 CMC, vol.40, no.3, pp.179-201, 2014

Effect of Gravitational Field and Temperature Dependent Properties on Two-Temperature Thermoelastic Medium with Voids under G-N Theory

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Abstract: This investigation is aimed to study the two dimensional problem of thermoelastic medium with voids under the effect of the gravity. The modulus of elasticity is taken as a linear function of the reference temperature and employing the two-temperature generalized thermoelasticity. The problem is studied in the context of Green-Naghdi (G-N) theory of types II and III. The normal mode analysis method is used to obtain the exact expressions for the physical quantities which have been shown graphically by comparison between two types of the (G-N) theory in the presence and the absence of the gravity, the temperature dependent properties and the two-temperature effect.

Keywords: Energy dissipation, Gravity, Green-Naghdi theory, Normal mode analysis, Temperature dependent, Thermoelasticity, Two-Temperature, Voids.

1 Introduction

The generalized thermoelasticity theories have been developed with the aim of removing the paradox of infinite speed of heat propagation inherent in the classical coupled dynamical thermoelasticity theory investigated by Biot (1956). In the generalized theories, the governing equations involve thermal relaxation times and they are of hyperbolic type. The extended thermoelasticity theory by Lord and Shulman (1967) which introduces one relaxation time in the thermoelastic process and the temperature-rate-dependent theory of thermoelasticity by Green and Lindsay (1972) which takes into account two relaxation times are two well established generalized theories of thermoelasticity. Recently, Green and Naghdi (1991, 1992 and 1993) developed a generalized theory of thermoelasticity which involves thermal displacement gradient as one of the constitutive variables in contrast to the classical coupled thermoelasticity which includes temperature gradient as one of

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the constitutive variables. An important feature of this theory is that it does not accommodate dissipation of thermal energy. On this theory the characterization of the material response for a thermal phenomenon is based on three types of the constitutive response functions. The nature of those three types of constitutive response functions is such that when the respective theories are linearized, type I is the same as the classical heat conduction equation (based on Fourier's law), whereas type II, the internal rate of production of entropy is taken to be identically zero, implying no dissipation of thermal energy. This model is known as the theory of thermoelasticity without energy dissipation. Type III involves the previous two models as special cases, and admits dissipation of energy in general, in this model, introducing the temperature gradient and thermal displacement gradient as the constitutive variables. Chandrasekharaiah (1996a, 1996b) solved some problems in thermoelasticity without energy dissipation. Sharma and Chauhan (1999) investigated a problem concerning thermoelastic interactions without energy dissipation due to body forces and heat sources. Othman and Song (2007) have investigated a reflection phenomenon of the plane waves from an elastic solid half-space under hydrostatic initial stress without energy dissipation. Othman et al. (2013a) studied the temperature dependence and the rotation on generalized thermoelasticity with voids under (G-N) theory.

Theory of elastic materials with voids is one of the most important generalizations of the classical theory of elasticity. This theory is concerned with elastic materials consisting of a distribution of small porous (voids) in which the void volume is included among the kinematic variables. Practically, this theory is useful for investigating various types of the geological and the biological materials for which elastic theory is inadequate. A nonlinear theory of elastic material with voids was developed by Nunziato and Cowin (1979). Cowin and Nunziato (1983) developed a theory of linear elastic materials with voids. Puri and Cowin (1985) studied the behavior of the plane waves in a linear elastic material with voids. The domain of influence theorem in the linear theory of elastic materials with voids was discussed by Dhaliwal and Wang (1994). Dhaliwal and Wang (1995) developed a heat flux dependent theory of thermoelasticity with voids. Ciarletta and Scarpetta (1995) discussed some results on thermoelasticity for dielectric materials with voids. Marin in (1997a, 1997b) studied uniqueness and domain of influence results in thermoelastic bodies with voids.

The effect of gravity on the wave propagation in an elastic solid medium was first considered by Bromwich in (1898), treating the force of gravity as a type of body force. Sezawa in (1927) studied the dispersion of elastic waves propagated on curved surfaces. In (1965) Love extended the work of Bromwich which investigated the influence of gravity on superficial waves and showed that the Rayleigh wave velocity is affected by the gravity field. Recently Othman et al. (2013b, 2013c and 2014) and Othman and Lotfy (2013) have studied many problems using the effect of the gravitational field on thermoelasticity.

Material properties, such as the modulus of elasticity and the thermal conductivity, may be affected by temperature dependent. The temperature dependence of the material properties is neglected when the temperature variation from the initial temperature is low, while the temperature dependence of the material properties is considered when the temperature changes very high. The reactor vessels, turbine engines, space vehicles and refractory industries are affected by high temperature changes. If the temperature dependence of material properties is neglected, this is due to significant errors as discussed by Noda in (1986). Othman and Song in (2008) studied the reflection of the magneto-thermoelastic waves with two relaxation times under the effect of temperature dependent elastic moduli. In (2011) Othman discussed the state-space approach to the generalized thermoelastic problem with the temperature dependent properties and internal heat sources.

The two temperature theory of thermoelasticity proposes that the heat conduction in deformable media depends upon two distinct temperatures, the conducting temperature θ and the thermodynamic temperature T according to Chen and Gurtin in (1968) also Chen and Gurtin (1969) and. While under certain conditions, these two temperatures can be equal, in time independent problems, however, in particular those involving wave propagation θ and T are generally distinct according to Warren and Chen in (1973). Youssef in (2006) studied the theory of the two-temperature generalized thermoelasticity. The propagation of harmonic plane waves in the media described by the two-temperature theory of thermoelasticity is investigated by Puri and Jordan in (2006).

The present article is proposed to determine the components of displacement, the stresses, the temperature distribution and the volume fraction field in a homogenous, linear, isotropic, thermoelastic solid with voids in the case of absence and presence of the gravity, the temperature dependent and the two temperature effects. The model was illustrated in the context of (G-N) theory of types II and III. The normal mode analysis is used to obtain the exact expressions for physical quantities. The distributions of considered variables are represented graphically.

2 Formulation of the problem and basic equations

Consider a linear homogeneous isotropic thermoelastic medium with voids and a half-space ($y \geq 0$) the rectangular Cartesian coordinate system (x, y, z) having originated on the surface $z = 0$. For two dimensional problem we assume the dynamic displacement vector as $\mathbf{u} = (u,\ v,\ 0)$. All quantities considered will be a function of the time variable t, and of the coordinates x and y, the governing equations in the displacement and thermal fields in the absence of body forces and heat sources under the two-temperature generalized thermoelasticity theory as Youssef in (2006).

