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Forced Vibration of the Pre-Stressed and Imperfectly Bonded Bi-Layered Plate Strip Resting on a Rigid Foundation

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Abstract: Within the scope of the piecewise homogeneous body model with utilizing of the three dimensional linearized theory of elastic waves in initially stressed bodies the influence of the shear-spring type imperfection of the contact conditions between the layers of the pre-stressed bi-layered plate strip resting on the rigid foundation, on the frequency response of this plate strip is investigated. The corresponding mathematical problem is solved numerically by employing FEM and numerical results illustrating the influence of the parameter characterizing the degree of the mentioned imperfectness, on the frequency response of the normal stress acting on the interface planes between the layers and between the plate and rigid foundation are presented and discussed. In particular, it is established that an increase in the value of the shear-spring parameter the absolute values of the compressed normal stress decrease, but the values of the stretched normal stress increase and this parameter has an influence also on the character of the action of the initial stresses on the frequency response under consideration.

Keywords: bi-layered plate-strip with finite length, frequency response, initial stress, shear-spring type imperfectness

1 Introduction

It is known that there are many factors on which the dynamical behavior of the layered elastic systems depends significantly. Two of them are following ones: (a) the imperfectness of the contact on the interface planes between the layers, (b)

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the static initial stresses in the layers which exist before the additional dynamical loading.

Note that the factor (a), i.e. the imperfectness of the contact between layers may arise as a defect caused by the corresponding technological process, as well as may be considered as constructional requirements. Moreover, the mentioned imperfectness can arise during the exploitation of the element of constructions fabricated from the layered materials as a result of actions of various type external impacts.

The mentioned factor (b) above, i.e. the initial stresses in the layers of the layered systems may also arise as a result of a technological process employing under its assembly and as a result of the action of the environmental temperature. Moreover, stresses caused by the exploitation of the external static loading can be also taken as initial stresses with respect to the additional dynamic loading. It should be noted that the influence of the initial stresses on the dynamical behavior of the deformed (including layered) systems cannot be taken into account within the framework of the classical linear theory of elastodynamics, because this influence regards the non-linear effects in the dynamics of the deformed systems. Therefore, the theoretical investigations of the noted influence require the use of the complicated geometrical non-linear equations of dynamics of the deformable body. However, according to the well-known mechanical considerations for the cases where the magnitude of the initial loading is greater than that of the additional dynamic loading these investigations can be carried out within the framework of the Three-Dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies (TDLTEWISB).

In the construction of the field equations of the TDLTEWISB, one considers two states of a deformable solid. The first is regarded as the initial or unperturbed state, and the second is a perturbed state with respect to unperturbed. By "the state of deformable solid" is meant both motion and equilibrium (as a particular case of motion). It is assumed that all values in a perturbed state can be represented as a sum of the values of the initial state and perturbations. The latter is also assumed to be small in comparison with the corresponding values in the initial state. It is also assumed that both initial (unperturbed) and perturbed states are described by the equations of non-linear solid mechanics. Due to the fact that perturbations are small, the relationships for the perturbed state in the vicinity of appropriate values for the unperturbed state are linearized and then the relationships for the unperturbed state are subtracted from them. The results are the equations of the TDLTEWISB. Since the equations contain the initial state variables, the TDLTEWISB describes the influence of the initial stresses on the perturbations.

For the determination of initial (unperturbed) state in relatively rigid materials the classical linear theory of elasticity is used. However the perturbed state is described by the geometrically non-linear exact equations of the theory of elasticity.

By linearizing these equations the aforementioned equations of the TDLTEWISB are obtained. This and other versions of the TDLTEWISB are analyzed in the monographs by Guz (1986a, 1986b, 2004) and in a paper by Akbarov (2012).

It should be noted that the investigations carried out up to now in this field can be divided into two groups. Investigations of the first group related to the wave propagation in the initially stressed bodies and systematic analyses of results of these investigations were made in monographs by Guz (1986a, 1986b, 2004). The review of the results obtained before 2002 was made in the paper by Guz (2002), but the review of the investigations obtained before 2007 was made in Akbarov (2007b). At the same time the review of the resent investigation was detailed in the paper by Akbarov (2012).

Investigations of the second group relate to the time harmonic Lamb's problem for the elastic systems consisting of the pre-stressed half-spaces and with the pre-stressed covering layer (see Akbarov (2006a, 2006b, 2007a), Akbarov, Emiroglu and Tasci (2005), Emiroglu, Tasci and Akbarov (2009), Akbarov and Guler (2007), Akbarov and Ilhan (2008, 2009, 2010)), as well as to the time harmonic dynamic stress field problem for the system consisting of the pre-stressed two layers resting on a rigid foundation (see Akbarov (2006c), Akbarov and Salmanova (2009), Akbarov, Zamanov and Suleimanov (2005) and others).

We note that, as in last three papers by Akbarov (2006c), Akbarov and Salmanova (2009), Akbarov, Zamanov and Suleimanov (2005), the subject of the present paper regards also the time-harmonic dynamic stress field in a bi-layered slab which rests on the rigid foundation. Moreover, we note that in the mentioned last three papers it was assumed that the length and width of the layers in the medium considered are infinite. Namely, this assumption simplifies the mathematical solutions of the corresponding problems. However, this assumption cannot be applicable for the cases where the thickness and length of the layers in the bi-layered systems have commensurable quantities. Consequently, in such cases the corresponding problems for plates with finite length were investigated in papers by Akbarov, Yildiz and Eröz (2011b) (for single-layer plate) and Akbarov, Yildiz and Eröz (2011a) (for bi-layered plate). However, in these papers it was assumed that between the layers, as well as between the plate and rigid foundation the complete contact conditions satisfy. However, in many cases, it is unrealistic to assume a perfectly bounded interface between the constituents of the elastic systems (see, an instance, a paper by Selvadurai and Nikopour (2012)). In order to apply the results of the theoretical investigations related to the forced vibration of the bi-layered slab (or plate) resting on the rigid foundation to practice real cases, it is necessary to take into account the factor (a) noted above, i.e. the imperfectness of the contacts between the constituents under these investigations.

Taking the foregoing discussions into account, in the present paper the forced vibration of the imperfectly bonded, pre-stressed bi-layered plate with finite length resting on the rigid foundation is studied within the scope of the piecewise homogeneous body model with the use of the TDLTEWISB. Numerical results are obtained by employing Finite Element Method (FEM). Under this study it is assumed that the shear-spring type imperfect contact conditions satisfy between the layers, but between the plate and the rigid foundation the complete contact conditions. It should be noted that the investigations carried out in this paper can be also considered as development of the investigations carried out in the paper by Akbarov, Yildiz and Eröz (2011a) for the case where between the layers of the plate the shear-spring type imperfect contact condition are satisfied.

2 Formulation of the Problem

Consider the bi-layered plate-strip with geometries shown in Fig.1 and determine the positions of the points of that by the Lagrangian coordinates in the Cartesian system of coordinates $Ox_1x_2x_3$. We assume that the length of the plate is infinite in the direction of Ox_3 axis and all investigations are made for plane-strain state in the Ox_1x_2 plane. The layers of the plate occupy the regions

$$\begin{aligned} &\Omega_1 = \{\, -a \leq x_1 \leq +a\, ;\, -h_1 < x_2 \leq 0 \,\} \\ &\Omega_2 = \{\, -a \leq x_1 \leq +a\, ;\, -h \leq x_2 \leq -h_1 \,\} \end{aligned} \tag{1}$$

The values related to the upper (lower) layer occupying the region Ω_1 (Ω_2) will be indicated by upper index (1) ((2)).

