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On Uniform Approximate Solutions in Bending of Symmetric Laminated Plates

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Abstract: A layer-wise theory with the analysis of face ply independent of lamination is used in the bending of symmetric laminates with anisotropic plies. More realistic and practical edge conditions as in Kirchhoff's theory are considered. An iterative procedure based on point-wise equilibrium equations is adapted. The necessity of a solution of an auxiliary problem in the interior plies is explained and used in the generation of proper sequence of two dimensional problems. Displacements are expanded in terms of polynomials in thickness coordinate such that continuity of transverse stresses across interfaces is assured. Solution of a fourth order system of a supplementary problem in the face ply is necessary to ensure the continuity of in-plane displacements across interfaces and to rectify inadequacies of these polynomial expansions in the interior distribution of approximate solutions. Vertical deflection does not play any role in obtaining all six stress components and two in-plane displacements. In overcoming lacuna in Kirchhoff's theory, widely used first order shear deformation theory and other sixth and higher order theories based on energy principles at laminate level in smeared laminate theories and at ply level in layer-wise theories are not useful in the generation of a proper sequence of 2-D problems converging to 3-D problems. Relevance of present analysis is demonstrated through solutions in a simple text book problem of simply supported square plate under doubly sinusoidal load.

Keywords: Plates, bending, laminates, elasticity.

1 Introduction

In literature, one finds vast amount of investigations and several review articles [Rasoul, Siamak, Philip and Vinney (2012); Ahn, Basu and Woo (2011); Tessler, Sciuva and Gherlone (2010); Carrera and Brischetto (2009); Chen and Wu (2008); Demasi (2008); Carrera (2003); Reddy and Robbins (1994); Reddy (1984, 1990a); Noor and Burton (1989); Kapania and Raciti (1989); Hashin (1983)] to quote a

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few of them reported on the analysis of laminated composite plates. There exist many theories such as single layer theories, smeared laminate theories, layer-wise theories, zigzag theories, etc. in the analysis of bending problems. Emphasis in these theories is for accurate estimation of displacements and transverse stresses required to satisfy continuity across interfaces. In layer-wise and zigzag theories, the analysis is generally based on stationary property of relevant total potential and neighboring plies are coupled through these continuity conditions.

With reference to the exact solution of a 3-D problem, proper analysis of face ply is a prerequisite and dependent on 2-D approximate theories like Kirchhoff's theory [Kirchhoff (1850)], First Order Shear Deformation Theory (FSDT) (based on Hencky (1947)), etc. In FSDT, vertical deflection w_0 and in-plane displacements are coupled in governing differential equations and boundary conditions. Reactive (statically equivalent) transverse shears are combined with in-plane shear resulting in the approximation of the torsion problem. In Kirchhoff's theory, in-plane shear is combined with transverse shear that is implied in Kelvin and Tait's physical interpretation of contracted boundary condition [Love (1934)]. In fact, the torsion problem is associated with flexure problem whereas flexure problem (unlike directly or indirectly implied in energy methods) is independent of the torsion problem. Recently, it is shown Vijayakumar (2011a) that correction to Kirchhoff's w_0 by either Reissner's theory [Reissner (1945)] or FSDT corresponds to an approximate solution of a torsion problem. In fact, the solution of an associated torsion problem is to nullify the effect due to applied or reactive edge stress τ_{xy} in flexure problem (In this connection, pertinent observation is that Kirchhoff's theory and FSDT are valid in the case of hard and soft simply supported plates, respectively). This can be inferred from numerical results reported by Lewinski (1990) in which *higher order theories give decreasing values of vertical deflection*. FSDT, Reissner's theory, other shear deformation theories and reported higher order theories (other than Kirchhoff's theory) based on energy principles at the laminate level in smeared laminate theories and at the ply level in layer-wise theories are definitely not useful in the generation of a proper sequence of 2-D problems converging to 3-D problems. *It does not serve much purpose to compare results from these theories with those from the sequence of 2-D problems proposed in the present investigation.*

The present work is an extension of layer-wise theory [Vijayakumar (2011c)] with some useful modifications (note that the first sentence after equation (50) in the above article should read as "The first term in $\sigma_z^{(k-1)}$ is $z\sigma_z^{(k)}$ from Eq. (1') uncoupled from Eq. (1)."). Modifications are based on the present author's recently developed theory titled 'Poisson's theory of plates in bending'. In this theory, transverse stresses are determined first and in-plane displacements are obtained in terms of these stresses in the preliminary solution. These transverse stresses are independent of material constants (unlike in Kirchhoff's theory and FSDT) and their dependence on material constants is from higher order corrections. The edge support condition on w is replaced by zero ε_z derived from a constitutive relation. Vertical deflection w does not directly play any role in obtaining in-plane displacements and reactive transverse stresses.

In earlier investigations [Vijayakumar (2011b,c)], it is shown that the expansion of displacements in polynomials of thickness coordinate z is not adequate for proper estimation of face and neutral plane deflections. *This fact is overlooked till now in the analysis of even isotropic homogeneous plates through widely used FSDT and other shear deformation theories. A solution of a supplementary problem is required for obtaining neutral plane deflection which is higher than face deflection.* It is, however, observed that an error in the estimation of face deflection is much higher than that of neutral plane deflection. But it is desirable to provide uniform approximation to deformations in the entire plate. This is particularly necessary in the analysis of laminates embedded with piezoelectric actuators so as to describe a proper electric field due to actuators [wide for example, Gopinathan, Varadan and Varadan (2000)]. It appears that no suitable 2-D modeling is reported till now giving more or less the same percentage of approximation to a displacement variable in the entire domain of the plate. Proper higher order theories are only to reduce maximum error in the estimation of a physical variable to a desirable level but not (more or less) the same percentage of error throughout the domain.

In Kirchhoff's theory, in-plane displacements are coupled with vertical displacement due to zero face shear conditions. This coupling is the root cause for the sixteen decade old problem of Poisson-Kirchhoff boundary conditions paradox. To resolve this paradox, Poisson's theory mentioned earlier is used. If one neglects the contribution of in-plane shear in stress resultants $[V_x, V_y]$, in-plane distributions $[Q_x, Q_y]$ of integrated shears $[\tau_{xz}, \tau_{yz}]$ correspond to applied vertical shear loads along edges of the plate. These shear stresses denoted by $[\tau_{xz0}, \tau_{yz0}]$ along with linear $\sigma_z = z\, q/2$ satisfying face load condition forms the basis in defining the auxiliary bending problem. These transverse stresses do not participate in static in-plane equilibrium equations but they contribute to the secondary effects in estimation of in-plane displacements through integration of equilibrium equations. *The need for such a problem exists in the analysis of interior plies due to continuity of transverse stresses across interfaces.*

2 Primary Flexure Problem

For simplicity in presentation, a symmetric laminate bounded by $0 \le X \le a$, $0 \le Y \le b$ and $-h_n \le Z \le h_n$ with interfaces $Z = h_k$ in the Cartesian coordinate system (X, Y, Z) is considered. For convenience, coordinates X, Y and Z and displacements

U, V, and W in non-dimensional form $x = X/L$, $y = Y/L$, $z = Z/h_n$, $u = U/h_n$, $v = V/h_n$, $w = W/h_n$ and half-thickness ratio $\alpha = h_n/L$ with reference to a characteristic length L in the $X-Y$ plane are utilized. The material of each ply is homogeneous and anisotropic with monoclinic symmetry. Let $\alpha_k = h_k/h_n$ so that interfaces are given by $z = \alpha_k(k = 1,2,\ldots,n-1)$ in the upper-half of the laminate.

In the primary problem, the laminate is subjected to asymmetric load $\sigma_z = \pm q(x,y)/2$ and zero shear stresses along its top and bottom faces. Equilibrium equations in terms of stress components *in each ply* (denoted with superscript 'k' but generally omitted unless it is required for clarification) take the form

$$\alpha(\sigma_{x,x} + \tau_{xy,y}) + \tau_{xz,z} = 0 \qquad \rightleftarrows (x,y) \tag{1}$$

$$\alpha(\tau_{xz,x} + \tau_{yz,z}) + \sigma_{z,z} = 0 \tag{2}$$

in which the suffix after ',' denotes the partial derivative operator and $\rightleftarrows$ indicates the interchange in x and y. These three equations have to be solved along with three edge conditions (to be specified later) and six continuity conditions of three displacements and three transverse stresses across interfaces.

