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Exact Solutions of Finite Deformation for Everted Compressible Hyperelastic Cylindrical Tubes

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Abstract: The eversion problem for a class of compressible hyperelastic thin-walled cylindrical tubes is examined. The mathematical model is formulated as a second-order nonlinear ordinary differential equation based on the theory of nonlinear elasticity. The exact solution that describes the mechanism of the finite deformation of the everted cylindrical tube is obtained. Using numerical simulations, it is shown that the initial thickness of the tube plays a significant role in the eversion.

Keywords: Compressible hyperelastic material, cylindrical tube, eversion, finite deformation, exact solution.

1 Introduction

It is well known that rubber and rubber-like materials, the typical representations of hyperelastic materials, are widely used in petrochemical, aerospace and many other fields of real life. The inflation, bending, torsion and eversion of hyperelastic solids are important research topics in nonlinear continuum mechanics, which may be seen in Beatty (1987), Fu (2001), Attard (2003) et al. Recently, Yuan et al. (2005 2006) researched the static and dynamic problem of the cavity formation, growth and motion in a hyperelastic sphere under a tensile load. The authors (2008) also examined the dynamic inflation problem for a cylindrical tube composed of a class of incompressible Ogden materials. Ren (2008) considered the instability for the inflation and deflation of a thin-walled spherical rubber balloon.

This paper mainly focuses on the eversion problem of a special compressible hyperelastic cylindrical tube. Significantly, this problem can be formulated as boundary value problem of a nonlinear differential equation. For incompressible materials,

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Rivlin (1949) and Chadwick et al (1972) originally investigated a series of incompressible hyperelastic everted cylindrical tubes, and the authors mainly discussed the existence and uniqueness of the cylindrical everted solutions. Haughton and Orr (1995) considered the eversion of incompressible cylindrical tubes composed of Ogden materials, the authors investigated the stability of the eversion problem with the initial thickness ratio as a parameter. Lin (2006) used the WKB method to analyse the buckling of an everted incompressible Varga spherical shell. Zhao et al. (2012) examined the finite deformation problem for an everted cylindrical tube composed of a class of neo-Hookean materials, the authors found that the effect of the initial thickness was essential. Note that the exact solutions of these problems always can be obtained by the incompressibility constraint. However, for compressible materials, the exact solutions do not always exist, they depend on the forms of the strain-energy functions strictly. Carroll and Horgan (1990) obtained an exact solution of the equilibrium equation described the eversion in the case of Blatz-Ko materials, but neither boundary conditions nor end conditions were considered. Then Haughton and Orr (1997) considered the eversion problems of several isotropic compressible hyperelastic cylindrical tubes, and found that some qualitative results were similar to those for incompressible materials. Moreover, the authors (2003) applied the WKB method to the bifurcation analysis of everted cylindrical and spherical shells composed of Varga materials. Erdemir and Carroll (2007) obtained the solutions for radial inflationcompaction and radial oscillation for the everted hollow spheres of harmonic and compressible Varga materials.

Carroll (1988) examined several deformation fields for a class of universal harmonic materials solids, such as spherical or cylindrical expansion and compaction, bending of a rectangular block, eversion of a spherical or cylindrical sector. The author applied the semi-inverse method and obtained the closed form solutions for the corresponding problems. In recent decades for this kind of materials, interest was revived by Murphy (1993, 2011), Carroll (2005). In this paper, a special case of the harmonic materials proposed by Carroll (2005) are considered. The aim of this paper is to investigate the finite deformation for the everted cylindrical tubes. Firstly, in the context of nonlinear elasticity, the mathematical model that describes radially symmetric deformation of the everted cylindrical tube is formulated as a second-order nonlinear ordinary differential equation. Then, the exact solution describing the finite deformation of the tube is obtained. Finally, numerical simulations show the effects of the initial thickness and the material parameter on the finite deformation.

