On Improving the Celebrated Paris' Power Law for Fatigue, by Using Moving Least Squares

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Abstract: In this study, we propose to approximate the $a-n$ relation as well as the $da/dn-\Delta K$ relation, in fatigue crack propagation, by using the Moving Least Squares (MLS) method. This simple approach can avoid the internal inconsistencies caused by the celebrated Paris' power law approximation of the $da/dn-\Delta K$ relation, as well as the error caused by a simple numerical differentiation of the noisy data for $a-n$ measurements in standard fatigue tests. Efficient, accurate and automatic simulations of fatigue crack propagation can, in general, be realized by using the currently developed MLS law as the "fatigue engine" [da/dn versus ΔK], and using a high-performance "fracture engine" [computing the K-factors] such as the Finite Element Alternating Method.

In the present paper, the "fatigue engine" based on the present MLS law, and the "fracture engine" based on the SafeFlaw computer program developed earlier by the authors, in conjunction with the COTS software ANSYS, were used for predicting the total life of arbitrarily cracked structures.

By comparing the numerical simulations with experimental tests, it is demonstrated that the current approach can give excellent predictions of the total fatigue life of a cracked structure, while the celebrated Paris' Power Law may miscalculate the total fatigue life by a very large amount.

Keywords: crack growth rate, Paris' Power Law, Moving Least Squares.

1   Introduction

Modeling the fatigue behavior of cracked built-up structures is among the most important tasks for the structural integrity assessment of aircraft [Atluri (1998)]. In the past several decades, the development of high-performance computer modeling techniques has enabled the highly accurate computations of fracture mechanics

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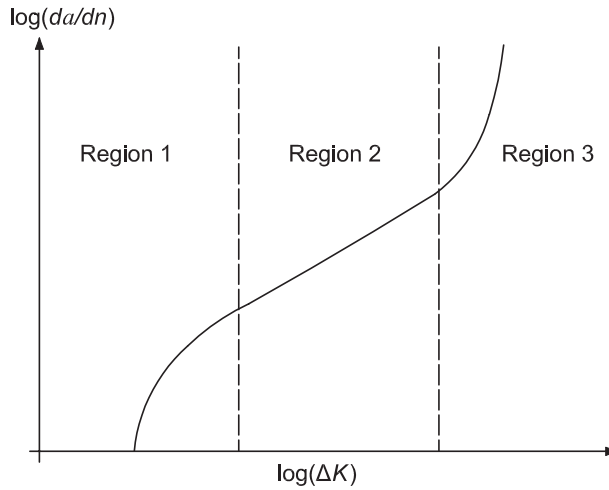[3] Vehicle Technology Directorate, US Army Research Laboratory.

Figure 1: Typical fatigue crack growth behavior for metals.

parameters of cracks, and efficient automatic simulations of non-collinear and non-planar mixed-mode crack growth in complex 2D & 3D structures. Among the best "fracture engines" which compute K-factors are the alternating methods developed by Atluri and co-workers, see [Nishioka and Atluri (1983); Wang and Atluri (1996); Park and Atluri (1998); Nikishkov, Park and Atluri (2001); Han and Atluri (2002); Dong and Aluri (2013a, 2013b)]. Such alternating methods have been embedded by the authors in software named SafeFlaw 2D & SafeFlaw 3D, which also simulate quite efficiently the non-planar mixed-mode crack propagation.

On the other hand, the development of fatigue crack growth laws, which relate the crack growth rate to fracture mechanics parameters (such as Stress Intensity Factors), is more likely to be an empirical art than a strict science. Having observed the linear dependence of $\log(da/dn)$ with respect to ΔK in a certain range of crack propagation, Paris et al. (1961) were the first to propose a power law relation between the crack growth rate and the range of SIF:

$$da/dn = C\Delta K^m. \tag{1}$$

As shown in Fig. 1, although the celebrated Paris' Law can give an acceptable approximation of the crack-growth-rates in the intermediate-SIF-range (region 2), it mostly overestimates da/dn in region 1 and underestimates da/dn in region 3. Forman et al. (1967) proposed to modify Paris' Law by further considering the

rapid crack growth near the fracture toughness K_{Ic}:

$$da/dn = \frac{C \Delta K^m}{(1-R) K_{Ic} - \Delta K},$$ (2)

where R is the stress ratio.

