{\Omega_1} W_i(x)dx & \int_{\Omega_1} f(x)W_i(x)dx & \int_{\Omega_1} (y - y_i)W_i(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2!} W_i(x)dx \\ \int_{\Omega_1} W_{i,\chi}(x)dx & \int_{\Omega_1} f(x)W_{i,\chi}(x)dx & \int_{\Omega_1} (y - y_i)W_{i,\chi}(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2!} W_{i,\chi}(x)dx \\ \int_{\Omega_1} W_{i,\chi\delta}(x)dx & \int_{\Omega_1} f(x)W_{i,\chi\delta}(x)dx & \int_{\Omega_1} (y - y_i)W_{i,\chi\delta}(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2!} W_{i,\chi\delta}(x)dx \end{vmatrix}}{\begin{vmatrix} \int_{\Omega_1} W_i(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha)W_i(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2!} W_i(x)dx \\ \int_{\Omega_1} W_{i,\chi}(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha)W_{i,\chi}(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2!} W_{i,\chi}(x)dx \\ \int_{\Omega_1} W_{i,\chi\delta}(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha)W_{i,\chi\delta}(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2!} W_{i,\chi\delta}(x)dx \end{vmatrix}} \tag{27}$$

$$f_{i,y} = \frac{\begin{vmatrix} \int_{\Omega_1} W_i(x)dx & \int_{\Omega_1} (x - x_i)W_i(x)dx & \int_{\Omega_1} f(x)W_i(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2!} W_i(x)dx \\ \int_{\Omega_1} W_{i,\chi}(x)dx & \int_{\Omega_1} (x - x_i)W_{i,\chi}(x)dx & \int_{\Omega_1} f(x)W_{i,\chi}(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2!} W_{i,\chi}(x)dx \\ \int_{\Omega_1} W_{i,\chi\delta}(x)dx & \int_{\Omega_1} (x - x_i)W_{i,\chi\delta}(x)dx & \int_{\Omega_1} f(x)W_{i,\chi\delta}(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2!} W_{i,\chi\delta}(x)dx \end{vmatrix}}{\begin{vmatrix} \int_{\Omega_1} W_i(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha)W_i(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2!} W_i(x)dx \\ \int_{\Omega_1} W_{i,\chi}(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha)W_{i,\chi}(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2!} W_{i,\chi}(x)dx \\ \int_{\Omega_1} W_{i,\chi\delta}(x)dx & \int_{\Omega_1} (x^\alpha - x_i^\alpha)W_{i,\chi\delta}(x)dx & \int_{\Omega_1} \frac{(x^\alpha - x_i^\alpha)(x^\beta - x_i^\beta)}{2!} W_{i,\chi\delta}(x)dx \end{vmatrix}} \tag{28}$$

From the formula derivation, we know that both RDSPH method and DSPH method use Taylor series expansion. First order RDSPH method and DSPH method would as same as CSPM when inner uniform particles are calculated. At the same time, those two methods have some differences. The Equations (21)–(22) and (26)–(28) reveal that the essential idea of RDSPH methods to solve discontinuous problems is neglecting the particles in different domains. In other words, there is no relationship between the particles in different domains, which distinguishes RDSPH from DSPH method. As we mentioned in section 3, the mass, density and distribution of particles in Ω_2 determine the contribution of particle i in the calculation of DSPH method.

If derivatives up to kth are retained in the Taylor series expansion, the resultant kernel and particle approximations of RDSPH method should have kth consistency, which is verified in section 5 by first and second order RDSPH method.

5 The accuracy of different methods

We obtained the functions and their derivate of first and second order RDSPH method in section 4, the accuracy and sensibility near interface and boundary as well as for non-uniform particles are verified in this section.