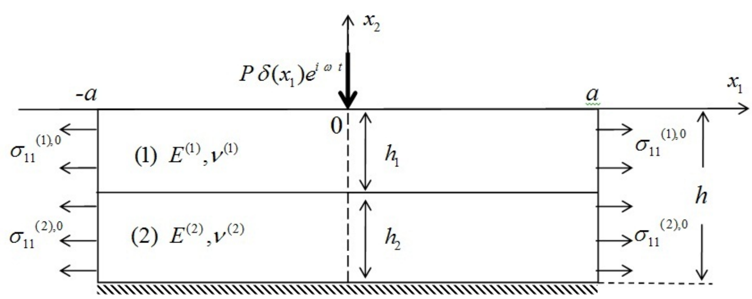

Figure 1: The geometry of the bilayered plate-strip for the considered problem

Before compounding with one another and with the rigid foundation, the layers are loaded separately with uniformly distributed normal forces acting at the ends of those, as a result of which a uniaxial homogeneous initial stress state appears in each of them. The values related to this initial state will be indicated with additional upper index 0.

Assume that the layers' materials are moderately rigid and foregoing initial stress state in the layers is determined within the scope of the classical linear theory of elasticity as follows

$$\sigma_{11}^{(m),0}=c_m \ ; \ m=1,2 \ , \ \sigma_{ij}^{(m),0}=0 \ \text{for} \ ij\neq 11, \tag{2}$$

where c_m is known constant for each layer. As we consider the case where the initial stress state in the layers is determined by the classical linear theory of elasticity, the distinction between the coordinates regarding the natural and the initial states is so slight that it need not be taken into account.

Thus, given the statements above, we assume that on the upper free face of the upper layer line-located time-harmonic dynamical force acts as shown in Fig.1. This is required to determine the dynamical response of the considered system to this load under the plane-strain state in Ox_1x_2 plane.

According to Guz (2004), the equations of motion of TLTEWISB for the small initial deformation considered are

$$\frac{\partial \sigma_{ij}^{(m)}}{\partial x_j}+\sigma_{11}^{(m),0}\frac{\partial^2 u_i^{(m)}}{\partial x_1^2}=\rho^{(m)}\frac{\partial^2 u_i^{(m)}}{\partial t^2} \ , \ i;\ j; m=1,2. \tag{3}$$

The materials of the layers are assumed to be isotropic and mechanical relations for those are written as follows

$$\sigma_{ij}^{(m)}=\lambda^{(m)}\theta^{(m)}\delta_{ij}+2\mu^{(m)}\varepsilon_{ij}^{(m)} \ , \ m=1,2 \tag{4}$$

where

$$\varepsilon_{ij}^{(m)}=\frac{1}{2}\left(\frac{\partial u_i^{(m)}}{\partial x_j}+\frac{\partial u_j^{(m)}}{\partial x_i}\right) \ , \ m=1,2. \tag{5}$$

In equations (3)-(5) and below conventional notation is used. According to the foregoing discussions, on the upper plane of the upper layer and on the ends of the plate the following boundary conditions satisfy

$$\sigma_{12}^{(1)}|_{x_2=0}=0 \ , \ \sigma_{22}^{(1)}|_{x_2=0}=-P\delta(x_1)e^{i\omega t}. \tag{6}$$

$$\left.\left(\sigma_{11}^{(m),0}\frac{\partial u_j^{(m)}}{\partial x_1}+\sigma_{1j}^{(m)}\right)\right|_{x_1=\pm a}=0 \ , \ m; j=1,2. \tag{7}$$

In Eq. (6), $\delta(x_1)$ denotes the Dirac's delta function.

Now we consider the formulation of the imperfect contact conditions on the interface plane between the lower and upper layers. It should be noted that, in general, the imperfectness of the contact conditions is identified by discontinuities of the displacements and forces across the mentioned interface. A review of the mathematical modeling of the various types of incomplete contact conditions for elastodynamics problems has been detailed in a paper by Martin (1992). It follows from this paper that for most models the discontinuity of the displacement $\mathbf{u}^+$ and force $\mathbf{f}^+$ vectors on one side of the interface are assumed to be linearly related to the displacement $\mathbf{u}^-$ and force $\mathbf{f}^-$ vectors on the other side of the interface. This statement, as in the paper by Rokhlin and Wang (1991), can be presented as follows:

$$[\mathbf{f}] = \mathbf{C}\mathbf{u}^- + \mathbf{D}\mathbf{f}^-, \quad [\mathbf{u}] = \mathbf{G}\mathbf{u}^- + \mathbf{F}\mathbf{f}^-, \tag{8}$$

where $\mathbf{C}$, $\mathbf{D}$, $\mathbf{G}$ and $\mathbf{F}$ are three-dimensional (3×3) matrices and the square brackets indicate a jump in the corresponding quantity across the interface. Consequently, if the interface is at $x_2 = -h_1$, then:

$$[\mathbf{u}] = \mathbf{u}|_{x_2=-h_1+0} - \mathbf{u}|_{x_2=-h_1-0} \quad , \quad [\mathbf{f}] = \mathbf{f}|_{x_2=-h_1+0} - \mathbf{f}|_{x_2=-h_1-0}\,. \tag{9}$$

It follows from (8) that we can write incomplete contact conditions for various particular cases by selection of the matrices $\mathbf{C}$, $\mathbf{D}$, $\mathbf{G}$ and $\mathbf{F}$. One such selection was made in the paper by Jones and Whitter (1967), according to which, it was assumed that $\mathbf{C} = \mathbf{D} = \mathbf{G} = \mathbf{0}$. In this case the following can be obtained from (9):

$$[\mathbf{f}] = \mathbf{0}, \quad [\mathbf{u}] = \mathbf{F}\mathbf{f}^-, \tag{10}$$

where $\mathbf{F}$ is a constant diagonal matrix. The model (10) simplifies significantly the solution procedure of the corresponding problems and is adequate in many real cases. Therefore, this model (i.e. the model (10)) is called a shear-spring type resistance model and has been used in many investigations carried out within the framework of classical elastodynamics by Jones and Whitter (1967), Berger, Martin and McCaffery (2000), Kepceler (2010) and by Akbarov and Ipek (2010, 2012). According to this statement, we also use the model (10) for the mathematical formulation of the imperfectness of the contact conditions and these conditions are written as follows.

$$\sigma_{i2}^{(1)}|_{x_2=-h_1} = \sigma_{i2}^{(2)}|_{x_2=-h_1} \ , i=1,2, \quad u_2^{(1)}|_{x_2=-h_1} = u_2^{(2)}|_{x_2=-h_1},$$

$$u_1^{(1)}|_{x_2=-h_1} - u_1^{(2)}|_{x_2=-h_1} = F\frac{h_1}{\mu^{(1)}}\sigma_{12}^{(1)}|_{x_2=-h_1}, \quad F>0 \tag{11}$$

Moreover, we assume that between the lower layer of the plate and the rigid foundation the complete clamped conditions occur

$$u_i^{(2)}|_{x_2=-h} = 0, \quad i=1,2, \tag{12}$$

where $h = h_1 + h_2$.

We will estimate below the degree of the shear-spring type imperfectness of the contact conditions (11) through the parameter F in (11). Note that the case where $F=0$ corresponds the full contact of the layers, but the case where $F=\infty$ to the full slipping contact between the layers of the plate.

This completes the formulation of the problem. It should be noted that in the case where $\sigma_{11}^{(m),0} = 0$ $(m=1,2)$, the problem formulation described above transforms into the corresponding one within the scope of the classical linear theory of elastodynamics.

3 Solution Method: Finite Element Formulation

Since the applied lineal located load is time-harmonic and the steady state is considered, all the dependent variables are also time-harmonic and can be represented as

$$\{\, u_i^{(m)}, \varepsilon_{ij}^{(m)}, \sigma_{ij}^{(m)} \,\} = \{\, \bar{u}_i^{(m)}, \bar{\varepsilon}_{ij}^{(m)}, \bar{\sigma}_{ij}^{(m)} \,\} e^{i\omega t}. \tag{13}$$

where the superposed bar denotes the amplitude of the corresponding quantity. Hereafter the bars will be omitted. Substituting expression (13) into the foregoing equations and conditions, with the change $(\partial^2 u_j^{(m)}/\partial t^2)$ and $P\delta(x_1)e^{i\omega t}$ by $(-\omega^2 u_j^{(m)})$ and $P\delta(x_1)$ respectively, we obtain the same equations and conditions for the amplitude of the sought values. To find the analytical solution of the formulated problem is impossible, therefore we attempt to solve this problem numerically by employing FEM.