And Donahue et al. (1972) proposed to replace ΔK with $\Delta K - \Delta K_{th}$ in Paris Law, in order to give a better prediction of da/dn near the threshold SIF (ΔK_{th}):

$$da/dn = C (\Delta K - \Delta K_{th})^m.$$ (3)

While both the Forman and Donahue's formulations in Eqs. (2)-(3) apply thresholds to characterize the asymptotic behavior of $da/dn - \Delta K$, other studies also take into account the underlying mechanisms of plasticity, crack-closure, and even microstructure, see the review by [Newman (1998)]. For example, Elber (1970) proposed to use ΔK_{eff} instead of ΔK to account for the effect of the crack closure. And crack growth rates can also be related to elastic-plastic fracture parameters such as J and T^* integrals instead of linear elastic SIFs, see [Rice (1968); Atluri (1982); Nishioka and Atluri (1982); Dowling and Begley (1976); El Haddad, Dowling, Topper and Smith (1980)].

However, considering the computational capabilities of even a present-day laptop computer, and the vast number of numerical methods available today, it is probably unnecessary to confine the development of fatigue laws to within the limited scope of power laws or very simple formulas, no matter what fracture mechanics parameters are used ($\Delta K, \Delta K_{eff}, \Delta J, \Delta T^*$, etc.). Moreover, with given discrete measurements and the attendant noisy data for the crack length (a) versus the number of cycles (n), computing da/dn using a secant method or a piecewise polynomial method may also cause discretization errors, see ASTM E647-13a.

In this study, we propose to approximate the $a - n$ relation as well as the $da/dn - \Delta K$ relation using Moving Least Squares (MLS) [Atluri (2004)] instead of power laws. This simple approach can avoid the internal inconsistencies caused by the power law approximation of $da/dn - \Delta K$ relation as well as the errors caused by simple numerical differentiation of the noisy discrete $a - n$ data. By applying this method for fatigue life prediction of cracked structures, it is found that the current MLS law can give highly accurate estimates of the total fatigue life, while the Paris' Law may miscalculate the total fatigue life by several multiples or even an order of magnitude. Although the current study only considers ΔK as the driving parameter with the assumption of perfect linear elasticity, other effects such as plasticity and crack closure can also be included by using ΔK_{eff} as the driving parameter with numerical techniques developed by [Newman et al. (1992,1999)].

With further development, it is expected that the currently developed MLS fatigue law can serve as a high-performance "fatigue engine", which can be combined the high-performance "fracture engine" of the alternating methods, to greatly improve the state-of-the-art of structural integrity assessment and damage tolerance of fixed as well as rotary-wing aircraft.

2   A Simple Moving Least Squares Fatigue Law

2.1   *Fundamentals of Moving Least Squares*

The Moving Least Squares (MLS) is generally considered to be one of the best methods to interpolate random data with a high accuracy, because of its completeness, smoothness, and locality. Its formulation is briefly reviewed here, while more detailed discussions on the MLS can be found in [Atluri (2004, 2005)].

Considering a one-dimensional space with n being the independent variable and a being the dependent variable, the MLS method starts by expressing $a(n)$ as polynomials:

$$a(n) = \mathbf{p}^T(n)\mathbf{b}(n), \tag{4}$$

where $\mathbf{p}^T(n)$ represents the monomial basis. In this study, we use a second-order interpolation, so that $\mathbf{p}^T(n) = [1, n, n^2]$. $\mathbf{b}(n)$ is a vector containing the coefficient functions of each monomial basis, which can be determined by minimizing the following weighted least squares objective function:

$$
\begin{aligned}
J[\mathbf{b}(n)] &= \sum_{I=1}^{m} w^I(n)[\mathbf{p}^T(n^I)\mathbf{b}(n) - \hat{a}^I]^2 \\
&= [\mathbf{Pb}(n) - \hat{\mathbf{a}}]^T \mathbf{W}(n)[\mathbf{Pb}(n) - \hat{\mathbf{a}}],
\end{aligned}
\tag{5}
$$

where $n^I, I = 1, 2, \cdots, m$ is a group of discrete nodes with $\hat{a}^I$ being the fictitious nodal value at node I. Differing from the traditional Least Squares Method, the MLS weight function $w^I(n)$ is a local function which vanishes outside the support size (r^I) of node I. In this study, a fourth order spline weight function is used:

$$
w^I(n) = \left\{ \begin{array}{ll} 1 - 6r^2 + 8r^3 - 3r^4 & r \leq 1 \\ 0 & r > 1 \end{array} \right. , \quad r = \frac{|n - n^I|}{r^I}. \tag{6}
$$