5.1 The accuracy near interface

The DSPH method is proposed initially to solve the problems near interface, so we start the testing from interface. The value of linear function (29), quadratic function (30) and thrice function (31) and their derivative are calculated by DSPH method, first order RDSPH method and second order RDSPH method, respectively. The one-dimensional model is studied first and the distribution of particles is uniform as shown in Figure 4. The numbers under the particles shows their serial number and the value of x in Equations (29)–(31). The kernel function is cubic spline function and the smoothing length h is chosen as 1.5.

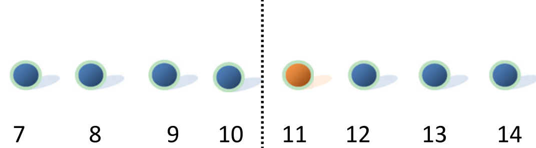


Figure 4: The distribution of particles with interface.

The results of $f(11)$ and $f'(11)$ for Equations (29)–(31) obtained by using different methods are shown in Table 1. RDSPH methods calculate both function and derivative accurately for linear function. Comparing with DSPH method, first order RDSPH method has worse results in derivative of quadratic function and thrice function, but has better results in the calculation of function. Second order RDSPH method shows its advantage in both of the calculation of function and derivative. Those results demonstrate the advantage of RDSPH method in the improvement of accuracy.

$$\begin{cases} f(x) = x & x \le 10 \\ f(x) = x+5 & x > 10 \end{cases} \tag{29}$$

$$\begin{cases} f(x) = x^2 & x \le 10 \\ f(x) = x^2+10x & x > 10 \end{cases} \tag{30}$$

$$\begin{cases} f(x) = x^3 & x \le 10 \\ f(x) = x^3+10x^2 & x > 10 \end{cases} \tag{31}$$

Table 1: Errors from different methods for particle 11 ($h = 1.5$).

		DSPH	first order RDSPH	second order RDSPH	Exact
Linear	Function	16.3115	16	16	16
	Derivative	1	1	1	1
Quadratic	Function	241.3443	230.8750	231	231
	Derivative	32.7	33.5000	32	32
Thrice	Function	2739.3	2535.3	2541	2541
	Derivative	614.2	650	581	583

Different smoothing lengths produce different results due to the distribution of particles in support domain. Smoothing length h is chose as 1.2 in Table 2 to compare with the results in Table 1. Similar conclusion can be found in table 2. However, it is shown better results by DSPH and first order RDSPH because the smaller support domain enhances the influence of particle 11 in equations (11), (12), (21) and (22). Both smooth length 1.2 and 1.5 have the same results in second RDSPH method, which illustrate that the varied smooth lengths have little influence on high accuracy method.

5.2 The accuracy near the boundary

Besides interface, the accuracy near boundary is also important in SPH simulation. The distribution of particles near boundary is shown in Figure 5 and particles 7

Table 2: Errors from different methods for particle 11 ($h = 1.2$).

		DSPH	first order RDSPH	second order RDSPH	Exact
Linear	Function	16.2279	16	16	16
	Derivative	1	1	1	1
Quadratic	Function	238.5295	230.9574	231	231
	Derivative	32.5755	33.1903	32	32
Thrice	Function	2684.3	2539.0	2541	2541
	Derivative	608.4717	635.7540	581	583

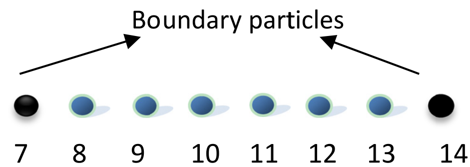


Figure 5: The distribution of particles near boundary.

and 14 are boundary particles. Furthermore, we should take notice that there is no interface in this model consequently we use continuous function in this part. The quadratic function (32) is chosen for this model. The reason is that all the methods have good results in the calculation of linear function and it is difficult to judge the better one from the calculation of thrice function since all of them have errors. The smoothing length h is chosen as 1.5. The values of Equation (32) for particles 12, 13 and 14 are calculated in Table 3, respectively. The errors from DSPH method are increasing when the particle is approaching the boundary. The error of derivative calculated by first order RDSPH method is increasing, but the error of function calculated by it is decreasing. The results from DSPH method and first order RDSPH method are same for particle 12, as mentioned in section 3 and 4, they are equal to CSPM when they calculate the inner uniform particles. Second order RDSPH method has higher precision and has no error near the boundary.