First, we introduce the dimensionless coordinate system by the following transformation

$$\hat{x}_1 = \frac{x_1}{h} \quad ,\hat{x}_2 = \frac{x_2}{h}.$$

For the FEM modeling of the boundary-value - contact problem which are obtained for the amplitude from equations (3)-(12) by substituting the expression (13) into them we propose the following functional

$$
\begin{aligned}
J(u^{(m)}) = {} & \frac{1}{2}\sum_{m=1}^{2}\iint_{\hat{\Omega}_m}\Big[T_{11}^{(m)}\frac{\partial u_1^{(m)}}{\partial \hat{x}_1}+T_{12}^{(m)}\frac{\partial u_2^{(m)}}{\partial \hat{x}_1}+T_{21}^{(m)}\frac{\partial u_1^{(m)}}{\partial \hat{x}_2}\\
& +T_{22}^{(m)}\frac{\partial u_2^{(m)}}{\partial \hat{x}_2}-\omega^2\rho^{(m)}\left(u_1^{(m)}u_1^{(m)}+u_2^{(m)}u_2^{(m)}\right)\Big]d\hat{x}_1 d\hat{x}_2\\
& \underline{+\frac{1}{2}\int_{-a/h}^{a/h}\frac{Fh_1}{h^2}\left(\frac{\partial u_1^{(1)}}{\partial \hat{x}_2}+\frac{\partial u_2^{(1)}}{\partial \hat{x}_1}\right)\Bigg|_{x_2=-h_1/h}\left(\frac{\partial u_1^{(1)}}{\partial \hat{x}_2}+\frac{\partial u_2^{(1)}}{\partial \hat{x}_1}\right)\Bigg|_{x_2=-h_1/h}d\hat{x}_1}\\
& +\int_{-a/h}^{+a/h}P\delta(\hat{x}_1)u_2^{(1)}|_{\hat{x}_2=0}d\hat{x}_1,
\end{aligned}
\tag{14}
$$

where $u^{(m)}=u^{(m)}(u_1^{(m)},u_2^{(m)})$. Note that the underlined term in the functional (14) characterizes the imperfectness of the contact conditions between the layers of the plate. In the expression (14) the following notation is used

$$
\begin{aligned}
& T_{11}^{(m)}=\sigma_{11}^{(m)}+\sigma_{11}^{(m),0}\frac{\partial u_1^{(m)}}{\partial x_1}=\omega_{1111}^{(m)}\frac{\partial u_1^{(m)}}{\partial x_1}+\omega_{1122}^{(m)}\frac{\partial u_2^{(m)}}{\partial x_2},\\
& T_{12}^{(m)}=\sigma_{12}^{(m)}+\sigma_{11}^{(m),0}\frac{\partial u_2^{(m)}}{\partial x_1}=\omega_{1212}^{(m)}\frac{\partial u_1^{(m)}}{\partial x_2}+\omega_{1221}^{(m)}\frac{\partial u_2^{(m)}}{\partial x_1},\\
& T_{21}^{(m)}=\sigma_{12}^{(m)}=\omega_{2112}^{(m)}\frac{\partial u_1^{(m)}}{\partial x_2}+\omega_{2121}^{(m)}\frac{\partial u_2^{(m)}}{\partial x_1},\\
& T_{22}^{(m)}=\sigma_{22}^{(m)}=\omega_{2211}^{(m)}\frac{\partial u_1^{(m)}}{\partial x_1}+\omega_{2222}^{(m)}\frac{\partial u_2^{(m)}}{\partial x_2},
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
& \omega_{1111}^{(m)}=\lambda^{(m)}+2\mu^{(m)}+\sigma_{11}^{(m),0}, \quad \omega_{1122}^{(m)}=\lambda^{(m)},\\
& \omega_{1212}^{(m)}=\mu^{(m)}, \quad \omega_{1221}^{(m)}=\mu^{(m)}+\sigma_{11}^{(m),0},\\
& \omega_{2112}^{(m)}=\mu^{(m)}, \quad \omega_{2121}^{(m)}=\mu^{(m)}, \quad \omega_{2211}^{(m)}=\lambda^{(m)},\\
& \omega_{2222}^{(m)}=\lambda^{(m)}+2\mu^{(m)}+\sigma_{11}^{(m),0}.
\end{aligned}
\tag{16}
$$

We attempt to prove the validity of the functional (14)-(16). For this purpose we consider the first variation of this functional which after some mathematical manipulations can be presented as follows:

$$\begin{aligned}
\delta J(u^{(m)}) = \sum_{m=1}^{2} \iint\limits_{\hat{\Omega}_m} \Bigg[& \left(\omega_{1111}^{(m)} \frac{\partial u_1^{(m)}}{\partial \hat{x}_1} + \frac{1}{2}\omega_{1122}^{(m)} \frac{\partial u_2^{(m)}}{\partial \hat{x}_2} + \frac{1}{2}\omega_{2211}^{(m)} \frac{\partial u_2^{(m)}}{\partial \hat{x}_2} \right) \frac{\partial\left(\delta u_1^{(m)}\right)}{\partial \hat{x}_1} \\
& + \left(\omega_{2222}^{(m)} \frac{\partial u_2^{(m)}}{\partial \hat{x}_2} + \frac{1}{2}\omega_{2211}^{(m)} \frac{\partial u_1^{(m)}}{\partial \hat{x}_1} + \frac{1}{2}\omega_{1122}^{(m)} \frac{\partial u_1^{(m)}}{\partial \hat{x}_1} \right) \frac{\partial\left(\delta u_2^{(m)}\right)}{\partial \hat{x}_2} \\
& + \left(\frac{1}{2}\omega_{1212}^{(m)} \frac{\partial u_2^{(m)}}{\partial \hat{x}_1} + \omega_{2112}^{(m)} \frac{\partial u_1^{(m)}}{\partial \hat{x}_2} + \frac{1}{2}\omega_{2121}^{(m)} \frac{\partial u_2^{(m)}}{\partial \hat{x}_1} \right) \frac{\partial\left(\delta u_1^{(m)}\right)}{\partial \hat{x}_2} \\
& + \left(\frac{1}{2}\omega_{1212}^{(m)} \frac{\partial u_1^{(m)}}{\partial \hat{x}_2} + \omega_{1221}^{(m)} \frac{\partial u_2^{(m)}}{\partial \hat{x}_1} + \frac{1}{2}\omega_{2121}^{(m)} \frac{\partial u_1^{(m)}}{\partial \hat{x}_2} \right) \frac{\partial\left(\delta u_2^{(m)}\right)}{\partial \hat{x}_1} \\
& - \omega^2 \rho^{(m)} \left(u_1^{(m)} \delta u_1^{(m)} + u_2^{(m)} \delta u_2^{(m)} \right) \Bigg] d\hat{x}_1 d\hat{x}_2 \\
& + \frac{1}{2} \int\limits_{-a/h}^{+a/h} \frac{F h_1}{h^2} \left(\frac{\partial u_1^{(1)}}{\partial \hat{x}_2} + \frac{\partial u_2^{(1)}}{\partial \hat{x}_1} \right)\Bigg|_{\hat{x}_2=-h_1/h} \left(\frac{\partial u_1^{(1)}}{\partial \hat{x}_2} + \frac{\partial u_2^{(1)}}{\partial \hat{x}_1} \right)\Bigg|_{\hat{x}_2=-h_1/h} d\hat{x}_1 \\
& - \int\limits_{-a/h}^{+a/h} P\delta(h\hat{x}_1)\delta u_2^{(1)}|_{\hat{x}_2=0} d\hat{x}_1
\end{aligned} \tag{17}$$