Substituting $\mathbf{b}(n)$ into Eq. (4), we can obtain the approximate expression of $a(n)$ as:

$$
\begin{aligned}
a(n) &= \mathbf{p}^T(n)\mathbf{A}^{-1}(n)\mathbf{B}(n)\hat{\mathbf{a}} = \boldsymbol{\alpha}^T(n)\mathbf{B}(n)\hat{\mathbf{a}} \\
&= \boldsymbol{\Phi}^T(n)\hat{\mathbf{a}} = \sum_{I=1}^{m} \Phi^I(n)\hat{a}^I,
\end{aligned}
\tag{7}
$$

where matrices $\mathbf{A}(n)$ and $\mathbf{B}(n)$ are defined by:

$$\mathbf{A}(n) = \mathbf{P}(n)^T \mathbf{W}(n)\mathbf{P}(n), \quad \mathbf{B}(n) = \mathbf{P}^T(n)\mathbf{W}(n). \tag{8}$$

$\Phi^I(n)$ is named as the MLS basis function for node I.

The derivatives of the basis functions can be computed using the following equations:

$$\begin{aligned}
\mathbf{\Phi}_{,n} &= \boldsymbol{\alpha}_{,n}^T \mathbf{B} + \boldsymbol{\alpha}^T \mathbf{B}_{,n}, \\
\boldsymbol{\alpha}_{,n}^T &= \mathbf{A}^{-1}\left(\mathbf{p}_{,n}^T - \mathbf{A}_{,n}\boldsymbol{\alpha}\right).
\end{aligned} \tag{9}$$

To give an example, we consider 21 uniformly distributed nodes ($n = 0, 500, 1000, \ldots, 10000$), with 2000 being the support size of each node. Fig. 2 plots the MLS basis function for the node at $n = 5000$ as well as its first order derivative. As can be seen, both the basis function and its first-order derivative are smooth functions which vanish outside the support range of each node.
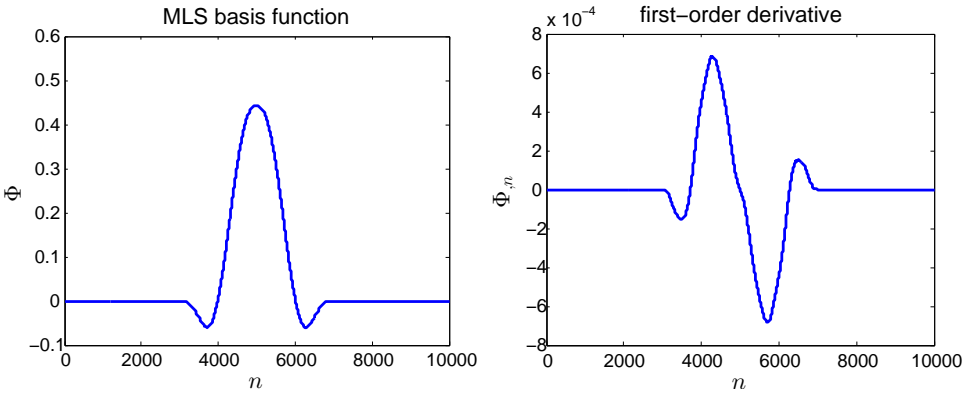

Figure 2: MLS basis function and its first order derivative.