$$f(x) = x^2 \tag{32}$$

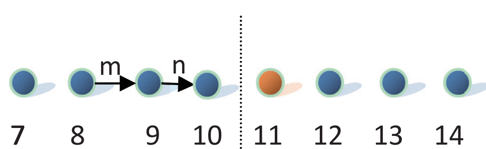


Figure 6: The non-uniform particles.

Table 3: Errors from different methods for boundary particles.

		DSPH	first order RDSPH	second order RDSPH	exact
12	Function	144.7541	144.7541	144	144
	Derivative	24.0000	24.0000	24	24
13	Function	167.8814	169.6154	169	169
	Derivative	25.5000	25.5769	26	26
14	Function	184.4318	195.8750	196	196
	Derivative	26.6000	26.5000	28	28

5.3 The sensibility on non-uniform particles

5.3.1 One-dimensional domain

Particles move non-uniformly by large deformations, which affect the simulation obviously. This section investigates the sensitivity of those methods for non-uniform particles. Only quadratic function (30) is tested in this part. The distribution of particles is shown in Figure 6, all the initial distances between neighbor particles are equal to 1. The numbers under the particles are serial number as initial value (x) in Equation (30). The positions of particles 11 and 12 are varied: the moving distance of particle 8 is m: $-0.5 < m < 0.5$ and the moving distance of particle 12 is n: $0 < n < 0.5$. So the non-uniform particles can be illustrated by the varied m and n, at the same time, the positions of other particles keep constant. The smoothing length is also chosen as 1.5.

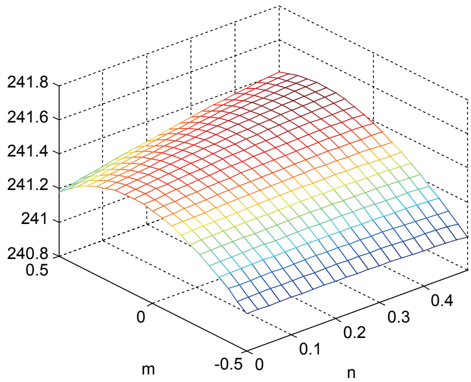
We get Figure 7 from DSPH method and Table 4 from RDSPH method because the particles in different domain have no influence on the calculation of RDSPH method. First order RDSPH has a better result than DSPH method in the calculation of function but has a worse result in derivative, which is accordant with the conclusion in Section 4.1. The result from second RDSPH is accurate.

Table 4: Errors from RDSPH methods for particle 11.

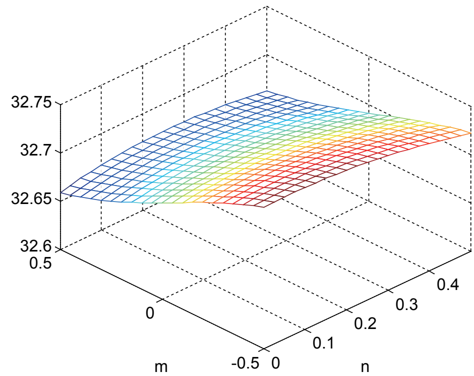
	first order RDSPH	second order RDSPH	Exact
Function	230.875	231	231
Derivative	33.5	32	32

5.3.2 Two-dimensional domain

When we solve the physical problems, the particles are always in two-dimensional (2D) space. So the accuracy and sensibility on non-uniform particles must be studied in 2D space. The distribution of particles in 2D is shown in Figure 8. Along



(a) Function



(b) Function's derivative

Figure 7: The results from DSPH method.