Using the expression (16) by direct verification it is proven that $\omega_{ijnm} = \omega_{mnji}$. Taking these equalities into account the following expression for $\delta J(u^{(m)})$ is obtained from equation (17):

$$\begin{aligned}
\delta J(u^{(m)}) = \sum_{m=1}^{2} \iint\limits_{\hat{\Omega}_m} \Bigg[& T_{11}^{(m)} \frac{\partial\left(\delta u_1^{(m)}\right)}{\partial \hat{x}_1} + T_{22}^{(m)} \frac{\partial\left(\delta u_2^{(m)}\right)}{\partial \hat{x}_2} + T_{21}^{(m)} \frac{\partial\left(\delta u_1^{(m)}\right)}{\partial \hat{x}_2} + T_{12}^{(m)} \frac{\partial\left(\delta u_2^{(m)}\right)}{\partial \hat{x}_1} \\
& - \omega^2 \rho^{(m)} \left(u_1^{(m)} \delta u_1^{(m)} + u_2^{(m)} \delta u_2^{(m)} \right) \Bigg] d\hat{x}_1 d\hat{x}_2 \\
& + \frac{1}{2} \int\limits_{-a/h}^{+a/h} \frac{F h_1}{h^2} \left(\frac{\partial u_1^{(1)}}{\partial \hat{x}_2} + \frac{\partial u_2^{(1)}}{\partial \hat{x}_1} \right)\Bigg|_{\hat{x}_2=-h_1/h} \left(\frac{\partial u_1^{(1)}}{\partial \hat{x}_2} + \frac{\partial u_2^{(1)}}{\partial \hat{x}_1} \right)\Bigg|_{\hat{x}_2=-h_1/h} d\hat{x}_1 \\
& - \int\limits_{-a/h}^{+a/h} P\delta(h\hat{x}_1)\delta u_2^{(1)}|_{\hat{x}_2=0} d\hat{x}_1
\end{aligned} \tag{18}$$

Using the equation

$$\delta \mathrm{J} = \sum_{m=1}^{2} \iint\limits_{\Omega_m} [\cdot]\, dx_1 dx_2 = \int_{-a}^{a} \left(\int_{-h_1}^{0} [\cdot]\, dx_2 \right) dx_1 + \int_{-a}^{a} \left(\int_{-h}^{-h_1} [\cdot]\, dx_2 \right) dx_1 = 0, \tag{19}$$

after performing some well-known transformations the equations $\partial T_{ij}^{(m)}/\partial x_i - \omega^2 u_j^{(m)} = 0$, boundary and contact conditions given for forces in (6), (7) and (11) are attained.

In this way the validity of the functional (14)-(16) is proven. Then the regions Ω_1 and Ω_2 given in (1) are divided into finite number rectangular Lagrange family of quadratic elements (Zienkiewicz and Taylor (1989)). The number of these finite elements is determined from the numerical convergence requirement. By employing usual procedure, we obtain the equation

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{a} = \mathbf{r} \tag{20}$$

from relation (19), where $\mathbf{K}$ is the stiffness matrix, $\mathbf{a}$ is the vector whose components are the values of the displacements at selected nodes, r is a force vector and $\mathbf{M}$ is a mass matrix. To reduce the size of the paper the explicit expressions for calculation of components of the matrices **K, M** and vector r are not given here. But, note that these expressions are derived from equations (15), (16), (17) and (19), by employing the known procedures. Thus, with the above-stated we exhaust the consideration of the FEM modeling of the formulated problem.

4 Numerical Results and Discussions

The main aim of the present numerical investigations is to study the influence of the shear-spring parameter Fon the frequency response of the bi-layered plate which rests on the rigid foundation. Before the analysis of the numerical results related to this aim we consider the verification of the algorithm and PC programs which are composed by the authors and used under obtaining these results.

Let us introduce the notation

$$\Omega = \frac{\omega h_1}{c_2^{(1)}}, \quad \eta_m = \frac{\sigma_{11}^{(m),0}}{\mu^{(m)}}, \quad e = \frac{E^{(1)}}{E^{(2)}} \tag{21}$$

where $c_2^{(1)} = \sqrt{\mu^{(1)}/\rho^{(1)}}$ and assume that $\rho^{(1)}/\rho^{(2)} = 1$, $\nu^{(1)} = \nu^{(2)} = 0.33$, $h_1 = h_2 = h/2$.

For validation of the used programs we consider the case where $\Omega = 0$, $\eta_1 = \eta_2 = 0$, $F = 0$ and $e = 1$. Note that this case under $\nu = 0.33$ for the plate with infinite length was considered in Uflyand (1963) and corresponding problem was solved by employing Fourier integral transformation method. According to the known mechanical consideration, the results obtained by the use of the present FEM approach in the particular case which is noted above must converge to the corresponding ones attained in Uflyand (1963) as $h/2a \to 0$. This prediction is proven by the graphs given in Fig.2 which show the distribution of the normal stress $\sigma_{22}h/P$ acting between the layer and absolute rigid foundation with respect to x_1/h. In Fig.2, the starred graph denotes the one given in Uflyand (1963). In this way, the validity and trustiness of the used algorithm and programs are proved.

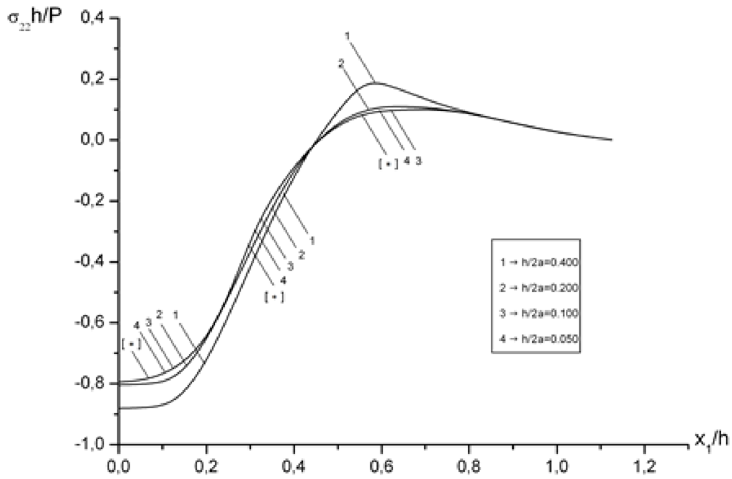

Figure 2: The dependencies between $\sigma_{22}h/P$ and x_1/h at the bottom surface for various $h/2a$ for the case where $E_1/E_2 = 1$, $\Omega = 0$, and $\eta_1 = \eta_2 = 0$. The notation [...] means Uflyand (1965)

Now we consider numerical results related the influence of the shear-spring parameter F on the distribution of the stress $\sigma_{22}h/P$ with respect to x_1/h under the static loading case, i.e. in the case where $\Omega = 0$. Graphs related to this distribution are given in Fig. 3 which are constructed in the cases where $x_2/h = -h_1/h$ (Fig. 3a) and $x_2/h = -1$ (Fig. 3b) under $e = 1.5$, $h/2a = 0.2$, $\eta_1 = \eta_2 = 0$. It follows from Fig. 3 that an increase in the values of the parameter F causes a decrease of the absolute values of the $\sigma_{22}h/P$ in the certain region $0 \le x_1/h \le (x_1/h)_*$. However, for the region $x_1/h > (x_1/h)_*$ an increase in the values of the parameter F causes an increase in the absolute values of the $\sigma_{22}h/P$. These results also show that in the quantitative sense the main effect of the mentioned above influence arise in the near vicinity of the point $x_1/h = 0$ and the absolute values of $\sigma_{22}h/P$obtained for the interface point on the plane $x_2/h = -h_1/h$ are greater significantly than those obtained on the plane $x_2/h = -1$. This result agrees also well with the known mechanical consideration.