2 Formulation and Solution

Here we are concerned with the exact solution describing the finite deformation of an everted cylindrical tube composed of a special case of the harmonic materials. Assume that the initial and the everted tubes occupy the following regions

$$0 < A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq Z \leq L, \tag{1}$$

$$0 < a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi, \quad -l \leq z \leq 0, \tag{2}$$

where $l > 0$. Interestingly, $r(A) = b$ and $r(B) = a$.

Under the assumption of axially symmetric deformation, the deformed configuration is given by

$$r = r(R), \quad \theta = \Theta, \quad z = -\lambda Z, \tag{3}$$

where $\lambda > 0$ is a constant to be determined. In this case, the principal stretches and the principal Cauchy stresses are as follows

$$\lambda_1 = -\frac{dr}{dR}, \quad \lambda_2 = \frac{r}{R}, \quad \lambda_3 = \lambda, \tag{4}$$

$$J\sigma_{ii} = \lambda_i W_i, \quad W_i = \partial W / \partial \lambda_i, \quad i = 1,2,3 \text{ no sum}, \tag{5}$$

where $W = W(\lambda_1, \lambda_2, \lambda_3)$ is the strain-energy function associated with a certain compressible hyperelastic material. In addition, for compressible hyperelastic materials, $\lambda_1\lambda_2\lambda_3 > 0$, this means that $r'(R) = dr/dR < 0$.

In the absence of body force, the equilibrium equations reduce to the following single equation

$$\frac{d\sigma_{11}}{dr} + \frac{1}{r}(\sigma_{11} - \sigma_{22}) = 0. \tag{6}$$

In terms of Eqs. (4) and (5), Eq.(6) can be rewritten as the following second-order nonlinear ordinary differential equation,

$$Rr''W_{11} + (\lambda_1 + \lambda_2)W_{12} - W_1 - W_2 = 0, \tag{7}$$

where $r'' = d^2r/dR^2$, $W_{ij} = \partial^2 W / \partial\lambda_i \partial\lambda_j (i, j = 1,2, i \neq j)$.

It is known that the strain energy function for a hyperelastic solid may be represented as a function of the principal invariants of stretch tensor, namely

$$W = W(i_1, i_2, i_3), \tag{8}$$

in which

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad i_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3, \quad i_3 = \lambda_1\lambda_2\lambda_3. \tag{9}$$

Carroll (1988) proposed three classes of strain energy functions based on the separation form of i_1, i_2, i_3, the form is as follows

$$W = W(i_1, i_2, i_3) = f(i_1) + g(i_2) + h(i_3), \tag{10}$$

where f, g and h are twice continuously differentiable functions. For each class, two of the functions are linear functions and the third is an arbitrary function. To investigate the implications of Shield's inverse deformation theorem for compressible finite elasticity, Carroll (2005) introduced two new classes of strain energy functions, including the following harmonic strain energy function

$$W(i_1, i_2, i_3) = 2\mu \left\{ f(i_1 - 3) - \zeta(i_2 - 3) - (1 - \zeta)(i_3 - 1) \right\}, \tag{11}$$

where μ is the shear modulus for infinitesimal deformation, $\zeta$is a non-dimensional parameter.

According to the linear constraint conditions for the above compressible material, it is easy to obtain

$$W(3,3,1) = f(0) - \zeta(3-3) - (1-\zeta)(1-1) = 0,$$

$$W_1(3,3,1) = 2\mu \left\{ f'(0) - 2\zeta - (1-\zeta) \right\} = 0,$$

$$W_{11}(3,3,1) = 2\mu f''(0) = \mu \frac{2(1-\upsilon)}{1-2\upsilon}, \tag{12}$$

where υ is the Poisson's ratio for infinitesimal deformation, f'denotes the derivative with respect to i_1.