y-axis, the distances between particles are same, all of which are 0.25; along x-axis, from 0 to 5 and from 0 to -5, the distances are 0.25, 0.5, 0.25 $\ldots$, respectively. Equation (33) is used in this part. Same as previous models, the kernel function is cubic spline function and the smoothing length h is 1.5.

$$\begin{cases} f(x,y) = x^2 + y^2 & x < 0 \\ f(x,y) = x^2 + y^2 + 10 & x \geq 0 \end{cases} \tag{33}$$

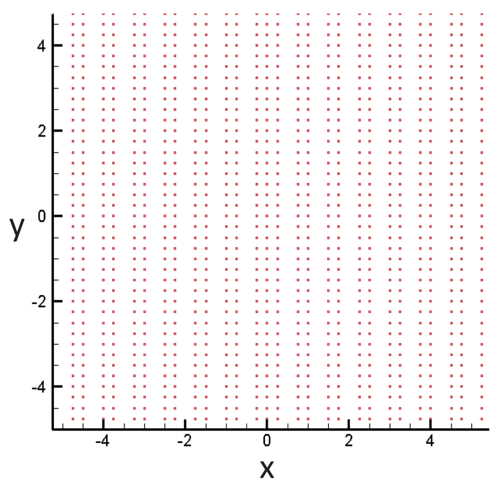
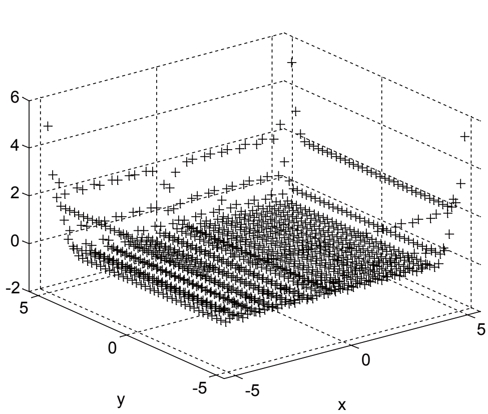
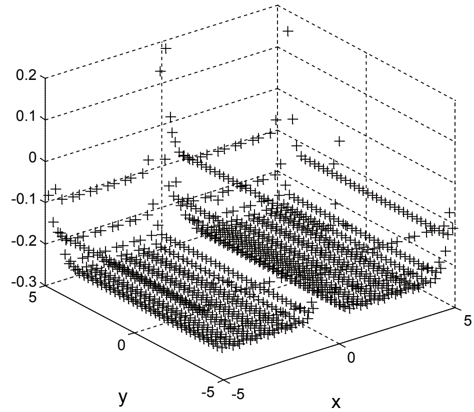


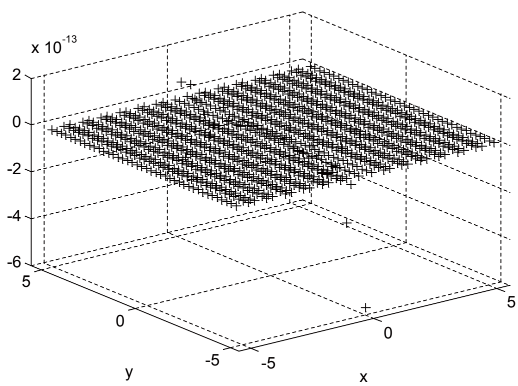
Figure 8: The model in 2D space.