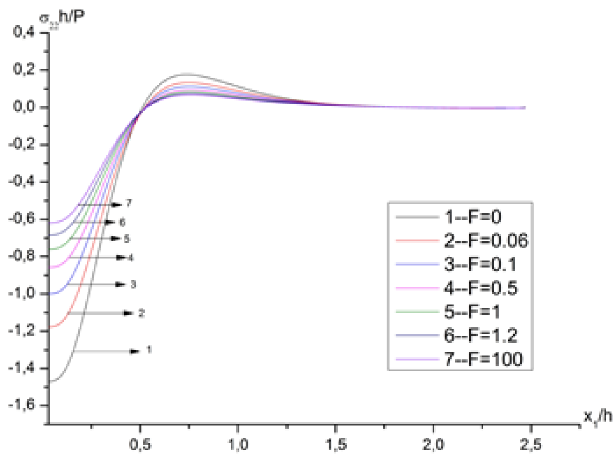

Figure 3a: The influence of F on the values of $\sigma_{22}h/P$ along Ox_1 axis at the interface plane for the case where $E_1/E_2 = 1.5$, $\Omega = 0$, $h/2a = 0.2$, and $\eta_1 = \eta_2 = 0$

Moreover, the results given in Fig. 3 show that the results obtained for $\sigma_{22}h/P$ converge to a certain asymptotic one with the parameter F and this asymptote relates to the full slipping case.

Consider graphs given in Fig. 4 which show the distribution of $\sigma_{22}h/P$ with respect to x_1/hon the interface plane $x_2/h = -h_1/h$ in the cases where $F = 1$ (Fig. 4a), 2 (Fig. 4b) and 3 (Fig. 4c) for various values of the parameter e. Note that, according to the well-known mechanical consideration it can be predicted that an increase in the values of the parameter e must cause a decrease in the absolute values of the stress $\sigma_{22}h/P$. This prediction is confirmed with the results given in Fig. 4 and again proves the validity of the algorithm and PC programs used for obtaining these results.

Thus, we begin the consideration of the frequency response of the bi-layered plate, i.e. the dependence between $\sigma_{22}h/P$ (calculated at $x_1/h = 0$) and dimensionless frequency Ω given in Eq. (21), and the influence of the parameter F on these dependencies. The graphs of the mentioned dependencies are given in Figs. 5 and 6 for the cases where the stress $\sigma_{22}h/P$ is calculated on the interfaces $x_2/h = -h_1/h$ and $x_2/h = -1$, respectively. Moreover, under construction of these graphs the

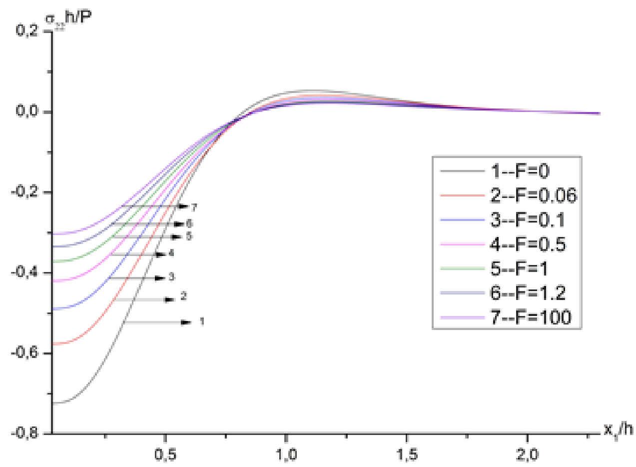

Figure 3b: The influence of F on the values of $\sigma_{22}h/P$ along Ox_1 axis at the bottom surface for the case where $E_1/E_2 = 1.5$, $\Omega = 0$, $h/2a = 0.2$, and $\eta_1 = \eta_2 = 0$

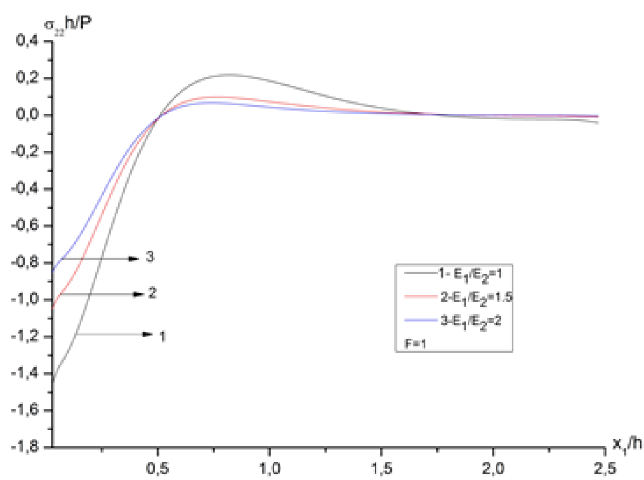

Figure 4a: The influence of E_1/E_2 on the values of $\sigma_{22}h/P$ along Ox_1 axis at the interface plane for the case where $F = 1$, $\Omega = 0$, $h/2a = 0.2$, and $\eta_1 = \eta_2 = 0$

cases where $h/2a = 0.5$ (Figs. 5a and 6a), 0.3 (Figs. 5b and 6b), 0.2 (Figs. 5c and 6c), 0.1 (Figs. 5d and 6d), and 0.05 (Figs. 5e and 6e) are considered. It

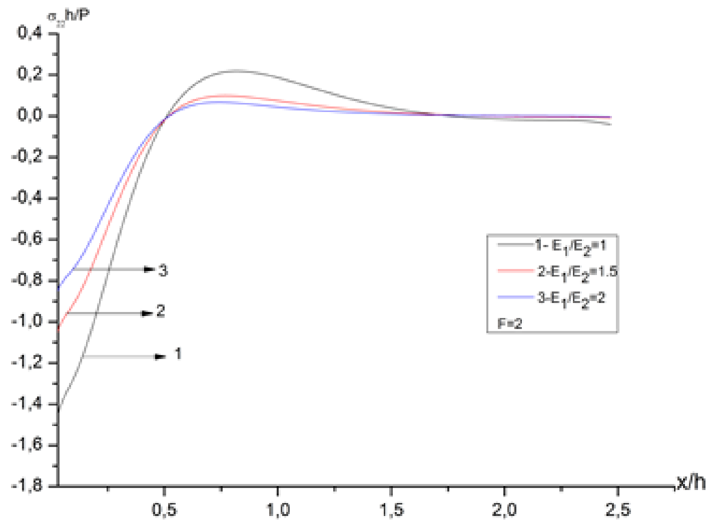

Figure 4b: The influence of E_1/E_2 on the values of $\sigma_{22}h/P$ along Ox_1 axis at the interface plane for the case where $F = 2$, $\Omega = 0$, $h/2a = 0.2$, and $\eta_1 = \eta_2 = 0$

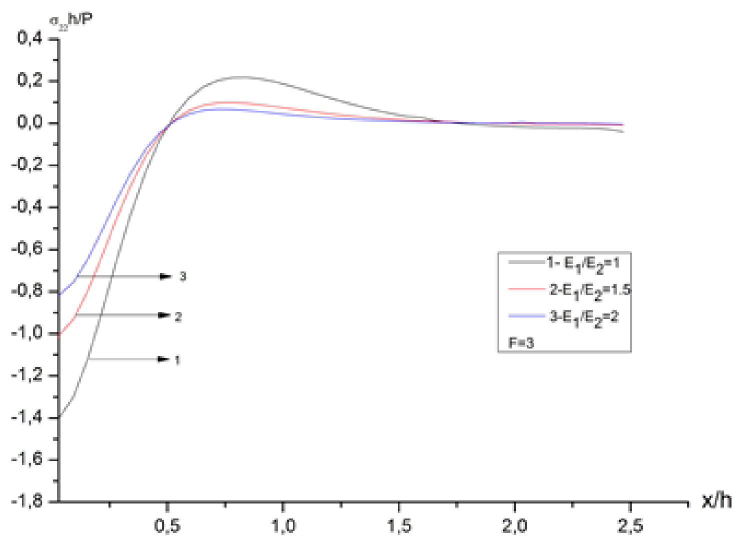

Figure 4c: The influence of E_1/E_2 on the values of $\sigma_{22}h/P$ along Ox_1 axis at the interface plane for the case where $F = 3$, $\Omega = 0$, $h/2a = 0.2$, and $\eta_1 = \eta_2 = 0$

follows from these figures that the influence of the shear-spring type parameter F on the frequency response of the normal stress σ_{22} is significant not only in the quantitative sense, but also in the qualitative sense. According to the numerical results, it can be concluded that an increase in the values of the parameter F, in general, causes to decrease the absolute values of the stress σ_{22}. At the same time, these results show that there are the cases (for example, the cases shown in Figs. 5a and 6a) where the parametric resonance of the system under consideration can occur under certain values of the parameter F.