2.2 Numerical Differentiation of Discrete $a - n$ Data Using MLS

Fatigue tests generally give discrete pairs of $a - n$ data which can often be noisy. Suppose one has measured crack sizes $a^1, a^2, \ldots, a^m$ at cycles $n^1, n^2, \ldots, n^m$, then by using $n^1, n^2, \ldots, n^m$ as MLS nodes, the $a - n$ relation can be approximated as a continuous function:

$$a(n) = \sum_{I=1}^{m} \Phi^I(n)\hat{a}^I, \tag{10}$$

where the fictitious nodal values $\hat{a}^1, \hat{a}^2, \ldots, \hat{a}^m$ and measured crack sizes $a^1, a^2, \ldots, a^m$ are related by[1]:

$$a^J = \sum_{I=1}^{m} \Phi^I \left(n^J \right) \hat{a}^I. \tag{11}$$

The crack growth rates can thus be computed by directly differentiating Eq. (10) with the aid of Eq. (9):

$$\frac{da}{dn}(n) = \sum_{I=1}^{m} \Phi^I_{,n}(n)\hat{a}^I. \tag{12}$$

In this study, we consider 7075-T6 aluminum alloy, and use the fatigue test data given in NASA/TM-2005-213907 report. Three test specimens were prepared following the ASTM E647-00 standard, i.e. AL-7-21, AL-7-22 and AL-7-23. A schematic plot of a middle tension M(T) specimen is given in Fig. 3, where $W = 102\ mm, B = 3.18\ mm$. Each specimen had different initial crack sizes, and was loaded with different magnitudes of forces, while having the same stress ratio ($R = 0.1$). Any data that satisfies $W - 2a \geq 1.25 P_{\max}/(B\sigma_{YS})$ were rejected to rule out the effects of inelastic material response. Fig. 4 gives the fitted $a - n$ curve as well as the thereafter derived crack growth rates da/dn.

2.3 *An Empirical* $da/dn - \Delta K$ *Relation by MLS Approximation*

In this study, ΔK is computed using the Finite Element Alternating Method, see [Park and Atluri (1998)]. It is found that computed SIFs almost exactly agree with the empirical formula suggested by ASTM E647-13a. The computed da/dn and ΔK at each crack increment from all the 3 tests are plotted in Fig. 5, which clearly demonstrates the same fatigue crack growth behavior as schematically shown in Fig. 1. A linear curve fitting in the intermediate-ΔK-range yields the celebrated Paris' Law, which however deviates from the measured crack growth rates as ΔK is near the threshold or near the fracture toughness values. In order to have a better prediction of da/dn, MLS is used to relate $\log(da/dn)$ to $\log(\Delta K)$. A log-log relation is retained in order to ensure that the predicted crack growth rates are positive.

However, a direct usage of all the measurements as nodal values for MLS is found to be undesirable. Firstly, the same ΔK from different tests may correspond to different da/dn. Secondly, because of the scattering of measured data, using a large number of MLS nodes leads to a local oscillatory behavior of $da/dn - \Delta K$ relation. Thus, it is proposed to use only a few nodes, and to have the fictitious nodal

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[1] Based on our experience, using measured crack sizes $a^I, I = 1, 2, \ldots, m$ directly, instead of $\hat{a}^I$, in Eq. (10) is simpler and makes very little difference in the fitted $a - n$ curve, because the values of a^I and $\hat{a}^I$ are very close.
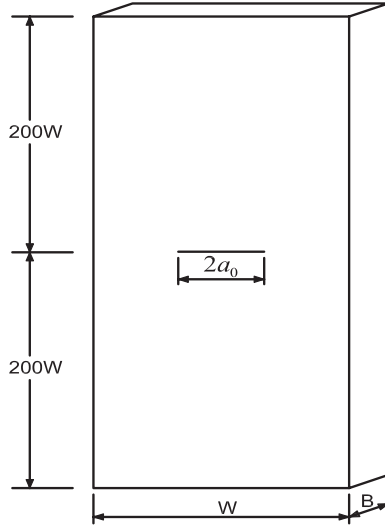
Figure 3: Schematic of a middle tension M(T) specimen by [Forth, Wright, and Johnson (2005)].