Following Green and Naghdi in (1993), Cowin and Nunziato (1983) the field equations and constitutive relations for a rotating linear homogenous, isotropic generalized thermoelastic solid with voids without body forces, heat sources and extrinsic equilibrated body force under the two-temperature generalized thermoelasticity theory in the context of (G-N) theory of type III, then the basic governing equations of a linear thermoelastic medium with voids under influence of gravitational field and two-temperature will be

$$\mu\,\nabla^2 u + (\lambda+\mu)\,\frac{\partial e}{\partial x} + b\,\frac{\partial \phi}{\partial x} - \beta\,\frac{\partial T}{\partial x} + \rho\, g\frac{\partial v}{\partial x} = \rho\,\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

$$\mu\,\nabla^2 v + (\lambda+\mu)\,\frac{\partial e}{\partial y} + b\,\frac{\partial \phi}{\partial y} - \beta\,\frac{\partial T}{\partial y} - \rho\, g\frac{\partial u}{\partial x} = \rho\,\frac{\partial^2 v}{\partial t^2}, \tag{2}$$

$$\alpha\,\nabla^2\phi - b\,e - \xi_1\,\phi - \omega_0\frac{\partial \phi}{\partial t} + m\,T = \rho\,\psi\frac{\partial^2 \phi}{\partial t^2}, \tag{3}$$

$$K\nabla^2\theta + K^*\frac{\partial}{\partial t}\nabla^2\theta - mT_0\,\frac{\partial \phi}{\partial t} = \rho\, C_e\frac{\partial^2 T}{\partial t^2} + \beta\, T_0\,\frac{\partial^2 e}{\partial t^2}. \tag{4}$$

$$\sigma_{ij} = [\lambda\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + b\,\phi - \beta\, T]\delta_{ij} + \mu\,(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}), \quad i,j,k = 1,2,3. \tag{5}$$

The strain-displacement relations

$$e_{ij} = \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}), \tag{6}$$

The thermodynamic temperature, T is related to the conductive temperature, θ as

$$T = (1 - d\,\nabla^2)\,\theta. \tag{7}$$

Where λ, μ are the lame's constants, $\alpha, b, \xi_1, \omega_0, m, \psi$ are the material constants due to presence of voids, $\beta = (3\lambda + 2\mu)\alpha_t$ such that α_t is the coefficient of thermal expansion, ρ is the density, C_e is the specific heat, K is the thermal conductivity, K^* is the material constant characteristic of the theory, T_0 is the reference temperature chosen so that $\left|(T-T_0)/T_0\right| << 1$, ϕ is the change in the volume fraction field, σ_{ij} are the components of the stress tensor, $\delta_{ij}$is the Kronecker delta, g is the acceleration due to the gravity, d is the two temperature parameter, When $K^* \to 0$ then (4) reduces to the heat conduction equation in (G-N) theory (of type II).

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \quad , \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

To investigate the effect of the temperature dependent properties on thermoelastic medium with voids, therefore we assume that

$$\lambda = \lambda_0 f(T), \quad \mu = \mu_0 f(T), \quad \beta = \beta_0 f(T), \quad \alpha = \alpha_0 f(T), \quad \omega_0 = \omega_{10} f(T),$$
$$\xi_1 = \xi_{10} f(T), \quad \psi = \psi_0 f(T), \quad m = m_0 f(T), \quad K = K_0 f(T), \quad b = b_0 f(T). \tag{8}$$

Where $\lambda_0, \mu_0, \beta_0, \alpha_0, \omega_{10}, \xi_{10}, \psi_0, m_0, k_0, b_0$ are constants, $f(T)$ is a given non-dimensional function of temperature. In the case of a temperature independent modulus of elasticity, $f(T) = 1$, such that $f(T) = (1 - \alpha^* T_0)$, where α^* is called the empirical material constant, in the case of the reference temperature independent of modulus of elasticity and thermal conductivity $\alpha^* = 0$. The governing equation can be put into a more convenient form by using the following non-dimensional variables

$$\begin{aligned}
&x' = \frac{\omega_1^*}{c_1} x, \quad y' = \frac{\omega_1^*}{c_1} y, \quad u' = \frac{\omega_1^*}{c_1} u, \quad v' = \frac{\omega_1^*}{c_1} v, \sigma'_{ij} = \frac{\sigma_{ij}}{\mu_0},\\
&T' = \frac{T}{T_0}, \quad \theta' = \frac{\theta}{T_0}, \quad \phi' = \frac{\omega_1^{*2} \psi_0}{c_1^2} \phi, \quad t' = \omega_1^* t, \quad P'_1 = \frac{P_1}{\mu_0},\\
&P'_2 = \frac{P_2}{T_0}, g' = \frac{g}{c_1 \omega_1^*}, c_1^2 = (\frac{\lambda_0 + 2\mu_0}{\rho}), \quad \text{and} \quad \omega_1^* = \frac{\rho C_e c_1^2}{k}.
\end{aligned} \tag{9}$$

In terms of non-dimensional quantities defined in equation (9) the governing equations (1) - (4) reduce to (dropping the prime for convenience)

$$\nabla^2 u + a_1 \tfrac{\partial e}{\partial x} + a_2 \tfrac{\partial \phi}{\partial x} - a_3 \tfrac{\partial}{\partial x}(1 - a_{12} \nabla^2)\, \theta + a_4 \tfrac{\partial v}{\partial x} = a_5 \tfrac{\partial^2 u}{\partial t^2}, \tag{10}$$

$$\nabla^2 v + a_1 \tfrac{\partial e}{\partial y} + a_2 \tfrac{\partial \phi}{\partial y} - a_3 \tfrac{\partial}{\partial y}(1 - a_{12} \nabla^2)\, \theta - a_4 \tfrac{\partial u}{\partial x} = a_5 \tfrac{\partial^2 v}{\partial t^2}, \tag{11}$$

$$\nabla^2 \phi - a_6\, e - a_7\, \phi - a_8 \frac{\partial \phi}{\partial t} + a_9 (1 - a_{12} \nabla^2)\, \theta = a_{10} \frac{\partial^2 \phi}{\partial t^2}, \tag{12}$$

$$\varepsilon_3 \nabla^2 \theta + \varepsilon_2 \tfrac{\partial}{\partial t} \nabla^2 \theta - a_{11} \tfrac{\partial \phi}{\partial t} = \tfrac{\partial^2}{\partial t^2}(1 - a_{12} \nabla^2)\, \theta + \varepsilon_1 \tfrac{\partial^2 e}{\partial t^2}. \tag{13}$$

Where $\varepsilon_1, \varepsilon_2$, and ε_3 are the coupling constants. Assuming the scalar potential function $\psi_1(x, y, t)$ and the vector potential function $\psi_2(x, y, t)$ in dimensionless form

$$u = \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y}, \quad \text{and} \quad v = \frac{\partial \psi_1}{\partial y} - \frac{\partial \psi_2}{\partial x}. \tag{14}$$

$$e = \nabla^2 \psi_1, \quad \text{and} \quad (\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}) = \nabla^2 \psi_2 \tag{15}$$

Using equation (15) in equations (10) - (13) to obtain

$$[b_1 \nabla^2 - a_5 \frac{\partial^2}{\partial t^2}]\, \psi_1 - a_4 \frac{\partial}{\partial x}\, \psi_2 + a_2\, \phi - a_3\, (1 - a_{12} \nabla^2)\, \theta = 0, \tag{16}$$

$$a_4 \frac{\partial}{\partial x}\, \psi_1 + [\nabla^2 - a_5 \frac{\partial^2}{\partial t^2}]\, \psi_2 = 0, \tag{17}$$

$$-a_6\, \nabla^2 \psi_1 + [\nabla^2 - a_7 - a_8 \frac{\partial}{\partial t} - a_{10} \frac{\partial^2}{\partial t^2}] \frac{\partial}{\partial t} + a_9\, (1 - a_{12} \nabla^2)\, \theta = 0, \tag{18}$$

$$-\varepsilon_1 \frac{\partial^2}{\partial t^2} \nabla^2 \psi_1 - a_{11} \frac{\partial \phi}{\partial t} + (\varepsilon_3 + \varepsilon_2 \frac{\partial}{\partial t}) \nabla^2 \theta - \frac{\partial^2}{\partial t^2} (1 - a_{12} \nabla^2)\, \theta = 0. \tag{19}$$