2.1 Displacement based models

In displacement based models, stress components are expressed in terms of displacements, via, six stress-strain constitutive relations and six strain-displacement relations. In the present study, these relations are confined to the classical small deformation theory of elasticity.

Upper face values of displacements $[u,v,w]^u$ and transverse stresses $[\tau_{xz},\tau_{yz},\sigma_z]^u$ in a ply are related to its lower face values $[u,v,w]_b$ and $[\tau_{xz},\tau_{yz},\sigma_z]_b$, respectively, through the solution of equations (1-2) together with three conditions specified later along constant x (and y) edges. Moreover, they have to satisfy continuity conditions across interfaces.

It is convenient to denote displacements $[u,v]$ as $[u_i]$, $(i = 1,2)$, in-plane stresses $[\sigma_x,\sigma_y,\sigma_y]$ and transverse stresses $[\tau_{xz},\tau_{yz},\sigma_z]$ as $[\sigma_i]$, $[\sigma_{3+i}]$, $(i = 1,2,3)$, respectively. With the corresponding notation for strains, strain-displacement relations are

$$[\varepsilon_1,\varepsilon_2,\varepsilon_3] = \alpha[u_{,x}, v_{,y}, u_{,y} + v_{,x}] \tag{3}$$

$$[\varepsilon_4,\varepsilon_5,\varepsilon_6] = [u_{,z} + \alpha w_{,x}, v_{,z} + \alpha w_{,y,}, w_{,z}] \tag{4}$$

2.2 Strain-stress and semi-inverted stress-strain relations

$$\varepsilon_i = S_{ij}\sigma_j \quad (i,j=1,2,3,6), \varepsilon_r = S_{rs}\sigma_s \quad (r,s=4,5) \tag{5}$$
$$\sigma_i = Q_{ij}[\varepsilon_j - S_{j6}\sigma_z] \quad (i,j=1,2,3) \tag{6}$$
$$\sigma_r = Q_{rs}\varepsilon_s \quad (r,s=4,5) \tag{7}$$

Along the wall of the laminate, $\varepsilon_6 = S_{6j}\sigma_j$ $(j=1,2,3)$ since σ_6 does not exist.

In the above equations, usual summation convention is used where repeated suffix indicates summation over its specified range of integer values.

With σ_i in Eq. (6), in-plane equilibrium equations (1) become

$$\alpha[Q_{1j}(\varepsilon_j - S_{j6}\sigma_z)_{,x} + Q_{3j}(\varepsilon_j - S_{j6}\sigma_z)_{,y}] + \tau_{xz,z} = 0 \tag{8a}$$
$$\alpha[Q_{2j}(\varepsilon_j - S_{j6}\sigma_z)_{,y} + Q_{3j}(\varepsilon_j - S_{j6}\sigma_z)_{,x}] + \tau_{yz,z} = 0 \tag{8b}$$

2.3 $f_n(z)$ functions and their use

In reducing 3-D problems into a sequence of 2-D problems, a complete set of coordinate functions $f_n(z)$, $(n=0,1,2,3,\ldots)$ are used with the associated 2-D variables. In the present work, they are chosen such that f_{2n+1} and f_{2n} are odd and even functions of z with reference to $z=0$ plane, respectively. They are generated with $f_0=1$ from recurrence relations, $f_{2n+1,zz} = f_{2n,z} = -f_{2n-1} (n \geq 1)$ such that $f_{2n}(\pm\alpha_k) = 0$. They are (up to $n=5$)

$$[f_1, f_2, f_3] = [z, 1/2(\alpha_k^2 - z^2), 1/2(\alpha_k^2 - z^3/3)] \tag{9}$$
$$[f_4, f_5] = [(5\alpha_k^4 - 6\alpha_k^2 z^2 + z^4)/24, z(25\alpha_k^4 - 10\alpha_k^2 z^2 + z^4)/120] \tag{10}$$

Displacements, strains and stresses are expressed in the form (with sum $n=0,1,2,\ldots$)

$$[w,u,v] = [f_{2n}w_{2n}, f_{2n+1}u_{2n+1}, f_{2n+1}v_{2n+1}] \tag{11}$$
$$[\varepsilon_x, \varepsilon_y, \gamma_{xy}, \varepsilon_z] = f_{2n+1}[\varepsilon_x, \varepsilon_y, \gamma_{xy}, \varepsilon_z]_{2n+1} \tag{12}$$

(In-plane distribution components $w_{2n} = -\varepsilon_{z2n-1}, n \geq 1$)

$$[\sigma_x, \sigma_y, \tau_{xy}, \sigma_z] = f_{2n+1}[\sigma_x, \sigma_y, \tau_{xy}, \sigma_z]_{2n+1} \tag{13}$$
$$[\gamma_{xz}, \gamma_{yz}, \tau_{xz}, \tau_{yz}] = f_{2n}[[\gamma_{xz}, \gamma_{yz}, \tau_{xz}, \tau_{yz}]_{2n} \tag{14}$$

In order to maintain the continuity of a 3-D variable across interfaces and keep the associated 2-D variable as a free variable, it is necessary to replace f_{2n+1} by f^*_{2n+1} given by

$$f^*_{2n+1} = f_{2n+1} - \beta_{2n-1}f_{2n-1}, n = 1,2,\ldots$$

In the above equation, $\beta_{2n-1}\alpha_k^2 = [f_{2n+1}(\alpha_k)/f_{2n-1}(\alpha_k)]$ so that $f^*_{2n+1}(\alpha_k) = 0$. With the above replacement of odd f_n functions, transverse stresses and the corresponding displacements become continuous across the interfaces if the variables associated with f_0 and f_1 are continuous across interfaces.

3 Auxiliary Bending Problems

At the first stage of the iterative method, primary transverse stresses $[\tau_{xz}, \tau_{yz}, \sigma_z]^k$ in the k^{th} ply are given by

$$[\tau_{xz}, \tau_{yz}, \sigma_z] = [\tau_{xz0}, \tau_{yz0}, z\sigma_{z1}]^k + [f_2\tau_{xz2}, f_2\tau_{yz2}, f_3\sigma_{z3}]^k \tag{15}$$

In the above equation, $[\tau_{xz0}, \tau_{yz0}]$ are $[\tau_{xz}, \tau_{yz}]_b$ in $(k+1)^{th}$ ply from interface continuity and $\sigma_{z1} = -\alpha(\tau_{xz0,x} + \tau_{yz0,y})$ from equation(2). The second expression consists of reactive stresses in the k^{th} ply.

In view of equation (2), $[\tau_{xz0}, \tau_{yz0}]^k$ are expressed in the form

$$[\tau_{xz0}, \tau_{yz0}]^k = \alpha[\psi_{0,x}, \psi_{0,y}]^k \tag{16}$$

so that $\psi_0(x,y)$ in the k^{th} ply is governed by

$$\alpha^2\Delta\psi_0 + \sigma_{z1} = 0 \tag{17}$$

in which Δ is the plane Laplace operator $(\partial^2/\partial x^2 + \partial^2/\partial y^2)$. Due to $\sigma_z = zq/2$ in the face ply, $\sigma_{z1} = q/2$ remains same throughout the laminate.

One finds that σ_{z1} is neglected in shear deformation theories in the analysis of cross-ply and angle-ply laminates. Its influence through constitutive relations in the case of isotropic plies is considered earlier [Vijayakumar (2011c)] by proper modification of σ_{z3} but neglected in the face plies. Moreover, transverse shear stress conditions along the edges, unlike in Kirchhoff's theory, are specified in the form of ply-wise parabolic distributions, though mathematically valid but not practical even in the face plies.

The equation (17) which is independent of material constants is to be solved with condition along x (and y) constant edges

$$\psi_0 = 0, \quad \alpha\psi_{0,x} = T_{xz0}(y) \qquad \rightleftarrows (x,y) \tag{18}$$

In the above equation, $T_{xz0}(y)$ is the prescribed distribution along the $x =$ constant wall of the laminate independent of material constants of plies. If it is independent of y, it becomes a more practical condition compared to the stress resultant which has no unique point-wise distribution of the relevant shear stress.