values determined by curve fitting. We consider that there are M set of measured data $\left(\Delta K^1, \frac{da}{dn}^1\right), \left(\Delta K^2, \frac{da}{dn}^2\right), ..., \left(\Delta K^M, \frac{da}{dn}^M\right)$ gathered from all the fatigue tests, and N MLS nodes $\log(\Delta K^{*1}), \log(\Delta K^{*2}), ... \log(\Delta K^{*N})$ are selected whose fictitious nodal values $\log\left(\widehat{\frac{da}{dn}}^{*1}\right), \log\left(\widehat{\frac{da}{dn}}^{*2}\right), ..., \log\left(\widehat{\frac{da}{dn}}^{*N}\right)$ are to be determined. The crack growth rate is thus related to the range of SIF by:

$$\log\left(\frac{da}{dn}\right) = \sum_{I=1}^{N} \Phi^I \left[\log(\Delta K)\right] \log\left(\widehat{\frac{da}{dn}}^{*I}\right). \tag{13}$$

And the fictitious nodal values can be determined by minimizing the following simple quadratic objective function:

$$J\left[\log\left(\widehat{\frac{da}{dn}}^{*I}\right)\right] = \sum_{J=1}^{M} \left[\sum_{I=1}^{N} \Phi^I \left[\log(\Delta K^J)\right] \log\left(\widehat{\frac{da}{dn}}^{*I}\right) - \log\left(\frac{da^J}{dn}\right)\right]^2 \tag{14}$$

In this study, 10 uniformly distributed (in the log scale) MLS nodes are used for demonstration, and the approximated $da/dn - \Delta K$ relation is plotted in Fig. 5. As can be seen, a very accurate prediction of crack-growth rates is obtained by using only a few MLS nodes, in contrast to the simple deviation of the Paris' Law.
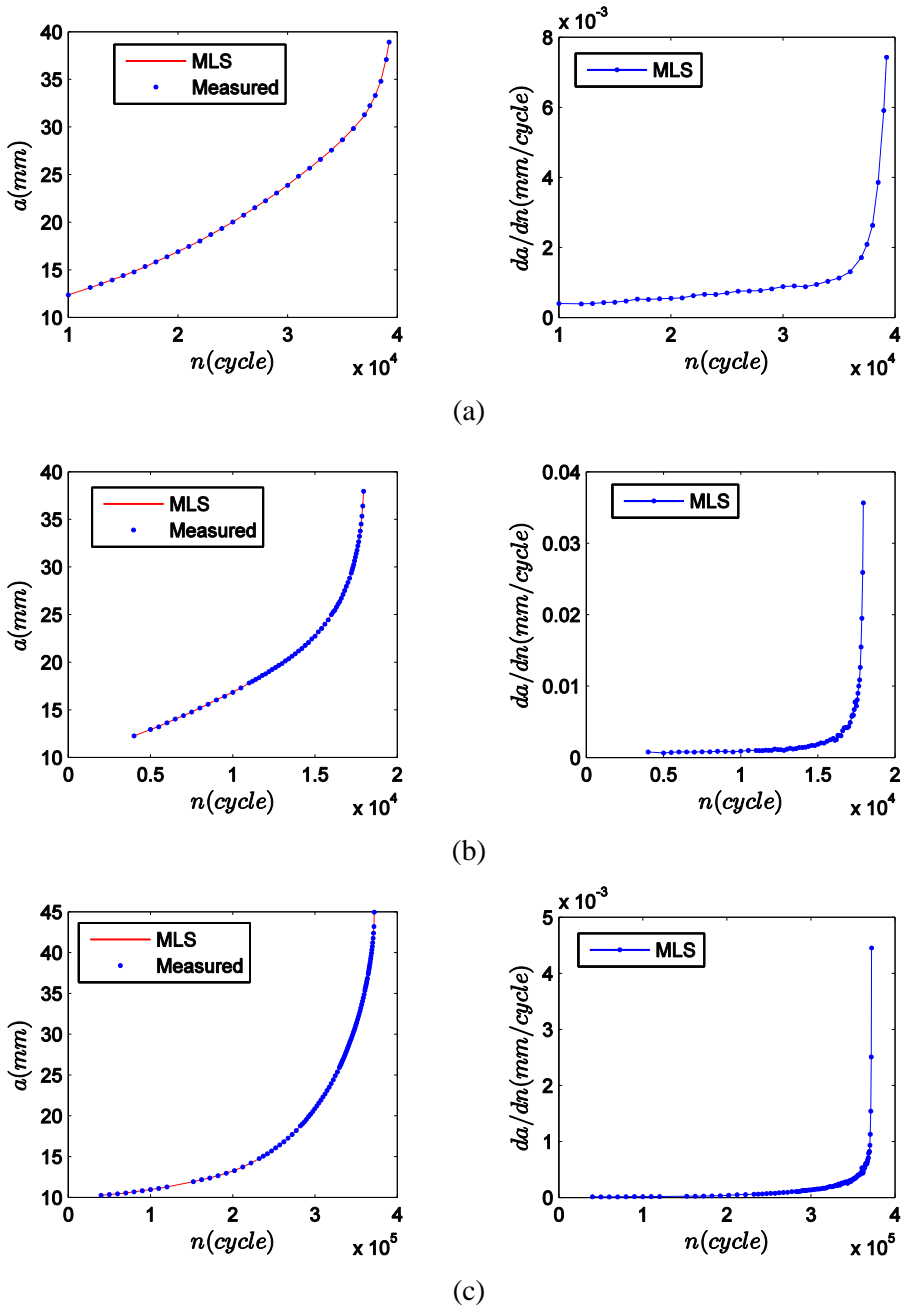
(a)