The components of stress tensor are

$$\sigma_{xx} = a_{13}\, [\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}] + 2 f(T) \frac{\partial u}{\partial x} + a_{14}\, \phi - a_{15}\, T, \tag{20}$$

$$\sigma_{yy} = a_{13}\, [\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}] + 2 f(T) \frac{\partial v}{\partial y} + a_{14}\, \phi - a_{15}\, T, \tag{21}$$

$$\sigma_{zz} = a_{13}\, [\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}] + a_{14}\, \phi - a_{15}\, T, \tag{22}$$

$$\sigma_{xy} = f(T)\, [\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}], \tag{23}$$

$$\sigma_{xz} = \sigma_{yz} = 0. \tag{24}$$

3 Normal mode analysis

The solution of the considered physical quantities can be decomposed in terms of the normal mode as the following form

$$[\psi_1, \psi_2, \phi, \theta, T, \sigma_{ij}](x, y, t) = [\psi_1^*, \psi_2^*, \phi^*, \theta^*, T^*, \sigma_{ij}^*](y) \exp(\omega\, t + i\, a\, x), \tag{25}$$

Where $[\psi_1^*, \psi_2^*, \phi^*, \theta^*, T^*, \sigma_{ij}^*](y)$ are the amplitude of the physical quantities, ω is the angular frequency, i $= \sqrt{-1}$ and a is the wave number in the x- direction.
Using (25) then (16) - (19) take the form

$$[D^2 - b_3]\, \psi_1^* - b_4 \psi_2^* + b_5\, \phi^* - [-b_7 (\mathrm{D}^2 - a^2) + b_6] \theta^* = 0, \tag{26}$$

$$b_2 \psi_1^* + [D^2 - b_8]\, \psi_2^* = 0, \tag{27}$$

$$-a_6 [D^2 - a^2]\, \psi_1^* + [D^2 - b_9]\, \phi^* + [a_9 - b_{10} (D^2 - a^2)] \theta^* = 0, \tag{28}$$

$$-b_{14} [D^2 - a^2]\, \psi_1^* - b_{15} \phi^* + [D^2 - b_{16}]\, \theta^* = 0. \tag{29}$$

Eliminating ψ_2^*, ϕ^* and T^* between (26) - (29), we obtain the differential equation

$$[D^8 - AD^6 + BD^4 - CD^2 + E]\ \psi_1^*(y) = 0, \tag{30}$$

In a similar manner we arrive at

$$[D^8 - AD^6 + BD^4 - CD^2 + E]\{\ \psi_1^*(y),\ \psi_2^*(y),\ \phi^*(y),\ \theta^*(y)\} = 0. \tag{31}$$

Equation (31) can be factored as

$$[(D^2 - k_1^2)(D^2 - k_2^2)(D^2 - k_3^2)(D^2 - k_4^2)]\{\psi_1^*, \psi_2^*(y), \phi^*(y), \theta^*(y)\} = 0, \tag{32}$$

Where $k_n^2\,(n = 1, 2, 3, 4)$ are the roots of the characteristic equation of the equation (31).

The general solution of the equation (32), which are bound at $y \to \infty$, is given by

$$\psi_1(x,y,t) = \sum_{n=1}^{4} R_n \exp(-k_n y + \omega t + iax), \tag{33}$$

$$\psi_2(x,y,t) = \sum_{n=1}^{4} H_{1n} R_n \exp(-k_n y + \omega t + iax), \tag{34}$$

$$\phi(x,y,t) = \sum_{n=1}^{4} H_{2n}\, R_n \exp(-k_n y + \omega t + iax), \tag{35}$$

$$\theta(x,y,t) = \sum_{n=1}^{4} H_{3n}\, R_n \exp(-k_n y + \omega t + iax), \tag{36}$$

$$T(x,y,t) = \sum_{n=1}^{4} H_{4n}\, R_n \exp(-k_n y + \omega t + iax). \tag{37}$$

Since $R_n(n = 1, 2, 3, 4)$ are some coefficients.

To obtain the components of the displacement vector, from (33) and (34) in (14)

$$u(x,y,t) = \sum_{n=1}^{4} M_{1n} R_n \exp(-k_n y + \omega t + iax), \tag{38}$$

$$v(x,y,t) = \sum_{n=1}^{4} G_{1n}\, R_n \exp(-k_n y + \omega t + iax), \tag{39}$$

From (38), (39), (35) and (37) into (20)-(23) to obtain the components of the stresses

$$\sigma_{xx}(x,y,t) = \sum_{n=1}^{4} H_{5n} R_n\ \exp(-k_n y + \omega t + iax), \tag{40}$$

$$\sigma_{yy}(x,y,t) = \sum_{n=1}^{4} H_{6n} R_n \exp(-k_n y + \omega t + i a x), \tag{41}$$

$$\sigma_{zz}(x,y,t) = \sum_{n=1}^{4} H_{7n} R_n \exp(-k_n y + \omega t + i a x), \tag{42}$$

$$\sigma_{xy}(x,y,t) = \sum_{n=1}^{4} H_{8n} R_n \exp(-k_n y + \omega t + i a x). \tag{43}$$

4 Boundary conditions

Consider the boundary conditions to determine the coefficients $R_n(n = 1,2,3,4)$,and suppress the positive exponentials to avoid the unbounded solutions at infinity. The coefficients R_1, R_2, R_3, R_4 can be defined from the boundary conditions on the surface at $y = 0$.

(1) The mechanical boundary conditions

$$\sigma_{yy} = -p_1 \exp(\omega t + i a x), \quad \sigma_{xy} = 0, \text{ and } \frac{\partial \phi}{\partial y} = 0. \tag{44}$$

(2) The thermal boundary condition:

The half-space subjected to thermal shock

$$T = p_2 \exp(\omega t + i a x). \tag{45}$$

Where p_1 is the magnitude of the applied force in the half-space and p_2 is the applied constant temperature to the boundary.

Substituting the expressions of the considered variables in the above boundary conditions, to obtain the following equations satisfied by the parameters.

$$\sum_{n=1}^{4} H_{6n} R_n = -p_1, \tag{46}$$

$$\sum_{n=1}^{4} H_{8n} R_n = 0, \tag{47}$$

$$\sum_{n=1}^{4} -k_n H_{2n} R_n = 0, \tag{48}$$

$$\sum_{n=1}^{4} H_{4n} R_n = p_2. \tag{49}$$

Invoking boundary conditions (44) and (45) at the surface $y = 0$ of the plate, then obtain a system of four equations, (46) - (49). After applying the inverse of matrix method, one can get the values of the four constants $R_n (n = 1,2,3,4)$.

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = \begin{pmatrix} H_{61} & H_{62} & H_{63} & H_{64} \\ H_{81} & H_{82} & H_{83} & H_{84} \\ -k_1 H_{21} & -k_2 H_{22} & -k_3 H_{23} & -k_4 H_{24} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{pmatrix}^{-1} \begin{pmatrix} -p_1 \\ 0 \\ 0 \\ p_2 \end{pmatrix}. \tag{50}$$

Hence obtain the expressions for the displacement components and the other physical quantities of the plate surface.