(a) DSPH



(b) first order RDSPH

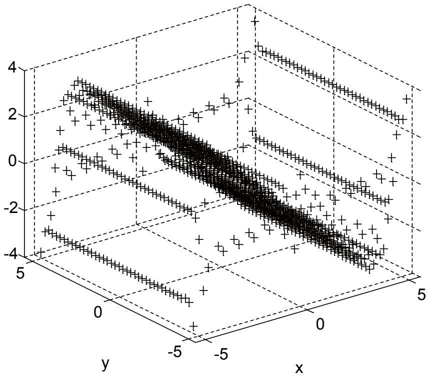


(c) second order RDSPH

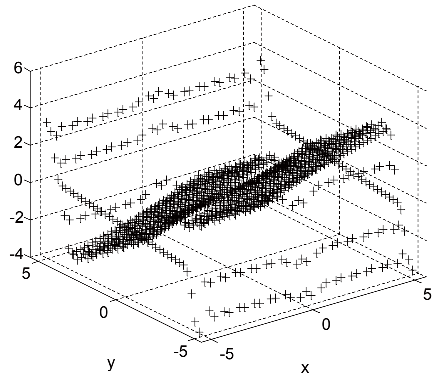
Figure 9: The errors from the calculation of $f(x,y)$.

The errors from DSPH method, first order RDSPH method and second order RD-SPH method in the calculation of $f(x,y)$ are shown in Figure 9 and the results are encouraging that both first order RDSPH method and second order RDSPH method have much better precision than DSPH method. The maximum error from DSPH reaches to 5.25 while the error from first order RDSPH is less than 0.2 and the results from second order RDSPH method are accurate. DSPH method has acceptable accuracy near interface but the errors near boundary cannot be neglected.

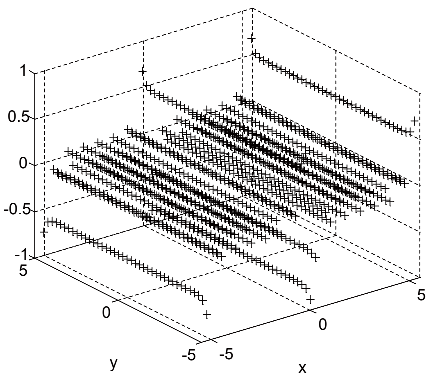
Figure 10 shows the errors from the first order partial derivative of $f(x,y)$ with respect to x and y. The errors from RDSPH method are also much smaller than DSPH method. DSPH method is sensitive to the non-uniform particles and the maximum error of derivative reaches to 4. First order RDSPH method has accurate



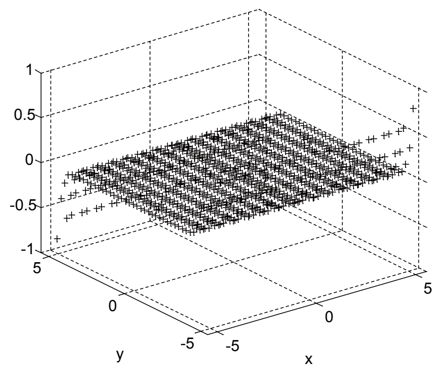
(a) DSPH method



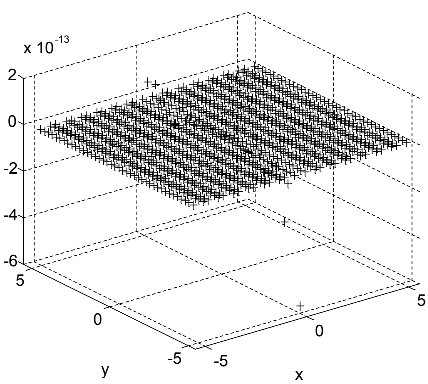
(b) DSPH method



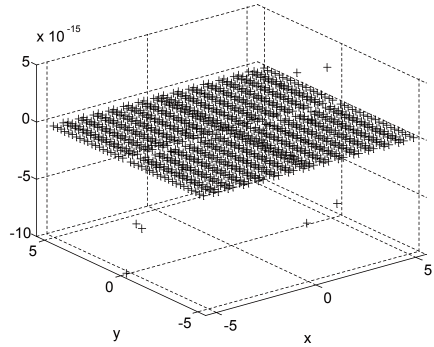
(c) first order RDSPH method



(d) first order RDSPH method



(e) second order RDSPH method



(f) second order RDSPH method

Figure 10: The errors from first order partial derivative of $f(x,y)$ about x (a, c, e) and y (b, d, f).

results and low sensitive to non-uniform particles inside the domain while has small errors near interface and boundary. The errors from partial derivative with respect to y is smaller than the errors from the derivative to x near interface in first order RDSPH, in other words, the direction could affect the errors of partial derivative when the particles have different distributions in different directions. Both first order RDSPH and DSPH methods have worse results near boundary and interface, which is consistent with the 1D results. The results from second order RDSPH are accurate.