(b)

(c)

Figure 4: Using Moving Least Squares to numerically differentiate discrete $a - n$ data given by [Forth, Wright, and Johnson (2005)]: (a) AL-7-21, (b) AL-7-22, (c) AL-7-23.
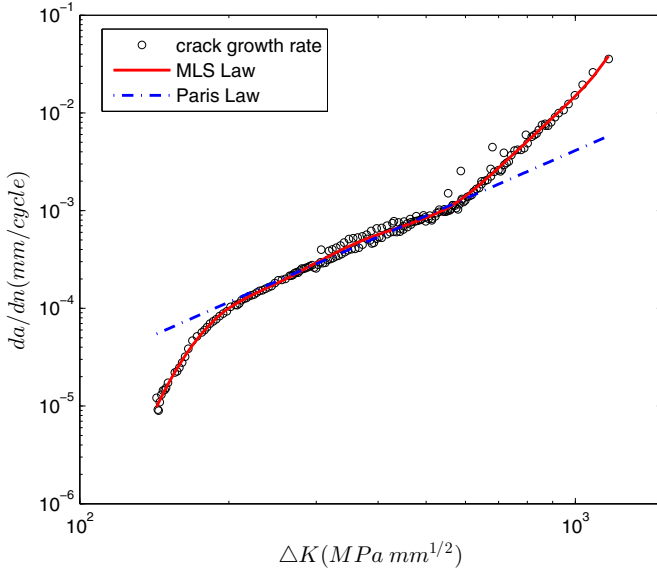
Figure 5: Approximating the $da/dn - \Delta K$ relation using Paris' Law as well as the Moving Least Squares (MLS).

3 Fatigue Crack Growth Simulation by the Currently Developed MLS Law

In this section, we verify the capability of a highly-accurate fatigue life assessment using the currently developed MLS fatigue law. This is done by implementing the MLS law into the well-established Finite Element Alternating Method [Park and Atluri (1998)]. The FEAM enables a fast and accurate computation of ΔK, as well as an automatic simulation of crack growth. With the newly developed MLS law, the fatigue life of a cracked structure can be computed by numerically evaluating:

$$n(a) = \int_{a_0}^{a} \frac{dn}{da} da = \int_{a_0}^{a} EXP \left\{ -\sum_{J=1}^{M} \Phi^I \left[\log(\Delta K) \right] \log \left(\widehat{\frac{da}{dn}}^{*I} \right) \right\} da, \tag{15}$$

where $EXP\{\cdot\}$ is the exponential function and a_0 is the initial crack size. Similarly, FEAM can also be combined with Paris' Law to simulate fatigue crack growth, the results of which are used for comparison in this study.