5 Numerical results and discussion

In order to illustrate the obtained theoretical results in the preceding section, following Dhaliwal and Singh in (1980) the magnesium material was chosen for purposes of numerical evaluations. The constants of the problem were taken as

$$\lambda = 2.17 \times 10^{10} N/m^2, \quad \mu = 3.278 \times 10^{10} N/m^2, \quad K = 1.7 \times 10^2 W/m\,\text{deg},$$
$$\rho = 1.74 \times 10^3 Kg/m^3, \quad \beta = 2.68 \times 10^6 N/m^2\,\text{deg}, \quad C_e = 1.04 \times 10^3 J/Kg\,\text{deg},$$
$$\omega_1^* = 3.58 \times 10^{11}/s, \quad \alpha_t = 1.78 \times 10^{-5} N/m^2, \quad T_0 = 298\,K.$$

The voids parameters are

$$\psi = 1.753 \times 10^{-15} m^2, \quad \alpha = 3.688 \times 10^{-5} N, \quad \xi_1 = 1.475 \times 10^{10} N/m^2,$$
$$b = 1.13849 \times 10^{10} N/m^2, \quad m = 2 \times 10^6 N/m^2\,\text{deg}, \quad \omega_0 = 0.0787 \times 10^{-3} N/m^2 s.$$

The comparisons were carried out for

$$p_1 = 0.5,\ p_2 = 2,\ \varepsilon_2 = 0.5,\ a = 0.1,\ \omega = -1.5,\ x = 2,\ t = 0.2,\ \text{and}\ 0 \le y \le 7.$$

The above numerical values was used for the distribution of the real parts for the displacement components u and v, the temperature distribution θ, the stress components σ_{xx}, σ_{xy} and the change in the volume fraction field ϕ with the distance y for (G-N) theory of types II and III, for these cases

(i) In the presence and the absence of the gravity effect in Figs. 1-6.

$[g = 0,\ 9.8\ \ with\ \ \alpha^* = 0.00051\ \ and\ \ d = 0.00015]$,

(ii) With and without the temperature dependent properties in Figs. 7-12.

$[\alpha^* = 0,\ 0.00051,\ g = 9.8\ with\ \ and\ \ d = 0.00015]$,

(iii) With and without the two temperature effect in Figs. 13-18.

$[d = 0.00015, 0 \;\; g = 9.8 \;\; with \;\; and \;\; \alpha^* = 0.00051]$.

In the graph the solid and dashed lines represent the solutions in the context of the (G-N) theory of type II and the lines with dot represent the derived solutions using (G-N) theory of type III.

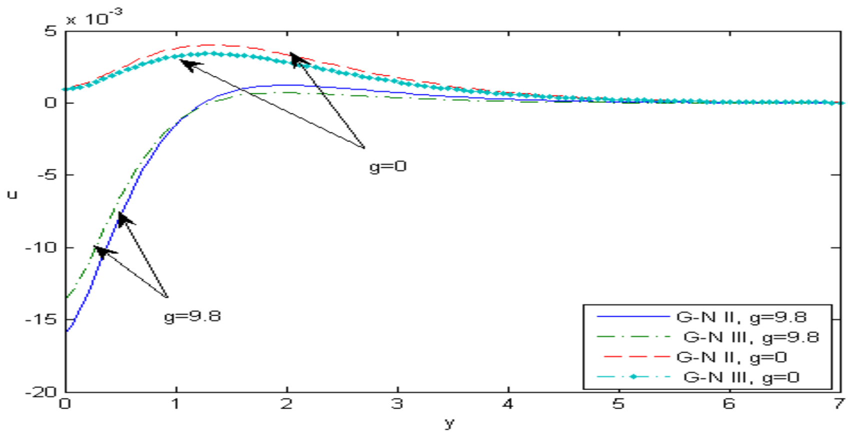

Figure 1: The displacement component u distribution against y with and without gravity.

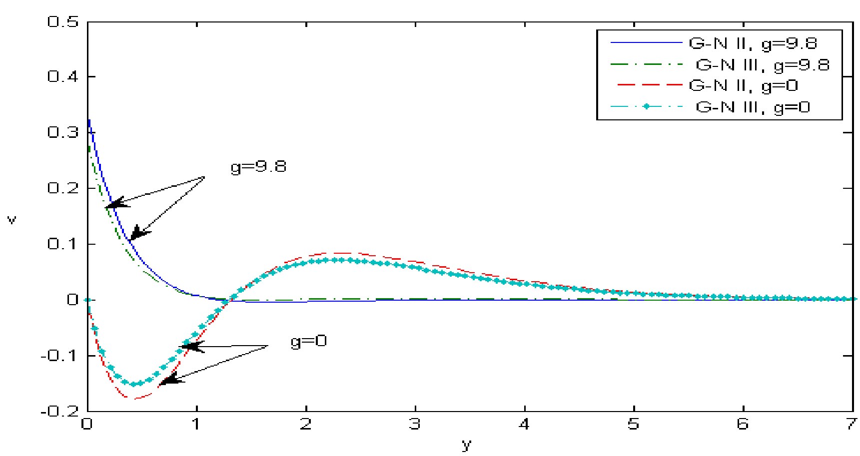

Figure 2: The displacement component v distribution against y with and without gravity.

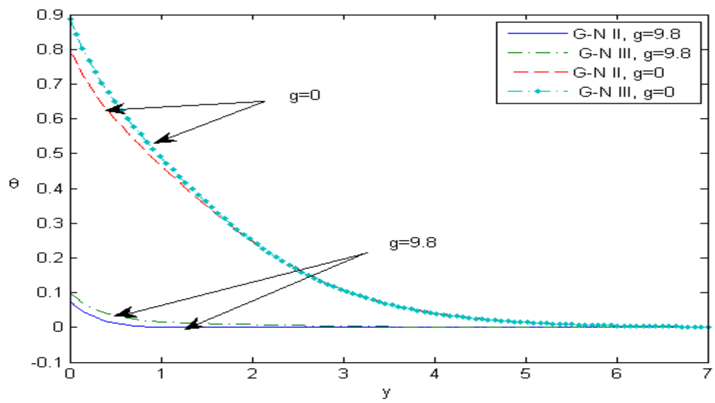

Figure 3: The distribution of the conductive temperature θ against y with and without gravity.

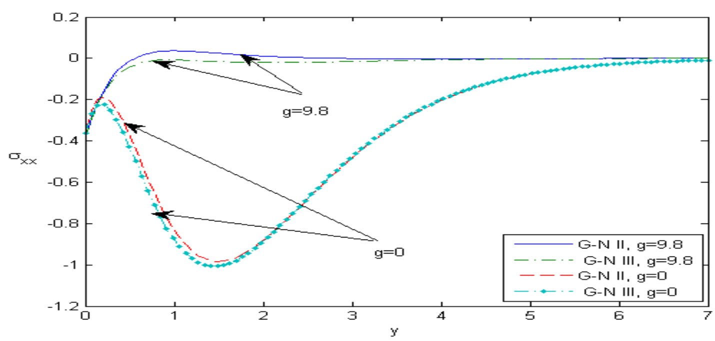

Figure 4: The distribution of the stress tensor component σ_{xx} against y with and without gravity.

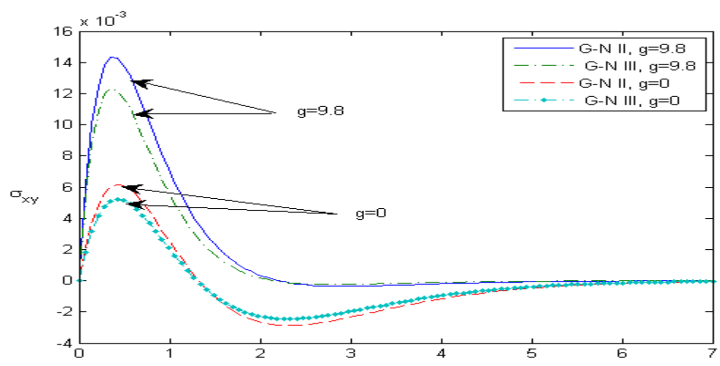

Figure 5: The distribution of the stress tensor component σ_{xy} against y with and without gravity.