It leads

$$f(0) = 0, f'(0) = 1 + \zeta, f''(0) = \frac{1-\upsilon}{1-2\upsilon}. \tag{13}$$

In terms of Eq. (13), here we take $f(i_1 - 3) = a_1(i_1 - 3) + a_2(i_1 - 3)^2$. So we assume that the strain energy function has the form

$$W(i_1, i_2, i_3) = 2\mu \left\{ a_1(i_1 - 3) + a_2(i_1 - 3)^2 - \zeta(i_2 - 3) - (1 - \zeta)(i_3 - 1) \right\}, \tag{14}$$

in which $a_1 = 1 + \zeta$, $a_2 = \frac{1-\upsilon}{1-2\upsilon}$.

Substituting Eq. (14) into the equilibrium equation (7), we obtain

$$r'' + \frac{r'}{R} - \frac{r}{R^2} = \frac{1}{R}\left(a_1/a_2 + (2 - \zeta/a_2)\lambda - 6\right). \tag{15}$$

Integrating twice produces the general solution

$$r(R) = \left(a_1/a_2 + \left(2 - \zeta/a_2\right)\lambda - 6\right)\left(\ln R - \frac{1}{2}\right)\frac{R}{2} + \frac{D_1}{2}R + \frac{D_2}{R}. \tag{16}$$

Since the inner and outer surfaces of the tube are traction-free, we get the following equations by using Eq. (16)

$$\begin{aligned} &\frac{B}{2}(-\zeta + (\zeta - 1)\lambda)D_1 + (4a_2 - \zeta + (\zeta - 1)\lambda)\frac{D_2}{B} + \\ &\frac{1}{4}(\zeta + (1 - \zeta)\lambda)G(\lambda)B + \frac{1}{2}(-\zeta + (\zeta - 1)\lambda)G(\lambda)B\ln B = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} &\frac{A}{2}(-\zeta + (\zeta - 1)\lambda)D_1 + (4a_2 - \zeta + (\zeta - 1)\lambda)\frac{D_2}{A} + \\ &\frac{1}{4}(\zeta + (1 - \zeta)\lambda)G(\lambda)A + \frac{1}{2}(-\zeta + (\zeta - 1)\lambda)G(\lambda)A\ln A = 0. \end{aligned} \tag{18}$$

Then we determine the integral constants D_1 and D_2, i.e.

$$D_1 = G(\lambda)\left(\frac{1}{2} + \frac{A^2\ln A - B^2\ln B}{B^2 - A^2}\right), \quad D_2 = \frac{1}{2}M(\lambda)G(\lambda)\frac{A^2B^2\ln(B/A)}{B^2 - A^2}. \tag{19}$$

where

$$G(\lambda) = a_1/a_2 + \left(2 - \zeta/a_2\right)\lambda - 6$$

and

$$M(\lambda) = (-\zeta + (\zeta - 1)\lambda)/(4a_2 - \zeta + (\zeta - 1)\lambda).$$

So the exact solution describing the finite deformation of the tube is given by

$$r(R) = \frac{G(\lambda)}{2}\left(R\ln(R/B) + R\frac{A^2\ln\left(A/B\right)}{B^2 - A^2} - \frac{M(\lambda)}{R}\frac{A^2B^2\ln\left(A/B\right)}{B^2 - A^2}\right), \tag{20}$$

where λ is to be determined.

Supposed that resultant load on the ends is zero, we have the following end condition proposed by Rivlin (1949)

$$N = 2\pi\int_a^b r\sigma_{33}dr = 0. \tag{21}$$

For convenience, we introduce the following notations

$$\delta = \frac{A}{B}, \quad m = \frac{R}{B}. \tag{22}$$

Applying the variable transformation, the end condition is rewritten as

$$\int_1^\delta \kappa\sigma_{33} r' dm = 0, \tag{23}$$

where

$$\kappa = \frac{r(R)}{B} = \frac{G(\lambda)}{2}\left(m\ln m + \left(m - \frac{M(\lambda)}{m}\right)\frac{\delta^2\ln\delta}{1-\delta^2}\right),$$

$$r'(R) = \frac{G(\lambda)}{2}\left(\ln m + 1 + \left(1 + \frac{M(\lambda)}{m^2}\right)\frac{\delta^2\ln\delta}{1-\delta^2}\right),$$

$$\sigma_{33} = \frac{1}{\lambda_1\lambda_2}W_3 = 2\mu\left(\frac{(2a_2(\lambda-3)+1+\zeta)m}{-r'\kappa} + (2a_2-\zeta)\left(\frac{m}{\kappa} - \frac{1}{r'}\right) - 1 + \zeta\right),$$