The present approach is firstly applied to model the fatigue behavior of M(T) specimens of NASA/TM-2005-213907 report used in section 2. As shown in Fig. 6-7, Paris' Law greatly underestimates the fatigue life of AL-7-23, this is because ΔK for this specimen is relatively small (near threshold region). Paris' Law also underestimates the crack growth rates of AL-7-22, where ΔK for this specimen is

relatively large (near fracture toughness region). In contrast, the MLS law can almost give an exact estimation of the fatigue behavior of both of these 2 specimens. Because ΔK of AL-72-21 stays in the intermediate power law range, both Paris' Law and MLS give good predictions of fatigue crack growth, and thus the results for AL-72-21 are not plotted here.
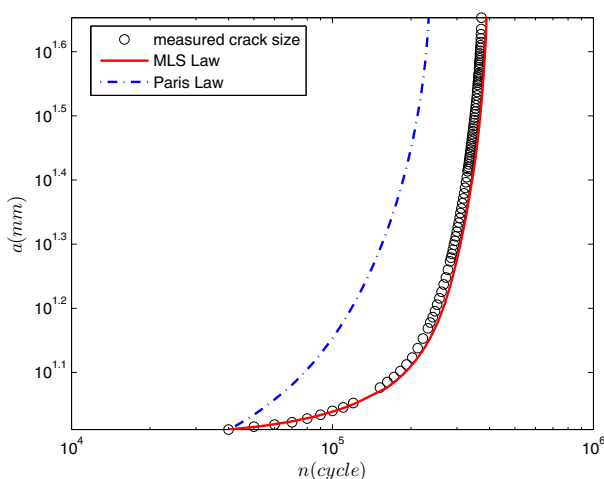

Figure 6: Predicted $a - n$ relation using Paris' Law and MLS Law, as compared to experimental results for specimen AL-7-23 in [Forth, Wright, and Johnson (2005)].

We further consider the open-hole crack experiment by [Stuart, Hill and Newman (2011)]. As shown in Fig. 8, each of the 3 dog-bone coupons is made of 2.03 *mm* thick 7075-T6 sheet. It has 381 *mm* in total length and is 88.9 *mm* wide at the gripped ends. The gripped ends taper to a central gage section (44.5 *mm* wide and 88.9 *mm* long) with a centrally located open hole which is 7.09 *mm* in diameter. The initial crack is 0.381 *mm* in size (including the notch & the pre-crack). The applied maximum gross tress is 47.2 *MPa*, with a stress ratio of 0.1.

This problem is solved by using the FEAM combined with Paris' as well as the present MLS fatigue law. As shown in Fig. 9, the Paris' Law underestimates the fatigue life by a factor of 3, while the MLS law can accurately simulate the fatigue growth of the crack near the open hole. Again, this is because the Paris' Law overestimates the crack growth rates in the near threshold region, while the currently developed MLS fatigue law gives very accurate predictions of crack growth rates in the threshold, intermediate, as well as the near fracture toughness regions.
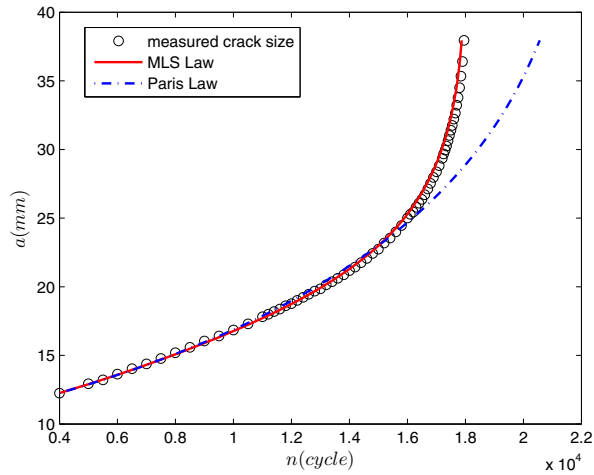
Figure 7: Predicted $a - n$ relation using Paris' Law and MLS Law, as compared to experimental results for specimen AL-7-22 in [Forth, Wright, and Johnson (2005)].
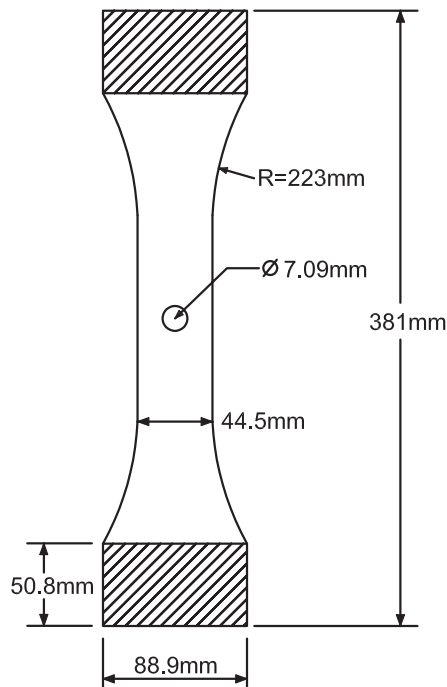