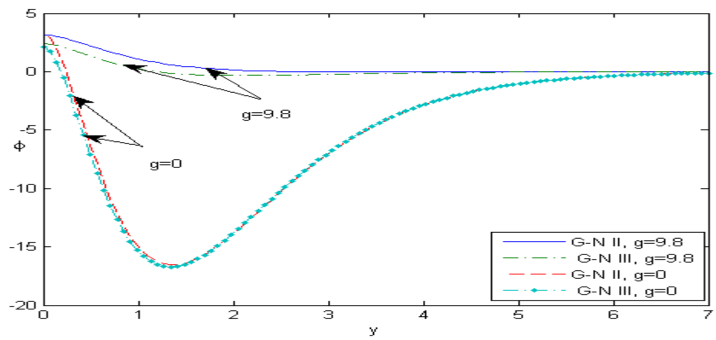

Figure 6: The distribution of the volume fraction field ϕ against y with and without gravity.

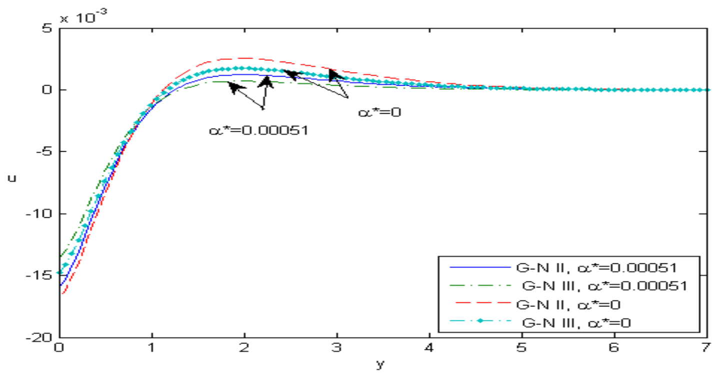

Figure 7: The displacement component u distribution against y with and without temperature dependent.

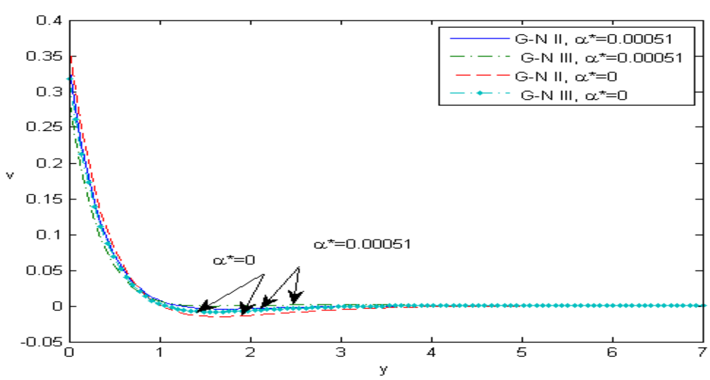

Figure 8: The displacement component v distribution against y with and without temperature dependent.

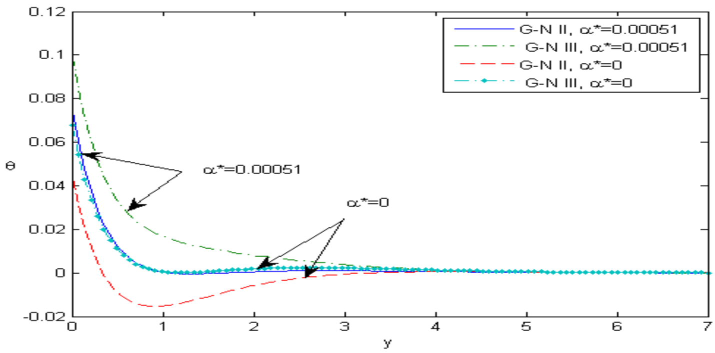

Figure 9: The distribution of the conductive temperature θ against y with and without temperature dependent.

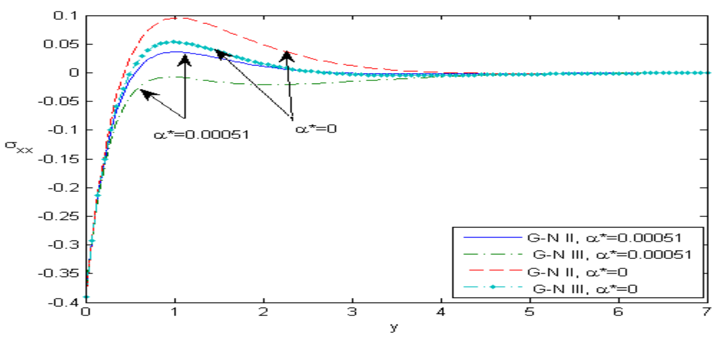

Figure 10: The distribution of the stress tensor σ_{xx} against y with and without temperature dependent.

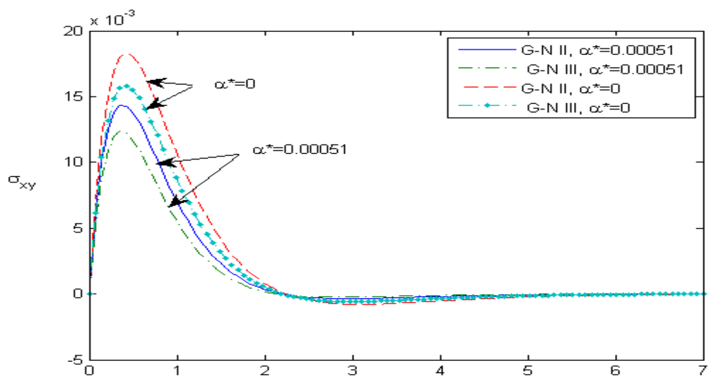

Figure 11: The distribution of the stress tensor σ_{xy} against y with and without temperature dependent.

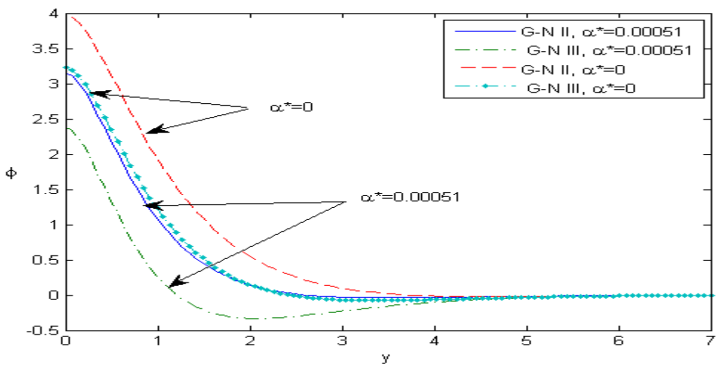

Figure 12: The distribution of the volume fraction field ϕ against y with and without temperature dependent.