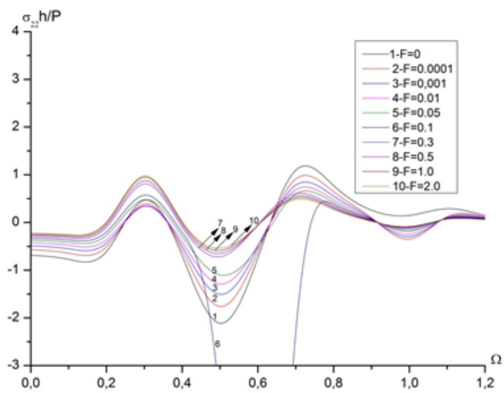

Figure 5a: The dependencies between $\sigma_{22}h/P$ (at $x_1/h = 0$, $x_2/h = -h_1/h$) and Ω for various F for the case where $E_1/E_2 = 1.5$, $h/2a = 0.5$, and $\eta_1 = \eta_2 = 0$

The results given in Figs. 5 and 6 allows us also to make conclusions on the influence of a decrease of the $h/2a$, i.e. of a decrease (an increase) of the length (thickness) of the plate strip on the frequency response under consideration. Thus, these conclusions can be formulated as follows:

- an increase in the values of $h/2a$ causes to decrease of the numbers of the local maximums and minimums of the $\sigma_{22}$with respect to dimensionless frequency Ω;
- the absolute maximum of the $\sigma_{22}$is obtained under certain value of the Ω (denote it by $\Omega*$) and the values of $\Omega*$increase with decreasing $h/2a$.

We recall that the results discussed above are obtained in the case where $\eta_1 = \eta_2 = 0$, i.e. in the case where there is not any initial stress in the layers of the plate strip. Now we analyze the results related to the influence of the mentioned initial stresses,

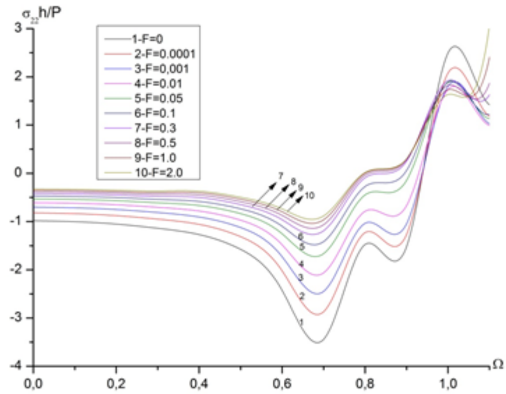

Figure 5b: The dependencies between $\sigma_{22}h/P$ (at $x_1/h = 0$, $x_2/h = -h_1/h$) and Ω for various F for the case where $E_1/E_2 = 1.5$, $h/2a = 0.3$, and $\eta_1 = \eta_2 = 0$

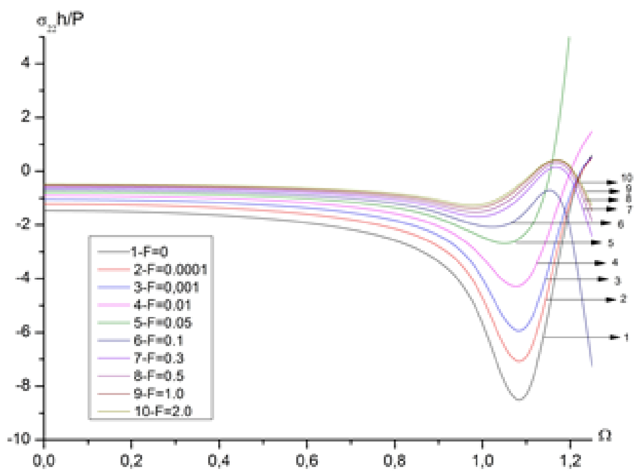

Figure 5c: The dependencies between $\sigma_{22}h/P$ (at $x_1/h = 0$, $x_2/h = -h_1/h$) and Ω for various F for the case where $E_1/E_2 = 1.5$, $h/2a = 0.2$, and $\eta_1 = \eta_2 = 0$

i.e. the influence of the parameters $\eta_1 = \eta_2 \neq 0$ on the distribution and frequency response of the stress σ_{22}.

Fig. 7 shows the influence of the $\eta_1 (= \eta_2)$on the distribution $\sigma_{22}h/P$ on the interfaces $x_2/h = -h_1/h$ (Fig. 7a) and $x_2/h = -1$(Fig. 7b) with respect to x_1/h

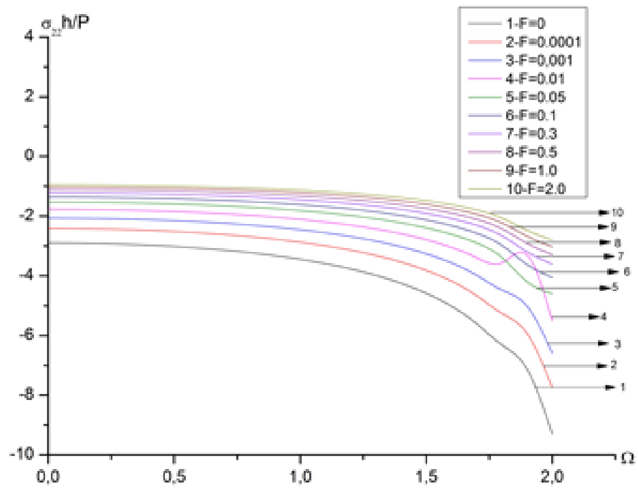

Figure 5d: The dependencies between $\sigma_{22}h/P$ (at $x_1/h = 0$, $x_2/h = -h_1/h$) and Ω for various F for the case where $E_1/E_2 = 1.5$, $h/2a = 0.1$, and $\eta_1 = \eta_2 = 0$

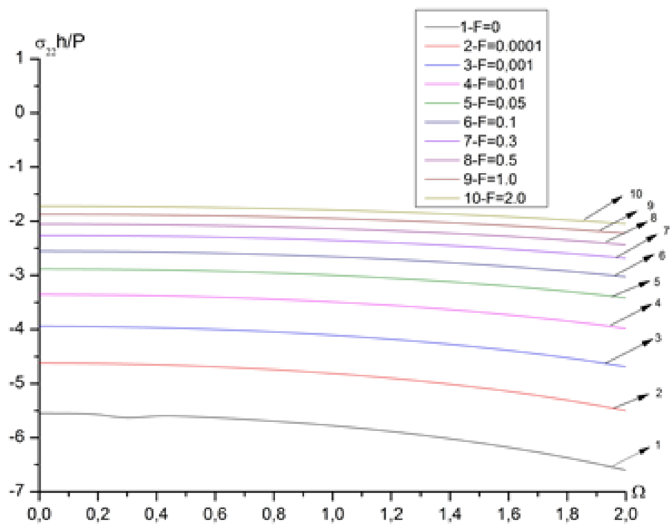

Figure 5e: The dependencies between $\sigma_{22}h/P$ (at $x_1/h = 0$, $x_2/h = -h_1/h$) and Ω for various F for the case where $E_1/E_2 = 1.5$, $h/2a = 0.05$, and $\eta_1 = \eta_2 = 0$