We have tested DSPH method in impact dynamics and used DSPH particle approximations in the calculation of Navier-Stokes equations including density, acceleration and energy [Yan, Xu, and Zhang (2013)]. We found that density was the most important parameter that should have high precision in simulation. Density is calculated by the velocity gradient, which knows from Navier-Stokes equations. The huge difference in velocity near the interface of two objects causes the inaccurate of density near the interface, which has a negative effect on whole simulation further. DSPH method improves the precision of density near interface dramatically. The RDSPH methods have similar application area, which need more validation in future work. To be mentioned, this method is applied easily in multi-dimension system and will be applied conveniently in physical problems.

6 Conclusions

This paper reviewed DSPH method and explored the idea to deal with discontinuous problem. DSPH method shows its advantage in the discontinuous function. It has accurate results when calculates constant function or derivative of linear function. In order to calculate higher-order functions and their derivate accurately, we proposed multidimensional RDSPH method including first order RDSPH and second order RDSPH based on DSPH method.

We tested those methods by 1D model firstly. Comparing with DSPH method, first order RDSPH method has worse results in derivative of quadratic function and thrice function, but has better results in the calculation of function near interface. For the calculation of inner uniform particles, DSPH shows same results with C-SPM. Second order RDSPH method shows high accuracy in calculations of both function and derivative.

In the end of this paper, a 2D model was set up and the results were encouraging. D-SPH method has low accuracy in non-uniform particles. Both first and second order RDSPH methods have much better results. Although the extra matrix calculation in RDSPH methods makes computing time slightly increasing, it is acceptable to gain the huge accuracy improvement. Furthermore, this method is easy to use as well. It is convenient to choose one of those methods according to the accuracy demand.

This paper has focused on the development of the accuracy of multidimensional DSPH method by proposing RDSPH method that is tested by numerical model. The RDSPH method should be tested in large deformation physical model with interface in future work.

Acknowledgement: The work described in this paper was supported by the National Natural Science Foundation of China 11272266 and 111 project (B07050).

References

Antoci, C.; Gallati, M.; Sibilla, S. (2007): Numerical simulation of fluid–structure interaction by SPH. *Computers and Structures*, vol. 85, pp. 879–890.

Antoci, C.; Gallati, M.; Sibilla, S. (2007): A new model to solve fluid–hypo-elastic solid interaction using the smoothed particle hydrodynamics (SPH) method. *European Journal of Mechanics B/Fluids*, vol. 30, pp. 184–194.

Belytschko, T.; Kongrauz, Y.; Dolbow, J.; Gerlach, C. (1998): On the completeness of meshfree particle methods. *International Journal of Numerical and Mechanical Engineering*, vol. 43, pp. 785–819.

Chaniotis, A. K.; Poulikakos, D.; Koumoutsakos, P. (2002): Remeshed smoothed particle hydrodynamics. *Journal of Computational Physics,* vol. 182, pp. 67–90.

Chen, J. K.; Beraun, J. E.; Carney, T. C. (1999): A corrective smoothed particle method for boundary value problems in heat conduction. *International Journal for Numerical Methods in Engineering*, vol. 46, pp. 231–252.

Chen, J. K.; Beraun, J. E. (2000): A generalized smoothed particle hydrodynamics method for nonlinear dynamic problems. *Computer Methods in Applied Mechanics and Engineering*, vol. 190, pp. 225–239.

Gingold, R. A.; Monaghan, J. J. (1977): Smoothed particle hydrodynamics: Theory and application to non-spherical stars. *Monthly Notices of Royal Astronomical Society*, vol. 181, pp. 375–389.