Fig. 1 shows the distribution of the displacement component u in the case of $g = 9.8$ and $g = 0$, in the context of both types II and III of (G-N) theory; it noticed that the distribution of u increased with the increase of the gravity for $y > 0$, the distribution of u is directly proportional to the gravity. Fig. 2 depicts the distribution of the displacement component v in the case of $g = 9.8$ and $g = 0$, in the context of both types II and III of (G-N) theory; it noticed that the distribution of v increased in $0 \leq y \leq 1.3$, then decreased in $1.3 \leq y \leq 7$ with the increase of the gravity value for both types II and III of (G-N) theory. Fig. 3 clarifies the distribution of the temperature θ in the case of $g = 0.9,\ 0$ in the context of both types II and III of (G-N) theory; it noticed the distribution of θ decreased with the increase of the value of the gravity in $0 \leq y \leq 7$ for both types II and III of (G-N) theory.

Fig. 4 depicts the distribution of the stress components σ_{xx} in the case of $g = 9.8,\ 0$ in the context of both types II and III of (G-N) theory; it noticed the distribution of σ_{xx} decreased in $0 \leq y \leq 0.2$ then increased in $0.2 \leq y \leq 7$ with the increase of the value of the gravity for both types II and III of (G-N) theory. Fig. 5 shows the distribution of the stress components σ_{xy} in the case of $g = 9.8,\ 0$ in the context of both types II and III of (G-N) theory; it noticed the distribution of σ_{xy}increased in $0 \leq y \leq 7$ with the increase of the value of the gravity for both types II and III of (G-N) theory.

Fig. 6 expresses the distribution of change in the volume fraction field ϕ in the context of both types II and III of (G-N) theory in the case of $g = 9.8,\ 0$, it noticed the distribution of ϕ increased in $0 \leq y \leq 7$ for type III of (G-N) theory, while for type II of (G-N) theory the distribution of ϕ decreased in $0 \leq y \leq 0.05$ then increased in $0.05 \leq y \leq 7$. It explained that all the curves converges to zero, and the gravity has an effective role in the distribution of all physical quantities in this physical problem for both types II and III of (G-N) theory.

Fig. 7 depicts the distribution of the displacement component u in the case of α^*=0.00051 and α^*= 0 in the context of both types II and III of (G-N) theory; it noticed that the distribution of u increased in $0 \leq y \leq 1$ then decreased in $1 \leq y \leq 7$ with the increase of α^* for both types II and III of (G-N) theory. Fig. 8 shows the distribution of the displacement component v in the case of α^*= 0.00051 and α^*=0 in the context of both types II and III of (G-N) theory; it noticed that the distribution of v decreased in $0 \leq y \leq 0.9$ then increased in $0.9 \leq y \leq 7$ with the increase of α^* for both types II and III of (G-N) theory.

Fig. 9 clarifies the distribution of the temperature θ in the case of α^*=0.00051,0 in the context of both types II and III of (G-N) theory; it noticed the distribution of θ increased with the increase of the value of α^* in $0 \leq y \leq 7$ for both types II and III of (G-N) theory.

Fig. 10 depicts the distribution of the stress components σ_{xx} in the case of α^*=0.00051, 0 in the context of both types II and III of (G-N) theory; it noticed the distribution of σ_{xx} decreased in $0 \leq y \leq 7$ with the increase of the value of α^* for both types II and III of (G-N) theory. Fig. 11 shows the distribution of the stress components σ_{xy} in the case of α^*= 0.00051, 0 in the context of both types II and III of (G-N) theory; it noticed the distribution of σ_{xy}decreased in $0 \leq y \leq 2.3$ then increased in $2.3 \leq y \leq 7$ with the increase of the value of $\alpha^*$for both types II and III of (G-N) theory.

Fig. 12 expresses the distribution of change in the volume fraction field ϕ in the context of both types II and III of (G-N) theory in the case of α^*= 0.00051, 0, it noticed the distribution of ϕ decreased in $0 \leq y \leq 7$ with the increase of the value of α^* for both types II and III of (G-N) theory. It explained that all the curves converges to zero, and the temperature dependent properties have an effective role in the distribution of all physical quantities in this problem for both types II and III of (G-N) theory.

Fig. 13 depicts the distribution of the displacement component u in the case of d = 0.00015 and d = 0 in the context of both types II and III of (G-N) theory; it noticed that the distribution of u increased in $0 \leq y \leq 7$ for (G-N) theory of both types II and III with the increase of the two-temperature parameter d.

Fig. 14 shows the distribution of the displacement component v in the case of d = 0.00015 and d = 0 in the context of both types II and III of (G-N) theory; it noticed that the distribution of v increased in $0 \leq y \leq 7$ for (G-N) theory of both types II and III with the increase of the two-temperature parameter d.

Fig.15 clarifies the distribution of the temperature θ in the case of d= 0.00015, 0 in the context of both types II and III of (G-N) theory; it noticed the distribution of θ decreased with the increase of the value of d in $0 \leq y \leq 7$ for type II of (G-N) theory, while the distribution of θ increased with the increase of the value of d in $0 \leq y \leq 7$ for (G-N) theory of type III.

Fig. 16 depicts the distribution of the stress components σ_{xx} in the case of d= 0.00015, 0 in the context of both types II and III of (G-N) theory; it noticed the distribution of σ_{xx} decreased in $0 \leq y \leq 7$ with the increase of the value of d for type III of (G-N) theory, but the distribution of σ_{xx}increased in $0 \leq y \leq 7$ with the increase of the value of d for (G-N) theory of type II.

Fig. 17 shows the distribution of the stress components σ_{xy} in the case of d= 0.00015, 0 in the context of both types II and III of (G-N) theory; it noticed the distribution of σ_{xy} for (G-N) theory of both types II and III decreased in $0 \leq y \leq 7$ with the increase of the value of d.

Fig. 18 expresses the distribution of change in the volume fraction field ϕ in the context of both types II and III of (G-N) theory in the case of d= 0.00015, 0 it noticed the distribution of ϕ increased in $0 \leq y \leq 7$ with the increase of the value of d for type II of (G-N) theory, while the distribution of ϕ for type III of (G-N) theory increased in $0 \leq y \leq 0.05$ then decreased in $0.05 \leq y \leq 7$ with the increase of the value of d. It explained that all the curves converges to zero, and the two-temperature effect has an effective role in the distribution of all physical quantities in this problem for both types II and III of (G-N) theory.

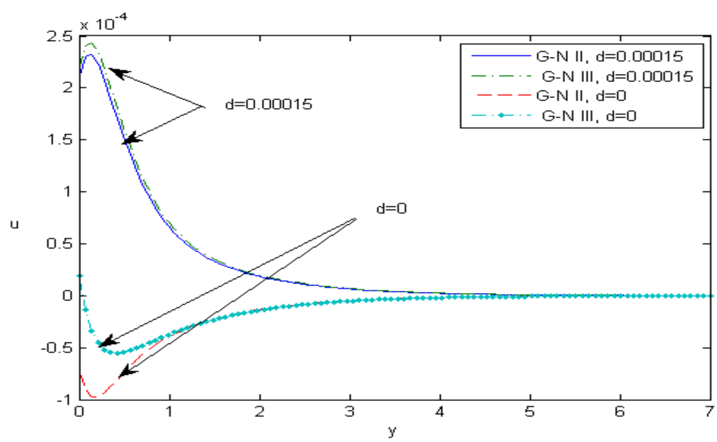

Figure 13: The displacement component u distribution against y with and without two temperature.