3.1 Homogeneous isotropic plate: Significance of function $\psi_0(x,y)$

The condition $\psi_0 = 0$ is different from the usual condition $w_0 = 0$ in the Kirchhoff's theory and FSDT. The function ψ_0 is related to normal strain ε_z. In the isotropic case, ψ_0 is proportional to $e_1(x,y) = (\varepsilon_{x1} + \varepsilon_{y1})$. e_1 is proportional to Δw_0 in the Kirchhoff's theory. Gradients of ψ_0 are proportional to gradients of transverse strains in FSDT. If the plate is free of applied transverse stresses, that is , the plate is subjected to bending and twisting moments only, $\psi_0(x,y)$, thereby, $\Delta w_0 \equiv 0$ from equation (2). (In Kirchhoff's theory, the tangential gradient of Δw_0 is proportional to the corresponding gradient of applied τ_{xy} along the edge of the plate implied from Kelvin and Tait's physical interpretation of the contracted transverse shear condition). This Laplace equation is not adequate to satisfy two in-plane edge conditions. One needs its conjugate harmonic functions to express in-plane displacements in the form

$$[u_1, v_1] = -\alpha z[w_{0,x} + \varphi_{0,y}, w_{0,y} - \varphi_{0,x}]$$

The function φ_0 was introduced earlier by Reissner (1945) as a stress function in satisfying equation (2).

After finding w_0 and φ_0 from solving in-plane equilibrium equations (1), correction w_{0c} to w_0 from zero face shear conditions is given by

$$w_{0c} = \int [\varphi_{0,y}\, dx - \varphi_{0,x}\, dy]$$

One should note here that $w(x,y) = w_0 + w_{0c}$ does not satisfy prescribed edge condition on w.

In FSDT, $[u_1, v_1]$ are obtained from solving Eqs. (1). w_0 has to be obtained from zero face shear conditions in the form

$$\alpha w_0 = \int [u_1\, dx + v_1\, dy]$$

In the case of non-homogeneous applied transverse stresses, w_0 is coupled with equations (1, 2) and edge conditions in FSDT.

With reference to the present analysis, it is relevant to note the following observation: In the preliminary solution, σ_z, $\alpha[\tau_{xz}, \tau_{yz}]$, $\alpha^2[\sigma_x, \sigma_y, \tau_{xy}]$ are of O(1). As such, estimation of in-plane stresses, thereby, in-plane displacements are not dependent on w_0. The support condition $w_0 = 0$ along the edge of the plate does not play any role in the estimation of in-plane displacements. In fact, Kirchhoff's theory gives only a lower bound for α for validity of small deformation theory. The validity of its application is a questionable proposition for small values of α even slightly higher than this lower bound. Edge condition on w_0 gives highly constrained solutions for displacements. Shear deformations are included in FSDT in which face shear conditions are not satisfied and Reissner's theory in which edge condition on w_0 is not satisfied. Both these theories give more or less same maximum vertical deflection. It shows that satisfying either edge condition on w_0 or face shear conditions have equal effect on deformations. In each of these theories, both conditions are satisfied only in the limit. Limiting processes, however, nullify the effect due to the applied or reactive edge stress τ_{xy} in the bending problem. Moreover, linear variation of σ_z in z is neglected in constitutive relations. The present analysis is to show that this linear σ_z results in more flexible plate deformations in conjunction with zero face shear conditions.

In view of the above observations, utility of ψ_0 in the auxiliary problem and gradients of $[\psi, \varphi]$ in representing $[u, v]$ like in the earlier work [Vijayakumar (2011b,c)] will be clear in the iterative procedure adapted in the present work. $\psi_0 = 0$ implies ε_0 along the supported edge. The edge condition $\varepsilon_z = 0$ is mathematically valid and more so at point supports along an edge of the plate. It is to be noted that ψ_0, thereby, $[\tau_{xz0}, \tau_{yz0}]$ remain same throughout the laminate. The condition $\varepsilon_z = 0$ along the supported edge does not ensure zero w_0. One can have a support to prevent the vertical deflections of intersections of faces with the wall of the laminate.

In the 3-D problem of homogeneous plates, $w_0(x,y,0)$ and $w_0(x,y,h)$ are neutral plane deflection w_{0N} and face deflection W_{0F}, respectively. Both of them are 2-D functions and they are from thickness-wise integration of ε_z in a 2-D problem. Determination of w_{0N} used as unknown variable is dependent on edge support condition and that of w_{0F} on zero face shear conditions. The present analysis is with reference to finding w_{0F} and the need to satisfy edge condition on w_{0N} does not exist.

3.2 In-plane displacements $(u_1, v_1)_c$ *in Auxiliary Problem*

The function ψ_0, thereby, $\sigma_z (= zq/2)$ do not participate in the static in-plane equilibrium equations (1) and are independent of ply material constants. They provide corrections in the semi-inverted in-plane stress-strain relations and secondary corrections in relevant transverse stresses through solutions of in-plane displacements. In the present analysis, displacements similar to those in Kirchhoff's theory (and FSDT) are denoted by $(w_0, u_1, v_1)_b$. In-plane distributions of reactive transverse stresses in the primary problem are considered to be transverse stresses in the auxiliary problem. As such, $(w_0, u_1, b_1)_b$ correspond to transverse stresses $[\tau_{xz}, \tau_{yz}] = f_2[\tau_{xz0}, \tau_{yz0}]$ and $\sigma_z = f_3 \sigma_{z1}$. Transverse stresses $[\tau_{xz0}, \tau_{yz0}, \sigma_{z1}]$ participate in the integrated equilibrium equations in the determination of corrections $(u_1, v_1)_c$ to bending displacements $(u_1, v_1)_b$ but not in static *in-plane* equilibrium equations. Hence, (u_1, v_1) are considered as real in the former but are absent in the latter case. Correspondingly, ε_{z1} due to σ_{z1} from the constitutive relation in the auxiliary problem is virtual but real if one neglects σ_{z1} like in Kirchhoff's and shear deformation theories.

In view of linear σ_z, normal strains including $\varepsilon_z = z\varepsilon_{z1}(x,y)$ are linear in z from constitutive relations. As such, in-plane displacements (u,v) in the auxiliary problem are linear in z so that $(u,v) = z(u_1,v_1)$. Determination of (u,v) satisfying equilibrium equations requires $f_2[\tau_{xz2}, \tau_{yz2}]$ along with $\sigma_z = f_3\sigma_{z3}$. Hence, transverse stresses in each ply in the auxiliary problem are

$$\tau_{xz} = f_0\tau_{xz0} + [f_2\tau_{xz2}]^k \qquad \rightleftarrows (x,y) \tag{19}$$

$$\sigma_z = f_1 q/2 + [f_3\sigma_{z3}]^k \tag{20}$$

To keep corrective σ_{z3} as a free variable, f_3 is modified in the form

$$f_3^* = f_3(z) - \beta_1 f_1(z) \tag{21}$$

In the above equation, $\beta_1\alpha_k^2 = [f_3(\alpha_k)/f_1(\alpha_k)]$ so that $f_3^*(\alpha_k) = 0$. Above transverse stresses with $\sigma_z = z(q/2) + [f_3^*\sigma_{z3}]^k$ are continuous across interfaces. Normal stress σ_z in Eq. (15) becomes

$$\sigma_z = [q/2 - \beta_1\sigma_{z3}]z + f_3(z)\sigma_{z3} \tag{22}$$

In-plane distributions (u_1, v_1) are modified in the form with

$$\gamma_{xz0} = S_{44}\tau_{xz0} + S_{45}\tau_{yz0} \tag{23}$$

$$u_1^* = u_1 + \gamma_{xz0} - \alpha w_{0,x} \qquad \rightleftarrows (x,y),(u,v) \tag{24}$$

Modified transverse shear stresses from stress-strain and strain-displacement relations are

$$\tau_{xz0}^* = Q_{44}u_1 + Q_{45}v_1 + \tau_{xz0} \qquad \rightleftarrows (x,y),(u,v),(4,5) \tag{25}$$