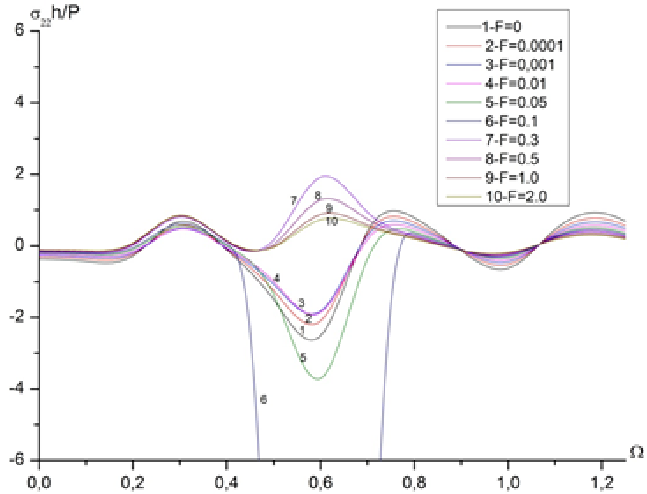

Figure 6a: The dependencies between $\sigma_{22}h/P$ (at $x_1/h = 0$, $x_2/h = -1$) and Ω for various F for the case where $E_1/E_2 = 1.5$, $h/2a = 0.5$, and $\eta_1 = \eta_2 = 0$

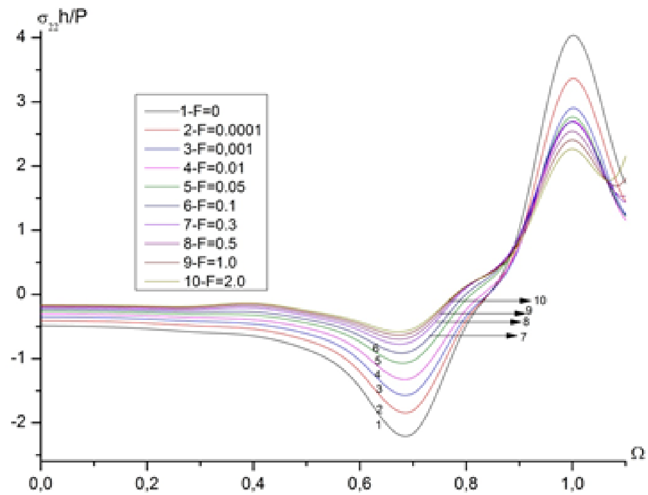

Figure 6b: The dependencies between $\sigma_{22}h/P$ (at $x_1/h = 0$, $x_2/h = -1$) and Ω for various F for the case where $E_1/E_2 = 1.5$, $h/2a = 0.3$, and $\eta_1 = \eta_2 = 0$

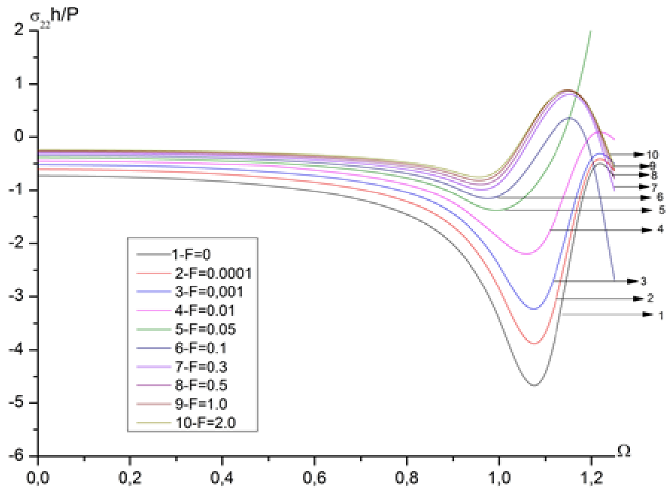

Figure 6c: The dependencies between $\sigma_{22}h/P$ (at $x_1/h = 0$, $x_2/h = -1$) and Ω for various F for the case where $E_1/E_2 = 1.5$, $h/2a = 0.2$, and $\eta_1 = \eta_2 = 0$

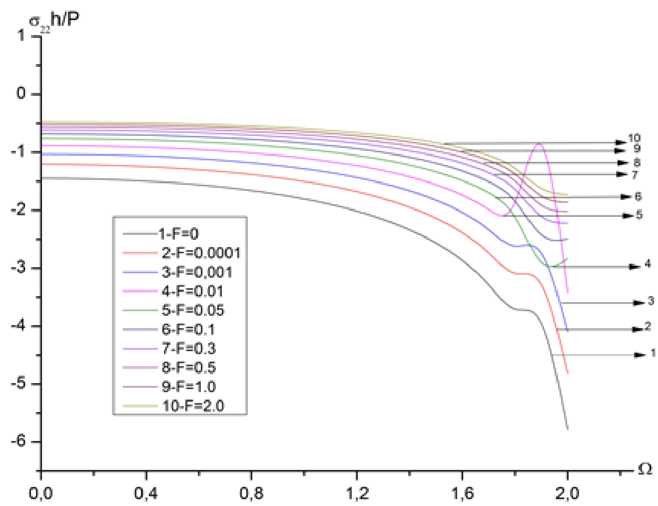

Figure 6d: The dependencies between $\sigma_{22}h/P$ (at $x_1/h = 0$, $x_2/h = -1$) and Ω for various F for the case where $E_1/E_2 = 1.5$, $h/2a = 0.1$, and $\eta_1 = \eta_2 = 0$.

under $e = 1.5$, $h/2a = 0.2$, $F = 1.0$, and $\Omega = 0.9$. It follows from the Fig. 7 that

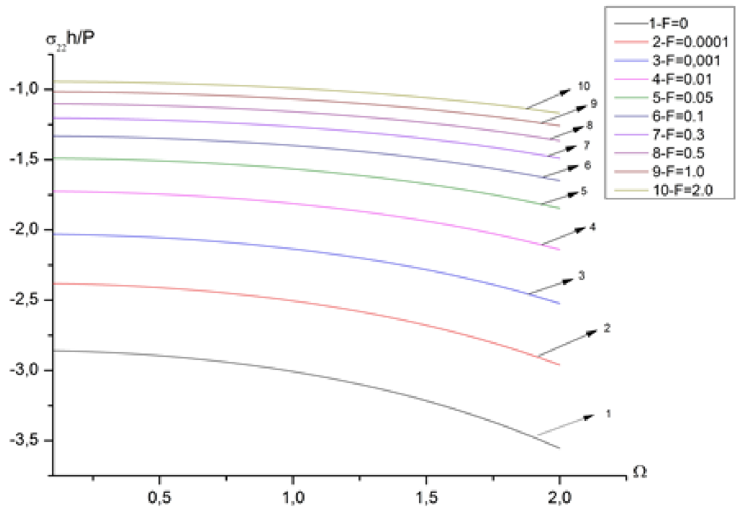

Figure 6e: The dependencies between $\sigma_{22}h/P$ (at $x_1/h = 0$, $x_2/h = -1$) and Ω for various F for the case where $E_1/E_2 = 1.5$, $h/2a = 0.05$, and $\eta_1 = \eta_2 = 0$