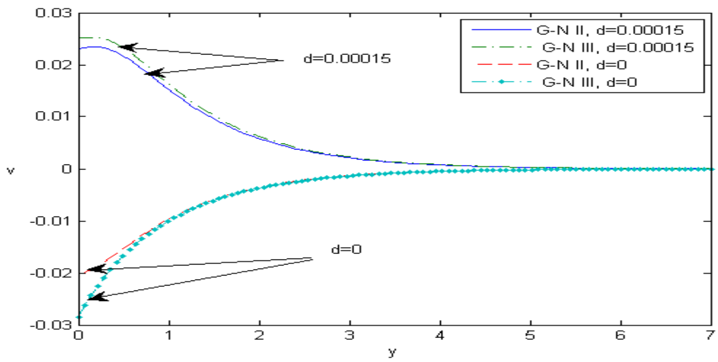

Figure 14: The displacement component v distribution against y with and without two temperature.

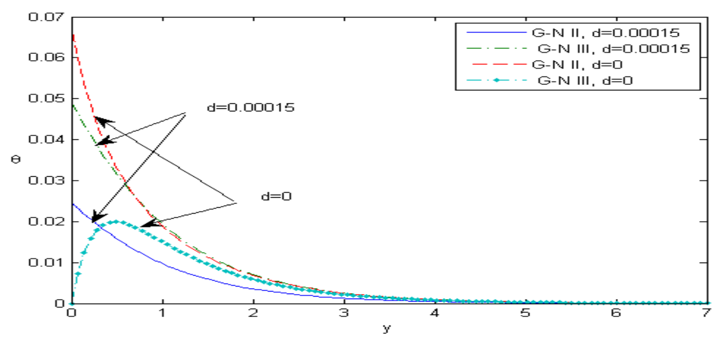

Figure 15: The distribution of the conductive temperature θ against y with and without two temperature.

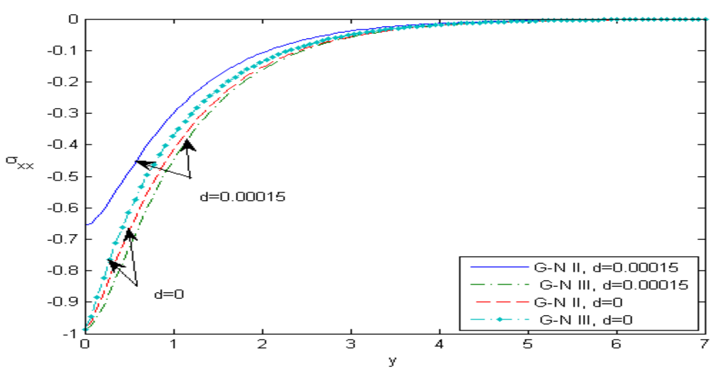

Figure 16: The distribution of the stress tensor component σ_{xx} against y with and without two temperature.

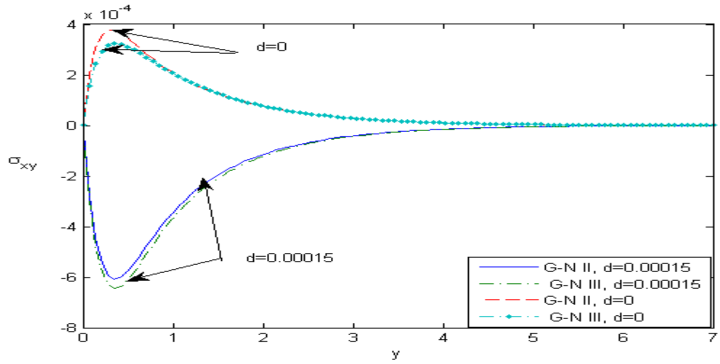

Figure 17: The distribution of the stress tensor component σ_{xy} against y with and without two temperature.

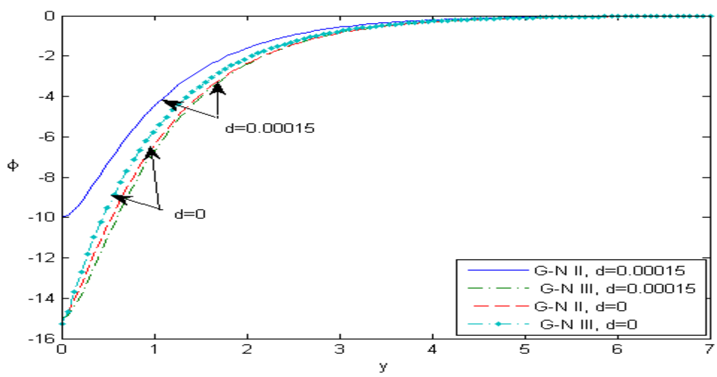

Figure 18: The distribution of the volume fraction field ϕ against y with and without two temperature.

6 Conclusion

In this article, we have studied the effect of the gravitational field and temperature dependent properties due to two-temperature for thermoelastic medium with voids. The analysis of the components of displacement, the stresses, the temperature distributions and the change in the volume fraction field due to the gravity, the temperature dependent properties, and the two-temperature effect for thermoelastic medium with voids is an interesting problem of thermo-mechanical. The gravitational influence, the temperature dependent properties and the two-temperature effect are significant in the current model since the amplitudes of these quantities are varying (increasing or decreasing) under the effect of the used fields. The normal mode analysis technique has been used which is applicable to a wide range of problems in thermoelasticity. The value of all physical quantities converges to zero with an increase in the distance *y*and all functions are continuous. Finally it deduced that the deformation of a body depends on the nature of the applied forces and thermal effects as well as the type of boundary conditions.

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Appendix

$$a_1 = \frac{\lambda_0 + \mu_0}{\mu_0}, \quad a_2 = \frac{b_0\, c_1^2}{\omega_1^{*2}\, \psi_0\, \mu_0}, \quad a_3 = \frac{\beta_0\, T_0}{\mu_0}, \quad a_4 = \frac{\rho g\, c_1^2}{\mu_0 f(T)},$$

$$a_5 = \frac{\rho\, c_1^2}{\mu_0 f(T)}, \quad a_6 = \frac{b_0\, \psi_0}{\alpha_0}, \quad a_7 = \frac{\xi_{10}\, c_1^2}{\alpha_0\, \omega_1^{*2}}, \quad a_8 = \frac{\omega_{10}\, c_1^2}{\alpha_0\, \omega_1^{*}}, \quad a_9 = \frac{m_0\, T_0\, \psi_0}{\alpha_0},$$

$$a_{10} = \frac{\rho\, c_1^2 \psi_0}{\alpha_0}, \quad a_{11} = \frac{m_0\, c_1^2 f(T)}{\rho\, C_e\, \psi_0\, \omega_1^{*3}}, \quad a_{12} = \frac{d\, \omega_1^{*2}}{c_1^2}, \quad \varepsilon_1 = \frac{\beta_0 f(T)}{\rho\, C_e},$$

$$\varepsilon_2 = \frac{k^*\, \omega_1^*}{\rho\, C_e\, c_1^2}, \quad \varepsilon_3 = \frac{k_0 f(T)}{\rho\, C_e\, c_1^2}, \quad A^* = \frac{1}{f(T)} = \frac{1}{(1-\alpha^* T_0)}, \quad b_1 = a_1 + 1,$$

$$a_{13} = \frac{\lambda_0 f(T)}{\mu_0}, \quad a_{14} = \frac{b_0\, c_1^2 f(T)}{\mu_0\, \psi_0\, \omega_1^{*2}}, \quad a_{15} = \frac{\beta_0\, T_0 f(T)}{\mu_0}, \quad a_{16} = f(T),$$