Using Eq. (20), we can have

$$\frac{a}{B} = \frac{r(B)}{B} = \frac{G(\lambda)}{2}(1-M(\lambda))\frac{\delta^2\ln\delta}{1-\delta^2}, \tag{24}$$

$$\frac{b}{B} = \frac{r(A)}{B} = \frac{G(\lambda)}{2}\left(\delta\ln\delta + \left(\delta - \frac{M(\lambda)}{\delta}\right)\frac{\delta^2\ln\delta}{1-\delta^2}\right). \tag{25}$$

Then it leads

$$\frac{a}{b} = \delta. \tag{26}$$

That is to say, the thickness of the cylindrical tube is maintained after eversion.

From Eq. (23), we can get the relation between λ and δ. Substituting λ and δ into Eq. (20), we have the relations among the axial stretch rate, the initial thickness, and the inner (outer) radius.

Since the initial configuration of the tube is natural, the corresponding total energy is zero. It is necessary to compare the total potential energy of the tube after eversion. So we now carry out an energy analysis. For the compressible hyperelastic material (13), the total potential energy of the everted tube is given by

$$\begin{aligned} E &= 2\pi L\int_A^B WRdR \\ &= 4\pi\mu LB^2\int_\delta^1 \Big\{a_1(-r' + \frac{\kappa}{m} + \lambda - 3) + a_2(-r' + \frac{\kappa}{m} + \lambda - 3)^2 \\ &+ \zeta(r'\frac{\kappa}{m} + r'\lambda - \frac{\kappa}{m}\lambda + 3) + (1-\zeta)(r'\frac{\kappa}{m}\lambda + 1)\Big\} mdm. \end{aligned} \tag{27}$$

From Eq. (27) combining Eq. (23), we can get the relation between E and δ.

3 Numerical simulations

Figs.1-6 show the effects of the initial thickness δ, the material parameter ζ and the Possion's ratio υ on the finite deformation of the everted cylindrical tube.

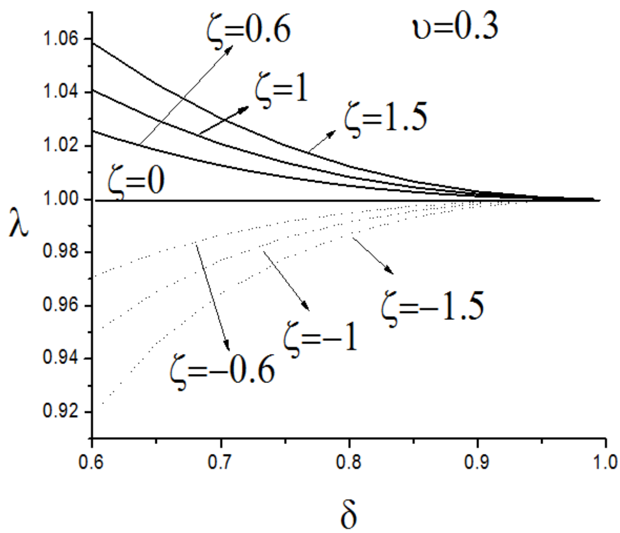

Figure 1: Curves of λ vs δ for various values of ζ.