Substitution of the above shear stresses in Eq. (2) with $\sigma_z = [q/2 - \beta_1\sigma_{z3}]z$ gives

$$\alpha[(Q_{44}u_1 + Q_{45}v_1)_{,x} + (Q_{54}u_1 + Q_{55}v_1)_{,y} = \beta_1\sigma_{z3} \tag{26}$$

For the use of $[u_1^*, v_1^*]$ in the integration of equilibrium equations (1, 2) and the need to keep τ_{xz0}^* equal to the prescribed τ_{xz0} along segments of the edge, it is necessary to express $[u_1, v_1]$ as

$$[u_1, v_1] = -\alpha[\psi_{1,x} + \varphi_{1,y}, \psi_{1,y} - \varphi_{1,x}] \tag{27}$$

Contributions of ψ_1 and w_0 in $[u_1, v_1]^*$ are one and the same in giving corrections to $w(x,y,z)$ and transverse stresses (in fact, contribution of w_0 is through strain-displacement relations in static equilibrium equations and through constitutive relations in through-thickness integration of equilibrium equations). Hence, w_0 in $[u_1, v_1]^*$ is replaced by ψ_1 (so as to be independent of w_0 used in strain-displacement relations) so that $[u_1, v_1, \varepsilon_{x1}, \varepsilon_{y1}, \gamma_{xy1}]^*$ are

$$[u_1, v_1]^* = -\alpha[(2\psi_{1,x} + \varphi_{1,y}), (2\psi_{1,y} - \varphi_{1,x})] + [\gamma_{xz0}, \gamma_{yz0}] \tag{28}$$

$$\varepsilon^*_{x1} = (\tilde{\varepsilon}_{x1} + \alpha\gamma_{xz0,x}) \qquad \rightleftarrows (x,y) \tag{29a}$$

$$\gamma^*_{xy1} = [\tilde{\gamma}_{xy1} + \alpha(\gamma_{xz0,y} + \gamma_{yz0,x})] \tag{29b}$$

In the above equations,

$$[\tilde{\varepsilon}_{x1}, \tilde{\varepsilon}_{y1}] = -\alpha^2[(2\psi_{1,xx} + \varphi_{1,xy}), (2\psi_{1,yy} - \varphi_{1,xy})]$$

$$\tilde{\gamma}_{xy1} = -\alpha^2[(4\psi_{1,xy} + \varphi_{1,yy} - \varphi_{1,xx})]$$

From integration of equilibrium equations (8, 2) using the above strains, reactive transverse stresses are

$$\tau_{xz2} = \alpha[Q_{1j}(\tilde{\varepsilon}_j - S_{j6}\sigma_{z1})_{,x} + Q_{3j}(\tilde{\varepsilon}_j - S_{j6}\sigma_{z1})_{,y}], (j = 1,2,3) \tag{30}$$

$$\tau_{yz2} = \alpha[Q_{2j}(\tilde{\varepsilon}_j - S_{j6}\sigma_{z1})_{,y} + Q_{3j}(\tilde{\varepsilon}_j - S_{j6}\sigma_{z1})_{,x}], (j = 1,2,3) \tag{31}$$

$$\alpha(\tau_{xz2,x} + \tau_{yz2,y}) + \sigma_{z3} = 0 \qquad \texttt{(here, } \sigma_{z3} \texttt{ is coefficient of } f_3\texttt{)} \tag{32}$$

The equation governing in-plane displacements (u_1, v_1), noting that σ_{z3} from Eq.(26) is negative of the one from Eq.(32) due to $(f_{3,zz} + f_1) = 0$ (since $f_{3,zz} = -f_1$ from $f(z)$ functions in Eq. (9)), is given by

$$\alpha\beta_1(\tau_{xz2,x} + \tau_{yz2,y}) = \alpha[(Q_{44}u_1 + Q_{45}v_1)_{,x} + (Q_{54}u_1 + Q_{55}v_1)_{,y}] \tag{33}$$

With the condition zero $\omega_z(= v_{,x} - u_{,y})$ required to decouple bending and torsion, the above equation consists of a harmonic equation $\Delta\varphi_1 = 0$ and a fourth order equation in ψ_1 to be solved with conditions along constant x (and y) edges

$$u^*_1 = 0 \texttt{ or } \sigma^*_1 = 0 \qquad \rightleftarrows (x,y), (u,v) \tag{34a}$$

$$v^*_1 = 0 \texttt{ or } \tau^*_{xy1} = 0 \qquad \rightleftarrows (u,v) \tag{34b}$$

$$\psi_1 = 0 \texttt{ or } \tau_{xz2} = 0 \qquad \rightleftarrows (x,y) \tag{34c}$$

(*If ψ_1 is zero in the above condition all along the closed contour of the edge, it can be shown that φ_1 is identically zero like in the simply supported (hard type) plate*)

Solutions for ψ_1 and φ_1 from Eqs.(33, 34) give in-plane displacements $[u_1, v_1]$. Edge conditions (34) are due to displacements in Eq. (28) which are independent of w_0 and used in the integration of equilibrium equations. *The displacements and transverse stresses thus obtained are dependent on material constants.*

These displacements and transverse stresses in each ply are designated with suffix '$_c$' for use in the further analysis so that

$$[u,v] = z[u_{1c}, v_{1c}]; \sigma_z = zq/2 + f_3(z)\sigma_{z3c}$$
$$[\tau_{xz}, \tau_{yz}] = [\tau_{xz0}, \tau_{yz0}] + f_2(z)[\tau_{xz2c}, \tau_{yz2c}] \tag{35}$$

4 Displacements (u_{1b}, v_{1b}) in a ply in the primary bending problem

In the absence of higher order in-plane displacement terms, transverse stresses in the bending problem with addition of solutions denoted by subscript 'c' from an auxiliary problem are

$$\tau_{xz2} = \tau_{xz0} + (\tau_{xz2c}) \qquad \rightleftarrows (x,y) \tag{36}$$
$$\sigma_{z3} = \sigma_{z1} + (\sigma_{z3c}) \tag{37}$$

Since the added terms are dependent on material constants, above transverse shear stresses are no longer applied stresses along the edges but they are completely known.

It is convenient in the further analysis to express $[\tau_{xz2}, \tau_{yz2}]$ as gradients of a function ψ so that

$$[\tau_{xy2}, \tau_{yz2}] = \alpha[\psi_{2,x}, \psi_{2,y}] \tag{38}$$

Displacements (u_{1b}, v_{1b}) are governed by Eqs. (8) without σ_{z1} since (u_{1c}, v_{1c}) are due to σ_{z1} in constitutive relations (it has to be neglected in Kirchhoff's and shear deformation theories due to the absence of relevant transverse stresses from the auxiliary problem). Hence,

$$\alpha[Q_{1j}\varepsilon_{j,x} + Q_{3j}\varepsilon_{j,y}] = \alpha\psi_{2,x} \tag{39a}$$
$$\alpha[Q_{2j}\varepsilon_{j,y} + Q_{3j}\varepsilon_{j,x}] = \alpha\psi_{2,y} \tag{39b}$$

Above equations have to be solved with edge conditions

$$u_1 = 0 \;\texttt{ or }\; \sigma_{x1} = zT_{x1}(y) \qquad \rightleftarrows (x,y),(u,v) \tag{40a}$$
$$v_1 = 0 \;\texttt{ or }\; \tau_{xy1} = zT_{xy1}(y) \qquad \rightleftarrows (x,y),(u,v) \tag{40b}$$

Due to the condition that the rotation $(v_{,x} - u_{,y}) = 0$ required to decouple bending and torsion problems, derivatives of in-plane strains are

$$[\varepsilon_{1,y}, \varepsilon_{1,x}, \varepsilon_{3,y}, \varepsilon_{1,x}] = \alpha^2[v_{,xx}, u_{,yy}, 2u_{,yy}, 2v_{,xx}] \tag{41}$$

With the use of the above relations, equations (39) and edge conditions become independent of cross derivative terms $[u_{xy}, v_{xy}]$. Equations (39) together with two edge conditions (40) form a fourth order system in each ply for finding $(u_1, v_1)_b$ in the primary bending problem. Like in the auxiliary problem, these displacements are also independent of lamination.