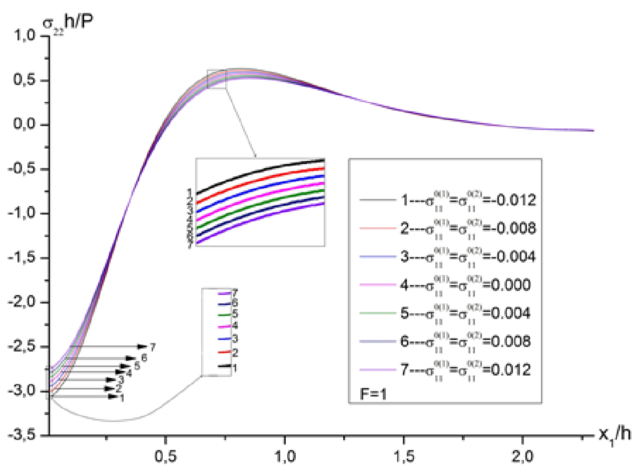

Figure 7a: The dependencies between $\sigma_{22}h/P$ and x_1/h at the interface plane for various $\sigma_{11}^{(1),0}$ and $\sigma_{11}^{(2),0}$ for the case where $E_1/E_2 = 1.5$, $h/2a = 0.2$, and $\Omega = 0.9$

under $0 \le x_1/h \le x_1{*}/h$ the initial stretching (compressing) causes to decrease (to increase) of the absolute values of the σ_{22}. But under $x_1{*}/h < x_1/h \le x_1{**}/h$,

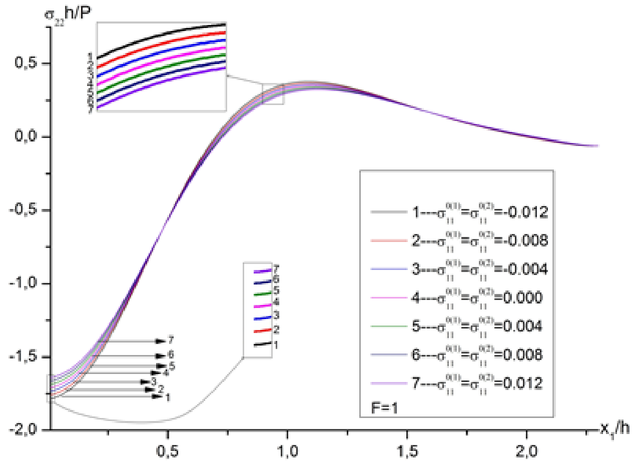

Figure 7b: The dependencies between $\sigma_{22}h/P$ and x_1/h at the bottom surface for various $\sigma_{11}^{(1),0}$ and $\sigma_{11}^{(2),0}$ for the case where $E_1/E_2 = 1.5$, $h/2a = 0.2$, and $\Omega = 0.9$

vice versa, the initial stretching (compressing) causes an increase (a decrease) of the absolute values of the σ_{22}. The values of x_1*/h and x_1**/h can be easily determined from the Fig. 7 and are different for $x_2/h = -h_1/h$ and $x_2/h = -1$. Note that, it also follows from the Fig. 7 that the character of the influence of the initial stresses on the σ_{22} agrees with the corresponding one obtained in Akbarov (2006a, 2006b, 2007a, 2007b), Akbarov, Emiroglu and Tasci (2005), Akbarov and Guler (2007), Akbarov and Ilhan (2008).

Now we consider graphs given in Fig. 8 which show the influence of the $\eta_1(=\eta_2)$on the frequency response of the σ_{22}. These graphs are constructed for various η_1 under $x_2/h = -h_1/h$(Fig. 8a) and $x_2/h = -1$(Fig. 8b) in the case where $F = 1.0$, $e = 1.5$, $h/2a = 0.3$, and $x_1/h = 0$. Analyses of the graphs and their compression of the corresponding ones given in Akbarov, Yildiz and Eröz (2011a) show that in the case where there exist the shear spring type imperfectness between the layers (as an example, in the case where $F = 1.0$) the influence of the initial stresses on the frequency response under consideration becomes more considerable than that in the case where there exists the full contact between the layers. According to Fig. 8, it can be concluded that excepting some particular cases for $\Omega < \Omega'$ ($\Omega > \Omega'$) the initial stretching of the layers causes a decrease (an increase), but the initial compressing an increase (a decrease) of the absolute values of the σ_{22}. Note that the values of the Ω' depend on the values of the initial stresses and can be easily determined from Fig. 8. Moreover, Fig. 8 shows that under $\eta_1 = \eta_2 = 0$,

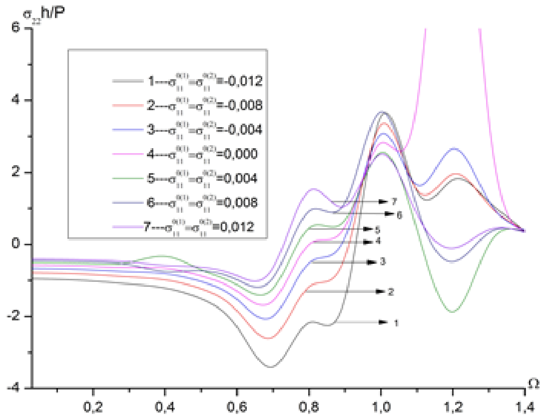

Figure 8a: The dependencies between $\sigma_{22}h/P$ (at $x_1/h = 0$, $x_2/h = -h_1/h$) and Ω for various $\sigma_{11}^{(1),0}$ and $\sigma_{11}^{(2),0}$ for the case where $F = 1$, $E_1/E_2 = 1.5$, and $h/2a = 0.3$

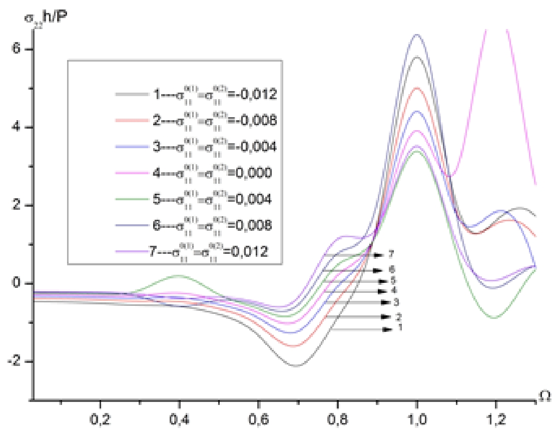

Figure 8b: The dependencies between $\sigma_{22}h/P$ (at $x_1/h = 0$, $x_2/h = -1$) and Ω for various $\sigma_{11}^{(1),0}$ and $\sigma_{11}^{(2),0}$ for the case where $F = 1$, $E_1/E_2 = 1.5$, and $h/2a = 0.3$

i.e. under absence of the initial stresses in the layers, in the case where $\Omega \approx 1.2$ the resonance of the system under consideration takes place. It follows from the results given in Fig. 8 that the initial stretching of the layers prevents this resonance, but the compressing of the layers passes after corresponding resonance mode.

5 Conclusions

In the present paper, within the scope of the piecewise homogeneous body model with utilizing of the TDLTEWISB, the influence of the shear-spring type imperfection of the contact conditions between the layers of the pre-stressed bi-layered plate strip resting on the rigid foundation has been investigated. The corresponding mathematical problem is solved numerically by employing FEM and the numerical results illustrating the influence of the parameter characterized the degree of the mentioned imperfectness, on the frequency response of the normal stress acting on the interface planes between the layers and between the plate and rigid foundation are presented and discussed. According to these results, it can be drawn the following main conclusions:

- by increasing the value of the shear-spring parameter the absolute values of the compressed normal stress decrease, but the values of the stretched normal stress increase;
- an increase in the length of the plate under fixed thickness of that, or a decrease in the values of the thickness of the plate under fixed length of that causes to decrease the number of the local maximums and minimums of the mentioned normal stress with respect to the frequency of the external force;
- the absolute maximum of the noted above normal stress is obtained under "certain value" of the frequency and this "certain value" increases with the thickness of the plate under fixed length, or with decreasing of the length under fixed thickness of that;
- the imperfectness of the contact between the layers of the plate influences also on the character of the action of the initial stresses on the frequency response under consideration. So, if the resonance of the system under consideration occurs in the certain frequency under absence of the initial stresses, then the initial stretching of the layers prevents this resonance, but the initial compressing of those passes after corresponding resonance mode.

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