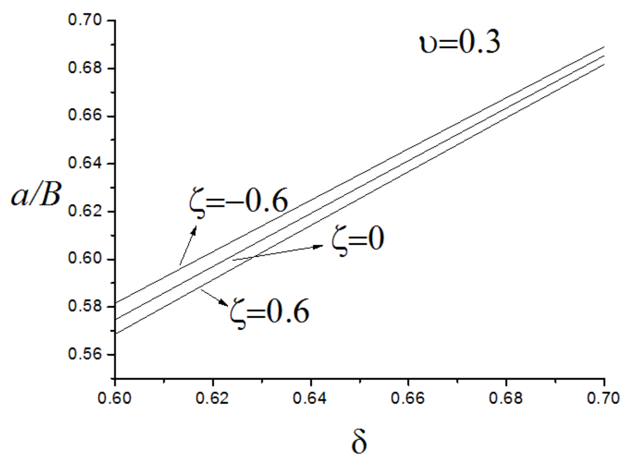

Figure 2: Curves of a/B vs δ for different values of ζ.

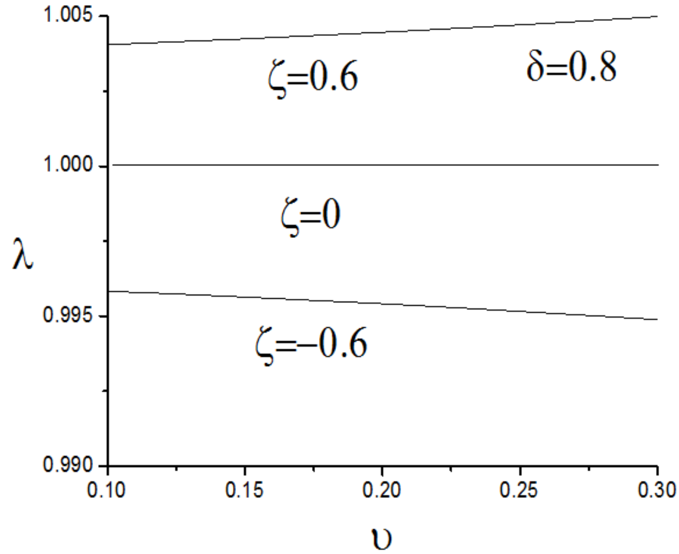

Figure 3: Curves of λ vs υ for different values of ζ.

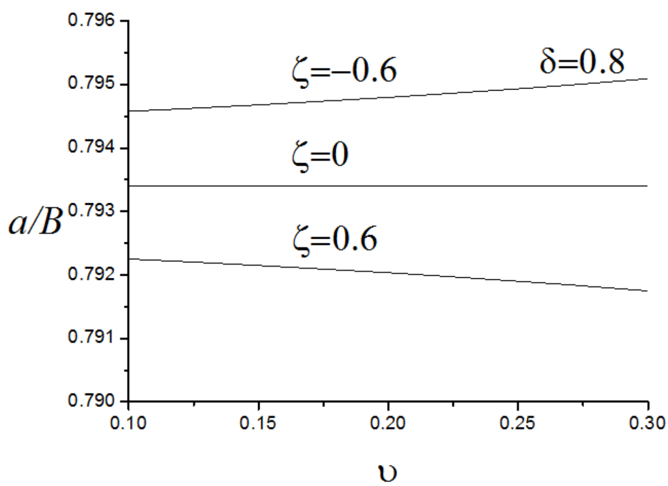

Figure 4: Curves of a/B vs υ for different values of ζ.

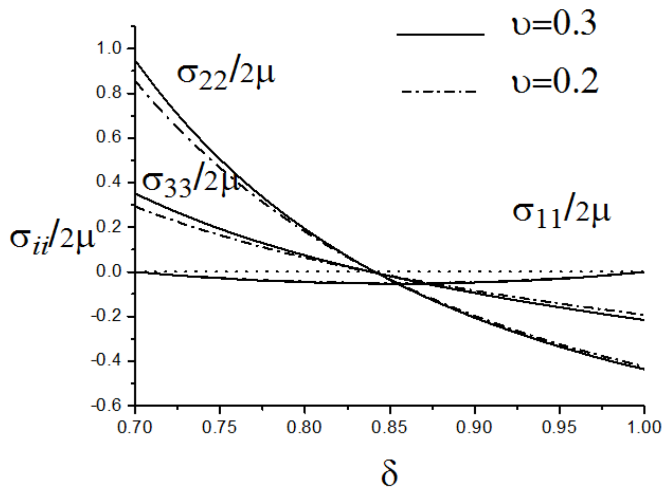

Figure 5: The stress distributions for various values of υ.