Jiang, T.; Tang, Y. S.; Ren, J. L. (2014): A corrected 3D parallel SPH method for simulating the polymer free surface flows based on the XPP model. *CMES:Computer Modeling in Engineering & Sciences*, vol. 101, no. 4, pp. 249–297.

Johnson, G. R.; Stryk, R. A. Beissel, S. R. (1996): SPH for high velocity impact computations. *Computer Methods in Applied Mechanics and Engineering*, vol. 139, pp. 347–373.

Korzilius, S. P.; Schilders, W. H. A.; Anthonissen, M. J. H. (2013): An improved corrective smoothed particle method approximation for second-order derivatives. *8th International Spheric Workshop*, Trondhein, Norway, pp. June 4–6.

Lucy, L. B. (1977): Numerical approach to testing the fission hypothesis. *Astronomical Journal*, vol. 82, pp. 1013–1024.

Liu, M. B.; Liu, G. R. (2003): Smoothed Particle Hydrodynamics: A meshfree particle method. World Scientific, Singapore.

Liu, M. B.; Liu, G. R.; Lam, K. Y. (2003): A one dimensional meshfree particle formulation for simulating shock waves. *Shock Wave*, vol. 13, pp. 201–211.

Liu, M. B.; Liu, G. R. (2006): Restoring particle consistency in smoothed particle hydrodynamics. *Applied Numerical Mathematics*, vol. 56, pp. 19–36.

Liu, M. B.; Liu, G. R. (2010): Smoothed Particle Hydrodynamics (SPH): an Overview and Recent Developments. *Archives of Computational Methods in Engineering*, vol. 17, pp. 25–76.

Liu, M. B.; Chang, J. Z. (2010): Particle distribution and numerical stability in smoothed particle hydrodynamics method. *Acta Physica Sinica*, vol. 59, no. 6, pp. 3654–3662.

Mehra, V.; Chaturvedi, S. (2006): High velocity impact of metal sphere on thin metallic plates: a comparative smooth particle hydrodynamics study. *Journal of Computational Physics*, vol. 212, pp. 318–337.

Monaghan, J. J. (2012): Smoothed particle hydrodynamics and its diverse applications. *Annual Review of Fluid Mechanics*, vol. 44, pp. 323–346.

Oger, G.; Doring, M.; Alessandrini, B.; Ferrant, P. (2007): An improved SPH method: Towards higher order convergence. *Journal of Computational Physics*, vol. 225, pp. 1472–1492.

Parshikov, A. N.; Medin, S. A. (2002): Smoothed particle hydrodynamics using interparticle contact algorithms. *Journal of Computational Physics*, vol. 180, pp. 358–382.

Song, J. H.; Zhang, C. Y.; Liang, C. X.; Li, W. G. (2009): A New One dimensional Smoothed Particle in Simulating Discontinuous Hydrodynamics Method Problem. *Journal of Guangxi Normal Universityï¿½ï¿½Natural Science Edition*, vol. 27, no. 3, pp. 9–13.

Vignjevic, R.; Campbell, J. (1999): A penalty approach for contact in smoothed particle hydrodynamics. *International Journal of Impact Engineering*, vol. 23, pp. 945–956.

Xu, F.; Zhao, Y.; Yan, R.; Furukawa, T. (2013): Multidimensional discontinuous SPH method and its application to metal penetration analysis. *International Journal for Numerical Methods in Engineering*, vol. 93, pp. 1125–1146.

Yan, R.; Xu, F.; Zhang, Y. Q. (2013): Validation of DSPH method and its application to physical problems. *Explosion and shock waves*, vol. 33, no. 2, pp. 133-139.

Zhang, G. M.; Batra, R. C. (2004): Modified smoothed particle hydrodynamics method and its application to transient problems. *Computational Mechanic*, vol. 34, pp. 137–146.