5 Supplementary problem in the face ply

It is known that use of polynomial $f(z)$ functions satisfying zero face shear conditions in reducing 3-D problems to 2-D problems is not adequate to reflect true solutions of 3-D problems [Vijayakumar (2011b)]. It is due to the same rotations of normal to the deformed face and neutral planes, like in Kirchhoff's theory and FSDT, implying same deformations of these planes. Neutral plane deflection has to be higher than face deflection since elastic medium is on either side of the neutral plane whereas it is on one side of the face plane. This deficiency is rectified here as in the above referred work through the solution of a supplementary problem in which odd functions of z are replaced by $\sin(\pi z/2)$.

Transverse stresses in the face ply with $\tau_{xz2} = (\tau_{xz0} + \tau_{xz2c})$ and $\sigma_{z3} = (\sigma_{z1} + \sigma_{z3c})$ are

$$\tau_{xz} = \tau_{xz0} + f_2 \tau_{xz2} \qquad \rightleftarrows (x, y) \tag{42}$$

$$\sigma_z = z\sigma_{z1} + f_3 \sigma_{z3} \tag{43}$$

In-plane distributions of displacements corresponding to the above transverse stresses are

$$u_1 = (u_{1c} + u_{1b}) \text{ and } u_1^* = (u_{1c} + u_{1b})^* \qquad \rightleftarrows (u, v) \tag{44}$$

Corrective in-plane displacements in the supplementary problem are assumed in the form:

$$u_s = u_{1s} \sin(\pi z/2) \rightleftarrows (u, v) \tag{45}$$

In-plane stresses are

$$\sigma_{1si} = Q_{ij} \varepsilon_{1sj} \sin(\pi z/2)(i, j = 1, 2, 3) \tag{46}$$

With the above in-plane stresses along with $[\tau_{xz}, \tau_{yz}] = [\tau_{xz2}, \tau_{yz2}]_s \cos(\pi z/2)$ and $\sigma_{z3} = \sigma_{z3s} \sin(\pi z/2)$, integration of equilibrium equations give

$$\tau_{xz2s} = -(2/\pi)\alpha[Q_{1j}\varepsilon_{1sj,x} + Q_{3j}\varepsilon_{1sj,y}] \tag{47a}$$

$$\tau_{yz2s} = -(2/\pi)\alpha[Q_{2j}\varepsilon_{1sj,y} + Q_{3j}\varepsilon_{1sj,x}] \tag{47b}$$

$$\sigma_{z3s} = (2/\pi)^2\alpha^2[Q_{1j}\varepsilon_{1sj,xx} + 2Q_{3j}\varepsilon_{1sj,xy} + Q_{2j}\varepsilon_{1sj,yy}] \tag{48}$$

In-plane distributions u_{1s} and v_{1s} are added as corrections to the known in-plane displacements (u_1^*, v_1^*) so that (u, v) in the supplementary problem are

$$u = (u_1^* + u_{1s})\sin(\pi z/2) \qquad (u, v) \tag{49}$$

By equating σ_{z3s} (from integration of equations (1, 2) with 's' variables) with $\beta_1\sigma_{z3c}$ (from static equations with '*' variables), one equation governing $[u_{1s}, v_{1s}]$ is

$$(2/\pi)^2\alpha[Q_{1j}\varepsilon_{1sj,xx} + 2Q_{3j}\varepsilon_{1sj,xy} + Q_{2j}\varepsilon_{1sj,yy}] = \beta_1\sigma_{z3c} \tag{50}$$

By expressing $[u_{1s}, v_{1s}] = -\alpha[\psi_{1s,x}, (\psi_{1s,y}]$ due to the second equation $v_{1s,x} = u_{1s,y}$, equation (50) becomes a fourth order equation in ψ_{1s} to be solved with two in-plane conditions along constant x (and y) edges

$$(u_1^* + u_{1s}) = 0 \ \texttt{or} \ \sigma_{x1}^* + \sigma_{xs1} = 0 \qquad \rightleftarrows (x, y), (u, v) \tag{51a}$$

$$(v_1^* + v_{1s}) = 0 \ \texttt{or} \ \tau_{xy1}^* + \tau_{xys1} = 0 \qquad \rightleftarrows (u, v) \tag{51b}$$

6 Continuity of displacements and transverse stresses across interfaces

In-plane displacements and transverse stresses in the face ply from the above analysis are

$$u = zu_1 + (u_1^* + u_{1s})\sin(\pi z/2) \qquad (u, v) \tag{52}$$

$$\tau_{xz} = \tau_{xz0} + f_2\tau_{xz2} + (\tau_{xz2}^* + \tau_{xz2s})(\pi/2)\cos(\pi z/2) \qquad (x, y) \tag{53}$$

$$\sigma_z = z(q/2) + [f_3 - \sin(\pi z/2)]\beta_1\sigma_{z3} - \sigma_{z3s}\sin(\pi z/2) \tag{54}$$

In the interior plies, displacements $[u_1, v_1]$, thereby, $[u_1^*, v_1^*]$ in the neighboring plies are obtained from solution of sixth order system of equations (1, 2) governing $[u_1, v_1]$. They are dependent on material constants but independent of lamination. Displacements $[u_1, v_1]_s$ and transverse stresses $[\tau_{xz2}, \tau_{yz2}, \sigma_{z3}]_s$ are obtained from continuity conditions across interfaces.

Continuity of (u, v) across interfaces is simply assured through the following recurrence relations

$$[u_{1s}^{(k)} - u_{1s}^{(k+1)}]\sin\frac{\pi}{2}\alpha_k = \alpha_k[u_1^{(k+1)} - u_1^{(k)}] + [\alpha_k + \sin\frac{\pi}{2}\alpha_k][u_1^{(k+1)} - u_1^{(k)}]^* \tag{55}$$

Since $[\tau_{xz0}, \tau_{yz0}]$ and $\sigma_{z1} = zq/2$ are same throughout the laminate, recurrence relations for $[\tau_{xz2}, \tau_{yz2}, \sigma_{z3}]$ are

$$\{\tau_{xz2s}^{(k)} - \tau_{xz2s}^{(k+1)} + [\tau_{xz2}^{(k)} - \tau_{xz2}^{(k+1)}]^*\}\frac{\pi}{2}\cos\frac{\pi}{2}\alpha_k = f_2^{(k+1)}(\alpha_k)\tau_{xz2}^{(k+1)} \qquad \rightleftarrows (x,y) \tag{56}$$

$$\{\sigma_{z3s}^{(k)} - \sigma_{z3s}^{(k+1)} + \beta_1[\sigma_{z3}^{(k)} - \sigma_{z3}^{(k+1)}]\}\sin\frac{\pi}{2}\alpha_k = \beta_1[f_3^{(k+1)}(\alpha_k)\sigma_{z3}^{(k+1)} - f_3^{(k)}(\alpha_k)\sigma_{z3}^{(k)}] \tag{57}$$

With ε_{z1} from constitutive relation, vertical deflection $w(x,y,z)$ is given by

$$w = w_0 - f_2\varepsilon_{z1} + w_{0s}(\pi/2)\cos(\pi z/2) \tag{58}$$

Vertical deflections w_0 and w_{0s} in the above equation are obtained from integration of shear strain- displacement relations (whereas ε_{z1} which does not participate in determination of $(u_1, v_1)_b$ is obtained from the constitutive relation (5) in the interior of each ply). They are

$$\alpha w_0 = \int [(\varepsilon_{40} - u_1)\,dx + (\varepsilon_{50} - v_1)\,dy] \tag{59}$$

$$\alpha w_{0s} = \int [(\varepsilon_{40} - u_{1s})\,dx + (\varepsilon_{50} - v_{1s})\,dy] \tag{60}$$

Due to zero face shear conditions, ε_{40} and ε_{50} are zero in the face ply and w_0 corresponds to face deflection. *To satisfy edge support condition, one needs only a support so as to prevent vertical movement of intersections of supported segments of the faces and wall of the plate.*

Continuity across interfaces gives the recurrence relation

$$\alpha[w_{0s}^{(k)} - w_{0s}^{(k+1)}]\frac{\pi}{2}\cos\frac{\pi}{2}\alpha_k = \alpha[w_0^{(k+1)} - w_0^{(k)}] - [f_2(\alpha_k)\varepsilon_{z1}]^{(k+1)} \tag{61}$$

Combined with the 12^{th} order system in the auxiliary problem (required to eliminate parabolic distribution of applied vertical shear and assumptions in Kirchhoff's theory), the present analysis constitutes of the 16^{th} order system in the face ply and a sixth order system in the interior plies in obtaining preliminary solutions of a primary problem. Solutions for displacements and transverse stresses from this system of equations are initial solutions in the iterative procedure in solving 3-D problems to generate a proper sequence of sets of 2-D equations.