$$b_2 = \mathrm{i}\, a\, a_4, \quad b_3 = a^2 + \frac{a_5\, \omega^2}{b_1}, \quad b_4 = \frac{b_2}{b_1}, \quad b_5 = \frac{a_2}{b_1}, \quad b_6 = \frac{a_3}{b_1}, \quad b_7 = \frac{a_3\, a_{12}}{b_1},$$

$$b_8 = a^2 + a_5\, \omega^2, \quad b_9 = a^2 + a_7 + a_8\, \omega + a_{10}\, \omega^2, \quad b_{10} = a_9\, a_{12},$$

$$b_{11} = \varepsilon_3 + \varepsilon_2 \omega, \quad b_{12} = a_{12}\, \omega^2, \quad b_{13} = b_{11} + b_{12}, \quad b_{14} = \frac{\varepsilon_1\, \omega^2}{b_{13}},$$

$$b_{15} = \frac{a_{11}\, \omega}{b_{13}}, \quad b_{16} = a^2 + \frac{\omega^2}{b_{13}}, \quad D = \frac{d}{dy},$$

$$\begin{aligned} A =& \frac{1}{(1 + b_7\, b_{14})} [b_9 + b_{16} + b_{10}\, b_{15} + b_3 + b_8 - a_6\, b_5 - b_5\, b_{10}\, b_{14} + b_7\, b_9\, b_{14} \\ &+ 2\, b_7\, b_{14}\, a^2 - a_6\, b_7\, b_{15} + b_7\, b_8\, b_{14} + b_6\, b_{14}], \end{aligned}$$

$$\begin{aligned} B =& \frac{1}{(1 + b_7\, b_{14})} [a_9\, b_{15} + b_9\, b_{16} + b_{10}\, b_{15}\, a^2 + b_3\, b_9 + b_3\, b_{16} + b_3\, b_{10}\, b_{15} + b_8\, b_9 \\ &+ b_8\, b_{16} - b_8\, b_{10}\, b_{15} + b_3\, b_8 - a_6\, b_5\, b_{16} - a_6\, b_5\, a^2 - a_9\, b_5\, b_{14} - 2\, b_5\, b_{10}\, b_{14}\, a^2 \\ &- a_6\, b_5\, b_8 - b_5\, b_8\, b_{10}\, b_{14} + 2\, b_7\, b_9\, b_{14}\, a^2 - 2\, a_6\, b_7\, b_{15}\, a^2 + b_7\, b_8\, b_9\, b_{14} \\ &+ 2\, b_7\, b_8\, b_{14}\, a^2 - a_6\, b_7\, b_8\, b_{15} + b_6\, b_9\, b_{14} + b_6\, b_{14}\, a^2 - a_6\, b_6\, b_{15} \\ &+ b_7\, b_{14}\, a^4 + b_6\, b_8\, b_{14}], \end{aligned}$$

$$\begin{aligned} C =& \frac{1}{(1 + b_7\, b_{14})} [a_9\, b_3\, b_{15} + b_3\, b_9\, b_{16} + b_3\, b_{10}\, b_{15}\, a^2 + a_9\, b_8\, b_{15} + b_8\, b_9\, b_{16} \\ &+ b_8\, b_{10}\, b_{15}\, a^2 + b_3\, b_8\, b_9 + b_3\, b_8\, b_{16} + b_3\, b_8\, b_{10}\, b_{15} + b_2\, b_4\, b_9 + b_2\, b_4\, b_{16} \\ &+ b_2\, b_4\, b_{10}\, b_{15} - a_6\, b_5\, b_{16}\, a^2 - a_9\, b_5\, b_{14}\, a^2 - b_5\, b_{10}\, b_{14}\, a^4 - a_6\, b_5\, b_8\, b_{16} \\ &- a_6\, b_5\, b_8\, a^2 - a_9\, b_5\, b_8\, b_{14} - 2\, b_5\, b_8\, b_{10}\, b_{14}\, a^2 + 2\, b_7\, b_8\, b_9\, b_{14}\, a^2 \\ &- 2\, a_6\, b_7\, b_8\, b_{15}\, a^2 + b_6\, b_9\, b_{14}\, a^2 - a_6\, b_6\, b_{15}\, a^2 + b_7\, b_9\, b_{14}\, a^4 \\ &- a_6\, b_7\, b_{15}\, a^4 + b_6\, b_8\, b_9\, b_{14} + b_6\, b_8\, b_{14}\, a^2 - a_6\, b_6\, b_8\, b_{15} + b_7\, b_8\, b_{14}\, a^4], \end{aligned}$$

$$\begin{aligned} E =& \frac{1}{(1 + b_7\, b_{14})} [a_9\, b_3\, b_8\, b_{15} + b_3\, b_8\, b_9\, b_{16} + b_3\, b_8 b_{10}\, b_{15}\, a^2 + b_2\, b_4 b_9\, b_{16} \\ &+ a_9\, b_2\, b_4\, b_{15} + b_2\, b_4\, b_{10}\, b_{15}\, a^2 - a_6\, b_5\, b_8\, b_{16}\, a^2 - a_9\, b_5\, b_8\, b_{14}\, a^2 \\ &- b_5\, b_8\, b_{10}\, b_{14}\, a^4 + b_6\, b_8\, b_9\, b_{14}\, a^2 - a_6\, b_6\, b_8\, b_{15}\, a^2 + b_7\, b_8\, b_9\, b_{14}\, a^4 \\ &- a_6\, b_7\, b_8\, b_{15}\, a^4]. \end{aligned}$$

$$H_{1n} = -\frac{b_2}{(k_n^2 - b_8)}, \quad n = 1, 2, 3, 4,$$

$$H_{2n}=\frac{-(1+b_7b_{14})k_n^4+(b_3+b_{16}+b_4H_{1n}+b_6b_{14}+2b_7b_{14}a^2)k_n^2-(b_3b_{16}+b_4b_{16}H_{1n}+b_7b_{14}a^4)}{(b_5+b_7b_{15})k_n^2-(b_5b_{16}+b_6b_{15}+b_7b_{15}a^2)},$$

$$H_{3n}=\frac{b_{14}k_n^4-(a_6\,b_{15}+b_9\,b_{14}+b_{14}\,a^2)\,k_n^2+(a_6\,b_{15}\,a^2+b_9\,b_{14}\,a^2)}{k_n^4-(b_9+b_{16}-b_{10}\,b_{15})\,k_n^2+(a_9\,b_{15}+b_9\,b_{16}+b_{10}\,b_{15}\,a^2)},\; n=1,2,3,4,$$

$$H_{4n}=[-a_{12}\,k_n^2+(1+a_{12}\,a^2)]\,H_{3n}, \quad n=1,2,3,4,$$

$$M_{1n}=(ia-k_nH_{1n}), \quad G_{1n}=-(k_n+iaH_{1n}), \quad n=1,2,3,4.$$

$$H_{5n}=a_{13}(M_{1n}-k_nG_{1n})+2\,i\,a\,a_{16}M_{1n}+a_{14}H_{2n}-a_{15}H_{4n}, \quad n=1,2,3,4,$$

$$H_{6n}=a_{13}(M_{1n}-k_nG_{1n})-2\,a_{16}G_{1n}+a_{14}H_{2n}-a_{15}H_{4n}, \quad n=1,2,3,4,$$

$$H_{7n}=a_{13}(M_{1n}-k_nG_{1n})+a_{14}H_{2n}-a_{15}H_{4n}, \quad n=1,2,3,4,$$

$$H_{8n}=a_{16}(-k_nM_{1n}+i\,a\,G_{1n}), \quad n=1,2,3,4.$$