Figure 8: Open-hole Al 7075-T6 coupons used in [Stuart, Hill and Newman (2011)] with one single mode-1 crack at the right side of the hole.
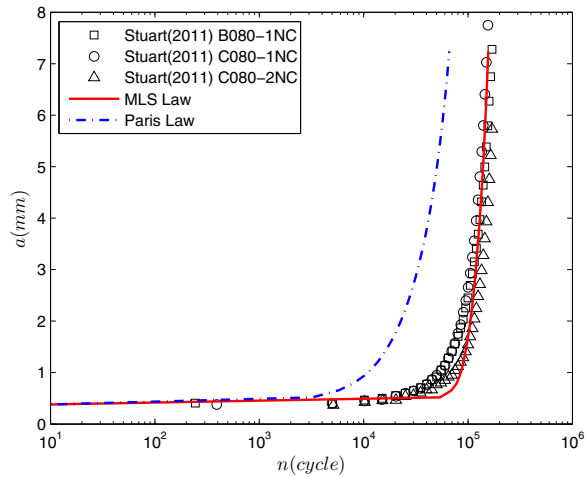
Figure 9: Predicted $a - n$ relation using Paris' Law and MLS Law, as compared to experimental results of 3 specimens in [Stuart, Hill and Newman (2011)].
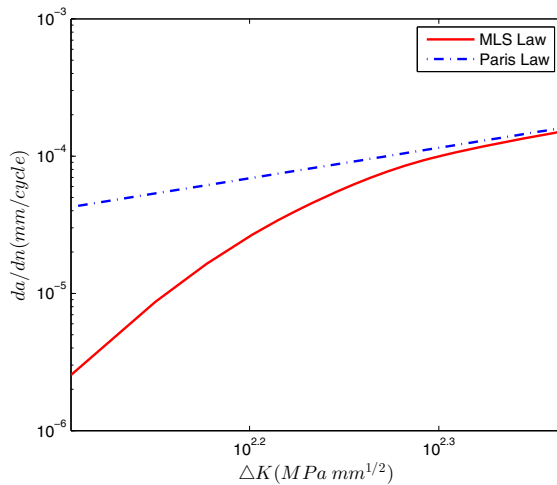

Figure 10: Predicted $da/dn - \Delta K$ relation using Paris's Law and MLS Law for the open-hole crack problem as given in [Stuart, Hill and Newman (2011)].

4 Conclusions

Using power laws such as the celebrated Paris' Law or other simple formulas to approximate the fatigue crack growth behavior was advantageous in the 1960s-1980s, which was the time when most of these fatigue laws were developed. However, with the rapid development of computers and numerical methods in the past half of a century, much better fatigue crack growth relations can be postulated today. This paper presents a preliminary demonstration of how much better predictions of crack growth rates and the total remaining fatigue life can be achieved by using the Moving Least Squares approximations. Other high-performance meshless approximating methods using Partitions of Unity, Shephard Functions, Radial Basis Functions, etc. [Atluri (2004)] will also be explored in the very near future. Moreover, effects of plasticity and crack closure can also be taken into account by simple corrections of $K-$factors, which will be considered in our future studies.

In the present paper, the "fatigue engine" based on the present MLS law, and the "fracture engine" based on the SafeFlaw computer program developed earlier by the authors, in conjunction with the COTS software ANSYS, were used for predicting the total life of arbitrarily cracked structures.

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