7 Higher order corrections in the ply from iterative procedure

The only error in the above analysis with reference to 3-D problems is in the transverse shear strain-displacement relations. Displacements $f_3[u_3, v_3]$ (thereby, ε_{z3})

consistent with $f_2[\tau_{xz2}, \tau_{yz2}]$ and reactive transverse stresses $(\tau_{xz4}, \tau_{yz4}, \sigma_{z5})$ have to be obtained from the first stage of the iterative procedure. Like in the auxiliary problem, it is necessary to keep σ_{z5} as a free variable by modifying f_5 in the form

$$f_5^*(z) = f_5(z) - \beta_3 f_3(z) \tag{62}$$

Here, $\beta_{3k}\alpha_k^2 = [f_5(\alpha_k)/f_3(\alpha_k)]$ so that $f_5^*(\alpha_k) = 0$. Denoting coefficient of f_3 in σ_z by σ_{z3}^*, it becomes

$$\sigma_{z3}^* = \sigma_{z3} - \beta_3\sigma_{z5} \tag{63}$$

Displacements (u_3, v_3) are modified such that they are corrections to face parallel plane distributions of the preliminary solution and are free to obtain reactive stresses τ_{xz4}, τ_{yz4}, σ_{z5} and normal strain ε_{z3}.

Transverse shear strains from constitutive relations are

$$\gamma_{xz2} = S_{44}\tau_{xz2} + S_{45}\tau_{yz2} \qquad (x,y),(4,5) \tag{64}$$

Corrective displacements are assumed in the form

$$\begin{aligned} &w = f_2(z)(w_2 - \varepsilon_{z1}) \\ &u_3^* = u_3 + \gamma_{xz2} - \alpha(w_2 - \varepsilon_{z1})_{,x} \qquad (x,y),(u,v) \end{aligned} \tag{65}$$

in which $u_3 = u_1 + u_{3c}$ with u_{3c} denoting correction due to transverse shear strain-displacement relation, w_2 is the corresponding correction to vertical displacement. Normal ε_{z1} is known from constitutive relation and given by

$$\varepsilon_{z1} = S_{6i}Q_{ij}\varepsilon_{j1} \quad (i,j = 1,2,3)$$

Transverse shear strains and stresses become

$$\gamma_{xz2}^* = u_3 + \gamma_{xz2} \qquad \rightleftarrows (x,y),(u,v) \tag{66}$$

$$\tau_{xz2}^* = Q_{44}u_3 + Q_{45}v_3 + \tau_{xz2} \qquad \rightleftarrows (x,y),(4,5) \tag{67}$$

The following equation governing (u_3, v_3, σ_{z5}) is derived from equations (2, 63, 67):

$$\alpha[(Q_{44}u_3 + Q_{45}v_3)_{,x} + (Q_{54}u_3 + Q_{55}v_3)_{,y} = \beta_3\sigma_{z5} - \alpha^2\Delta\sigma_{z1} \tag{68}$$

For the use of $[u_3^*, v_3^*]$ in integrating equilibrium equations (1, 2), $[u_3, v3]$ like in the earlier analysis are expressed as

$$[u_3, v_3] = -\alpha[\psi_{3,x} + \varphi_{3,y}, \psi_{3,y} - \varphi_{3,x}] \tag{69}$$

Contributions of ψ_3 and w_2 in $[u_3^*, v_3^*]$ are one and the same in giving corrections to $w(x,y,z)$ and transverse stresses so that w_2 in $[u_3^*, v_3^*]$ is replaced by ψ_3. (*There is no need to determine w_2 since it is a virtual one due to its non-participation in the static in-plane equilibrium equations. In-plane distribution of corrective displacement is given by ε_{z3} from Eq. (5)*). $[u_3^*, v_3^*]$ and the corresponding in-plane strains are

$$[u_3, v_3]^* = -\alpha[(2\psi_{3,x} + \varphi_{3,y}), (2\psi_{3,y} - \varphi_{3,x})] + [\gamma_{xz2}, \gamma_{yz2}] \tag{70}$$

$$\varepsilon_{x3}^* = (\tilde{\varepsilon}_{x3} + \alpha\gamma_{xz2,x}) \qquad (x,y) \tag{71a}$$

$$\gamma_{xy3}^* = [\tilde{\gamma}_{xy3} + \alpha(\gamma_{xz2,y} + \gamma_{yz2,x})] \tag{71b}$$

In the above equations,

$$[\tilde{\varepsilon}_{x3}, \tilde{\varepsilon}_{y3}] = -\alpha^2[(2\psi_{3,xx} + \varphi_{3,xy}), (2\psi_{3,yy} - \varphi_{3,xy})] \tag{72a}$$

$$\tilde{\gamma}_{xy3} = -\alpha^2[(4\psi_{3,xy} + \varphi_{3,yy} - \varphi_{3,xx})] \tag{72b}$$

From integration of equilibrium equations, reactive transverse stresses are

$$\tau_{xz4} = \alpha[Q_{1j}\varepsilon_{j,x}^* + Q_{3j}\varepsilon_{j,y}^*] \qquad (j = 1,2,3) \tag{73a}$$

$$\tau_{yz4} = \alpha[Q_{2j}\varepsilon_{j,y}^* + Q_{3j}\varepsilon_{j,x}^*] \qquad (j = 1,2,3) \tag{73b}$$

$$\alpha(\tau_{xz4,x} + \tau_{yz4,y}) + \sigma_{z5} = 0 \qquad (\texttt{here, } \sigma_{z5} \texttt{ is coefficient of } f_5) \tag{74}$$

Normal strain ε_{z3} from the constitutive relation (5) is given by

$$\varepsilon_{z3} = [S_{6j}\sigma_j + S_{66}(\sigma_z + \alpha^2\Delta\sigma_{z1})] \tag{75}$$

Noting that $f_{5,zz} = -f_3$ from $f(z)$ functions in Eqs.(9, 10), one equation governing in-plane displacements (u_3, v_3) from Eqs (68, 74) is given by

$$\alpha\beta_3(\tau_{xz4,x} + \tau_{yz4,y}) = \alpha^2[(Q_{44}u_3 + Q_{45}v_3)_{,xx} + (Q_{54}u_3 + Q_{55}v_3)_{,yy}] + \alpha^2\Delta\sigma_{z1} \tag{76}$$

With the condition zero $\omega_z = (v_{,x} - u_{,y})$ required to decouple bending and torsion, the above equation consists of a harmonic equation $\Delta\varphi_3 = 0$ and a fourth order equation in ψ_3 to be solved with conditions along constant x (and y) edges

$$u_3^* = 0 \texttt{ or } \sigma_3^* = 0 \qquad \rightleftarrows (x,y), (u,v) \tag{77a}$$

$$v_3^* = 0 \texttt{ or } \tau_{xy3}^* = 0 \qquad \rightleftarrows (u,v) \tag{77b}$$

$$\psi_3 = 0 \texttt{ or } \tau_{xz4} = 0 \qquad \rightleftarrows (x,y) \tag{77c}$$