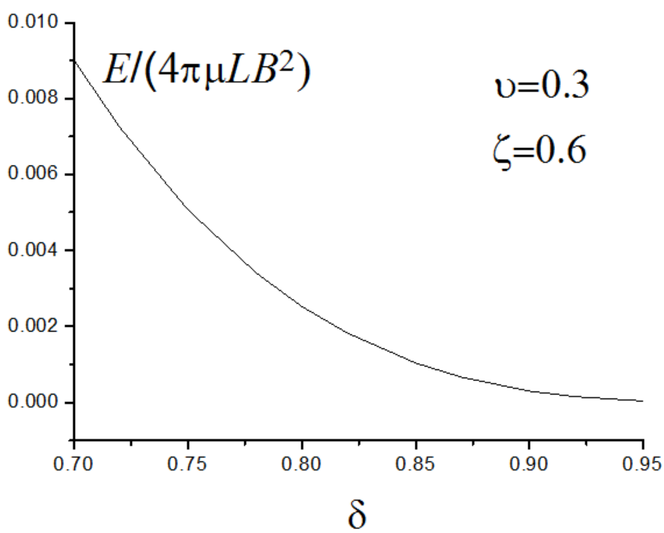

Figure 6: The total potential energy for various values of δ.

For the given value of υ, as shown in Figs.1, 2, if the material parameter $\zeta > 0$, the axial stretch rate λ decreases with the initial thickness δ, and $\lambda > 1$, if $\zeta < 0$, the axial stretch rate λ increases with the initial thickness δ, and $\lambda < 1$ if $\zeta = 0$ the axial stretch rate λ is maintained. The inner radius of the everted cylindrical tube increases with the initial thickness δ. In Figs.3, 4, for the given values of δ and ζ, it is shown that the Possion's ratio υ do not influence the inner radius and the axial stretch rate essentially.

Fig.5 shows the stress distributions. It is easy to see that $\sigma_{11} \leq 0$ throughout the everted tube and is zero at the surfaces of the tube which coincides with the boundary conditions. σ_{22} and σ_{33} decrease with the increasing initial thickness δ. The behaviors of the stresses are similar to those for the compressible materials obtained by Haughton and Orr (1997). Fig.6 shows the total potential energy corresponding to deformed equilibrium configuration of the tube. It can be seen that the eversion is an absorbing energy process, and the total energy decreases with the increasing initial thickness δ. Moreover, by the numerical results in Figs. 5, 6, it is also shown that the influences of the Possion's ratio υ on the everted stresses are significant, however, the influences of the material parameter ζ on the everted stresses and the total potential energy are not obvious.

4 Conclusions

In this work, the finite deformation of an everted thin-walled cylindrical tube composed of a class of compressible hyperelastic materials is examined. The results reveal that

1. The thickness of the cylindrical tube is maintained after eversion.

2. The influence of the material parameter ζ on the axial stretch is obvious, i.e. if $\zeta > 0$, the thinner the initial cylindrical tube is, the smaller the axial expansion is. If $\zeta < 0$, the thinner the initial cylindrical tube is, the smaller the axial compaction is. If $\zeta = 0$, the axial stretch is sustained.

3. The influence of the Possion's ratio υ on the everted thickness is not obvious, however, on the stress distributions is significant.

4. The initial inner portion of the tube is subjected to an axial tension and the initial outer portion to an axial compression.

5. The thicker the initial cylindrical tube is, the more the total absorbing energy is.

In particular, the results by numerical simulations are qualitatively similar to those for incompressible materials.

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