7.1 Supplementary problem in the face ply

Corrective displacements in the supplementary problems are assumed in the form:

$$w = w_{2s}(\pi/2)\cos(\pi z/2), \quad u_s = u_{3s}\sin(\pi z/2) \qquad \rightleftarrows (u,v) \tag{78}$$

$$\sigma_{3si} = Q_{ij}\varepsilon_{3sj} \quad (i,j = 1,2,3) \tag{79}$$

Analysis here is a repetition of the corresponding analysis in the auxiliary problem with suffixes 1, 2, 3 in equations (42-51) changing to 3, 4, 5. Leaving out the text and trivial equations, the necessary equations are listed below:

$$\tau_{xz2s} = -(2/\pi)\alpha[Q_{1j}\varepsilon_{3sj,x} + Q_{3j}\varepsilon_{3sj,y}] \tag{80a}$$

$$\tau_{yz2s} = -(2/\pi)\alpha[Q_{2j}\varepsilon_{3sj,y} + Q_{3j}\varepsilon_{3sj,x}] \tag{80b}$$

$$u = (u_3^* + u_{3s})\sin(\pi z/2) \qquad \rightleftarrows (u,v) \tag{81}$$

$$(2/\pi)^2\alpha^2[Q_{1j}\varepsilon^*_{sj,xx} + 2Q_{3j}\varepsilon^*_{sj,yy}]_{(3)} = \beta_3\sigma_{z5} \tag{82}$$

Corrective displacements are

$$w = w_{04}(x,y) - f_4\varepsilon_{z3} + w_{04s}(\pi/2)\cos(\pi z/2) \tag{83}$$

$$u = f_3u_3^* + (u_3^* + u_{3s})\sin(\pi z/2) \qquad \rightleftarrows (u,v) \tag{84}$$

$$\alpha w_{04} = \int[(\gamma_{xz2} - u_3)\,dx + (\gamma_{yz2} - v_3)\,dy] \tag{85a}$$

$$\alpha w_{04s} = \int[(\gamma_{xzs2} - u_{3s})\,dx + (\gamma_{yzs2} - v_{3s})\,dy] \tag{85b}$$

7.1.1 Continuity of displacements across interfaces

Here, continuity of $[\tau_{xz4}, \tau_{yz4}, \sigma_{z5}]$ across inter faces is not presented since these stresses require further correction due to the next higher order terms. Recurrence relations for continuity of displacements are

$$[u_{3s}^{(k)} - u_{3s}^{(k+1)}]\sin\frac{\pi}{2}\alpha_k = \alpha_k[u_1^{(k+1)} - u_1^{(k)}] + [f_3^{(k+1)}(\alpha_k) + \sin\frac{\pi}{2}\alpha_k][u_3^{(k+1)} - u_3^{(k)}]^* \tag{86a}$$

$$[v_{3s}^{(k)} - v_{3s}^{(k+1)}]\sin\frac{\pi}{2}\alpha_k = \alpha_k[v_1^{(k+1)} - v_1^{(k)}] + [f_3^{(k+1)}(\alpha_k) + \sin\frac{\pi}{2}\alpha_k][v_3^{(k+1)} - v_3^{(k)}]^* \tag{86b}$$

(Similar recurrence relation involving with w_{04} and w_{04s})

Successive application of the above equations starting from top ply ensures continuity of displacements across the interfaces.

In the face ply, the second set of equations from the above analysis consists of sixth and fourth order sets of equations governing (ψ_3, φ_3) and ψ_{3s}, respectively. Obviously, similar sets of equations govern $(\psi, \varphi)_{2n+3}$ and ψ_{2n+3s} $(n = 1, 2, 3, \ldots)$ corresponding to higher order displacement terms. In the interior plies, only a sixth order system of equation govern $(\psi, \varphi)_{2n+3}$.

8 Assessment of present analysis: Bench-mark problem

Relevance and significance of the present analysis of face ply is demonstrated through the solution of a simple text book problem of the bending of simply supported homogeneous square plate under doubly sinusoidal vertical load. In this example, we present a new analysis (mentioned in the Introduction) with a proper resolution of Poisson-Kirchhoff boundary conditions paradox.

In Kirchhoff's theory, reactive transverse shear stresses are gradients of $\Delta w_0(x,y)$. Here, $\Delta w_0(x,y)$ related to ε_z is replaced by $\psi_2(x,y)$ (without the consideration of an auxiliary problem) so that $[\tau_{xz2}, \tau_{yz2}]$ are expressed as

$$[\tau_{xz2}, \tau_{yz2}] = -\alpha[\psi_{2,x}, \psi_{2,y}]$$

The equation governing $\psi_2(x,y)$ from Eq. (2) and face load condition is

$$\alpha^2 \Delta\psi_2 = (3/2)q \tag{87}$$

The above equation is to be solved with edge condition either $\psi_2 = 0$ or the sum of its normal gradient and applied transverse shear stress is zero.

In-plane equilibrium equations (1) in terms of $[u_1, v_1]$ using $v_{1,x} = u_{1,y}$ required to decouple bending and torsion problems are

$$E^{'}\alpha^2 \Delta u_1 = \alpha\psi_{2,x} \qquad \rightleftarrows (x,y),(u,v) \tag{88}$$

in which $E^{'} = E/(1-\nu^2)$.

Solutions of Laplace equations in the equations (88) are conjugate harmonic functions since both $(u_{1,x} + v_{1,y})$ and $(v_{1,x} - u_{1,y})$ are zero. Edge conditions take the form

$$u_1 = 0 \;\; \texttt{or} \;\; E\alpha u_{1,x} = (1+\nu)T_x(y) \qquad \rightleftarrows (x,y),(u,v) \tag{89a}$$

$$v_1 = 0 \;\; \texttt{or} \;\; 2Gv_{1,x} = T_{xy}(y) \qquad \rightleftarrows (x,y),(u,v) \tag{89b}$$

Equation (87) with one edge condition and equations (88) with two edge conditions (89), form a sixth order system. Vertical deflection $w_0(x,y)$ from zero face shear conditions is given by

$$\alpha w_0 = -\int [u_1\, dx + v_1\, dy]$$

The above analysis may be titled as *'Poisson's theory of plates in bending'* but applied shear loads are associated with $f_2(z)$ distributions corresponding to statically equivalent stresses in Kirchhoff's theory.

Poisson-Kirchhoff boundary condition is indirectly resolved in the Technical Note [Vijayakumar (2009)]. In this Note, w_0 from equation (8) is w_{0F} and it is same as w_{0N} from equation (13). In Kirchhoff's theory and FSDT, determination of in-plane displacements is coupled with w_{0N}. Both these theories and Reissner's sixth order theory are based on parabolic distribution of transverse shear stresses. In Reissner's theory, average displacements include w_2 due to ε_{z1} from constitutive relation and the corresponding higher order displacements (u_3, v_3) consistent with w_2. As such, Reissner's theory gives more accurate estimation of stress components.

Applied vertical shear along the edge is satisfied in terms of stress resultant in Reissner's theory and contracted vertical shear in Kirchhoff's theory. Transverse shear strain from strain-displacement relation is associated with satisfaction of applied vertical stress in FSDT only along with shear correction factor. Reddy's shear deformation theory without shear correction factor [Reddy (1984)] is equivalent to Reissner's theory [Reissner (1985)] in terms of displacements. The above mentioned Poisson's theory eliminates the use of auxiliary function φ in Reissner's theory and coupling with w_{0N} in Kirchhoff's theory and FSDT.

8.1 Exact and approximate solutions

Use of polynomials in z without satisfying zero shear conditions along faces in plate element equations results in approximation to $w(x,y,z)$ *in series form* for face deflection whereas it is $w_0(x,y)$ for neutral plane deflection corresponding to the same order of approximation of the face deflection. The converse is true if $f_n(z)$ satisfy, a priori, face shear conditions. In the present analysis along with the solution of the supplementary problem, both face and neutral plane deflections are obtained in series form.

In the case of a simply supported plate, Poisson-Kirchhoff's boundary conditions paradox does not exist. With $q = q_0 \sin(\pi x/a)\sin(\pi y/b)$, $\nu = 0.3$, $\alpha = 1/6$ and $\beta = \sqrt{2}\alpha\pi$, one obtains from the exact solutions [Vijayakumar (2011b)]

$$(E/2q_0)w(a/2, a/2, 0) = 4.487$$
$$(E/2q_0)w(a/2, a/2, 1) = 4.166$$

With reference to face value, Kirchhoff's theory gives a value of 2.27 which is same for all face parallel planes.

Poisson's theory gives an additional value of 0.267 due to ε_{z1}. Face and neutral plane deflection are one and the same since $w(x,y,z)$ can be expressed as

$$w(x,y,z) = w_{0F}(x,y) + f_2(z)w_2 = w_{0N}(x,y) - z^2\varepsilon_{z1}/2$$

In the above expression, w_{0F} and w_{0N} are face and neutral plane deflections, respectively.

(Correction due to coupling with torsion is 1.45 [Vijayakumar (2011a)] whereas it is 1.423 from FSDT and other sixth order theories [Lewinski (1990)]. With reference to numerical values reported in Lewinsky's work, the above correction is less than 1.23 in Reissner's 12^{th} order and other higher order theories. It clearly shows that shear deformation and other higher order theories do not lead to the solutions of bending problems. Vertical deflection w_0 from FSDT corresponds to w_{0N} and its estimated value is 3.693 lower by about 17.7% from the exact value 4.487)

Higher order correction to w_0 uncoupled from torsion is 1.262 so that total correction to the value from Kirchhoff's theory is about 1.53 [Vijayakumar (2011b)]. Neutral plane deflection w_{0N} is corrected from the solution of a supplementary problem. This total correction over face deflection is about 0.658 giving a value of 4.458 which is very close to the exact value 4.487 (Hence, it is safe to conclude that second order corrections in transverse stresses from the iterative method serve the purpose of assessing data from Kirchhoff's theory and FSDT).

However, error in the estimated value $(= 3.8)$ of w_{0F} is relatively high compared to the accuracy achieved in neutral plane deflection. It is possible to improve estimation of w_{0F} by including ε_{z1} in (u_3^*, v_3^*) in equations (46) in the earlier work [Vijayakumar (2011b)] such that (τ_{xz2}, τ_{yz2}) are independent of ε_{z1}. In the present example, correction to face deflection changes to 1.431 giving 3.97 $(= 2.27 + 0.267 + 1.431)$ for face deflection which is under 4.7% from the exact value. The correction 1.431 to the face deflection is due to ε_{z3} from a constitutive relation. Determination of ε_{z3} in terms of (σ_{z3}, u_3, v_3) involves lengthy algebra and arithmetical work. Since this correction is mainly due to inclusion of σ_{z1} in the in-plane constitutive relations, it is much simpler to find its effect from solutions of the present auxiliary problems.

8.1.1 Utility of auxiliary problem

In the case of homogeneous isotropic plates, the equation corresponding to Eq. (33) is

$$E'\beta_1\alpha^4\Delta\Delta(2\psi_1 + \psi_0/G) + Ge_{1c} - \mu\beta_1\alpha^2\Delta\sigma_{z1} = 0 \tag{33'}$$

After some algebra with $G = E/2(1+\nu)$, $\mu = \nu/(1-\nu)$ and $\beta_1 = 1/3$ and $\psi_1 = c_{1c}\sin(\pi x)\sin(\pi y)$, equation governing c_{1c} becomes

$$[\beta^2 + \frac{3(1-\nu)}{4}]c_{1c} - \frac{(1+\nu)}{8}(2-\nu)(2q_0/E) = 0 \tag{90}$$

The solution of the above equation gives $c_{1c} = 0.2574(2q_0/E), (\nu = 0.3, \alpha = 1/6)$. Face deflection w_{0c} from the integration of transverse shear strain-displacement relations is

$$w_{0c} = \{[(1+\nu)/2\beta^2](2q_0/E) + c_{1c}\} \tag{91}$$

from which $w_{0c} = 1.389(2q_0/E)$ in the present example. The above solution for w_{0c} is in the absence of applied in-plane stresses along edges of the plate.

8.1.2 w_{0b} due to (u_{1b}, v_{1b})

In the case of applied in-plane stresses along the edges of the plate,

$$E^{'}\alpha^4\Delta\Delta\psi_{1b} = 3[q/2 + Ge_{1c}/\beta_1] \tag{92}$$

The above equation is expressed in the form

$$\alpha^4\Delta\Delta\psi_{1b} = \frac{3(1-\nu)(1+\nu)}{4}(2q/E) - \frac{3(1-\nu)}{2}\alpha^2\Delta\psi_{1c} \tag{93}$$

from which

$$c_{1b} = \left[\frac{3(1-\nu)(1+\nu)}{4}(2q/E) + \frac{3(1-\nu)}{2}\beta^2 c_{1c}\right]/\beta^4 \tag{94}$$

In the present example, estimated c_{1b} is $2.4182(2q_0/E), (= 2.27 + 0.1482)$. It is also equal to w_{0b} from the integration of strain-displacement relations.

Estimated face deflection $(E/2q_0)w_F = 4.0742(= 2.4182 + 0.267 + 1.389)$ which is fairly close to the exact value (4.166). *It shows that solution to w from the present analysis provides proper correction to the estimation of face deflection.* It is under-estimated by 2.2%.

8.1.3 Correction to neutral plane deflection

Additional corrections to in-plane displacements in face parallel planes are obtained by assuming

$$[u, v] = [\sin(\pi z/2)u_{s1}, \sin(\pi z/2)v_{s1}]$$

The above in-plane distributions $[u_{s1}, v_{s1}]$ are added as corrections to $[u_1^*, v_1^*]$ so that $[u, v]$ in the supplementary problem are

$$[u, v] = \sin(\pi z/2)[(u_1^* + u_{s1}), (v_1^* + v_{s1})]$$

Since in-plane distribution of σ_z from the integration of equilibrium equations with s variables is the same as σ_z from the static equilibrium equations with * variables, ψ_{s1} is related to e_{1c} by the equation

$$E^{'}(4/\pi)\alpha^4\Delta\Delta\psi_{s1} = Ge_{1c}\&(E/2q_0)c_{s1} = [\frac{1-\nu}{2}\pi^2 - \beta_1^2](E/2q_0)c_{1c} \tag{95}$$

so that $c_{s1} = 0.5478(2q_0/E)$ with $\nu = 0.3$ and $\alpha = 1/6$.

Neutral plane deflection $w_{0N} = 4.355(2q_0/E)$, $(= 2.4182 + 1.389 + 0.5478)$ is underestimated by 2.94% from the exact value (4.487).

Evaluation of neutral plane deflection involves u_{1c} and u_{1s} associated with $f_2(z)$ and $\cos(\pi z/2)$ distributions of transverse shear stresses, respectively. As such, the solution of the present auxiliary problem provides second order corrections to the solution of the primary problem. *The significant implication of this observation is that the solution of the present auxiliary problem is necessary so as to obtain a (more or less) uniform approximation to the deformation of face-parallel planes.* Moreover, applied transverse shear stress along an edge is independent of material constants.

9 Concluding remarks

Estimation of transverse stresses in the preliminary solution is independent of material constants through Poisson's theory in the analysis of primary bending problem defined from Kirchhoff's theory. Hence, it is unaltered in the analysis of bending of homogeneous and laminated plates with orthotropic and anisotropic material. Analysis of homogeneous plates along with estimation of higher order distributions of displacements is necessary in the layer-wise theory of laminates [Vijayakumar (2011c)] in which the analysis of face plies is independent of lamination. Laminate analysis is, however, with ply-wise parabolic distribution of transverse shear stresses along its edges.

Application of transverse shear stresses independent of material constants and thickness coordinate z along the edges of the plate is more practical particularly in laminates though higher order thickness-wise distributions are mathematically valid. Coupling of $w_0(x, y)$ in obtaining in-plane displacements which is the root cause for the Poisson-Kirchhoff boundary condition paradox is eliminated. In addition, inclusion of the tangential gradient of applied in-plane shear stress in transverse shear in Kirchhoff's theory is removed. Thickness-wise distributions of displacements in terms of polynomials in z are not adequate to reduce 3-D problems into a sequence of 2-D problems.

Determination of in-plane displacements and stress components in each ply are independent of the edge support condition on w. Dependence of transverse stresses on material constants is initially through the preliminary solution of a primary problem in conjunction with the auxiliary problem. Analysis at the first stage of the iterative method forms the basis for the generation of a proper sequence of 2-D problems converging to 3-D problems and provides a (more or less) uniform approximation of the deformation of face parallel planes.

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