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A Combined Sensitive Matrix Method and Maximum Likelihood Method for Uncertainty Inverse Problems

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Abstract: The uncertainty inverse problems with insufficiency and imprecision in the input and/or output parameters are widely existing and unsolved in the practical engineering. The insufficiency refers to the partly known parameters in the input and/or output, and the imprecision refers to the measurement errors of these ones. In this paper, a combined method is proposed to deal with such problems. In this method, the imprecision of these known parameters can be described by probability distribution with a certain mean value and variance. Sensitive matrix method is first used to transform the insufficient formulation in the input and/or output to a resolvable one, and then the mean values of these unknown parameters can be identified by maximizing the likelihood of the measurements. Finally, to quantify the uncertainty propagation, confidence intervals of the obtained solutions are calculated based on linearization and Monte Carlo methods. Two numerical examples are presented to demonstrate the effectiveness of the present method.

Keywords: Inverse problems; Uncertainty; Sensitivity matrix method; Maximum likelihood method; Confidence interval

1 Introduction

Inverse problems are usually defined as problems to determine input through given output, in contrast with the forward problems which are concern with the estimation of output from known input. The inverse problem on a structure is of great interest in mechanical engineering, aerospace engineering, civil engineering and so on. The examples include identification of material properties [Pagnacco, Moreau and Lemosse (2007)], reconstruction of external loads [Jiang and Hu (2008)], updating of structure model [Chang, Chang and Xu (2000)], design of structure shape [Tiow,

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Yiu and Zangeneh (2002)] and many other applications[Liu and Han (2003); Hon, Li and Melnikov (2010); Zhang, Liu, Han and Chen (2012)]. In these works, most of the existing inverse methods focus on deterministic inverse problems in which all of the known input, output and system parameters can be given in certain values. In practical engineering, however, uncertainty widely exists in the material property, external load and geometrical dimension etc. Thus, to obtain the feasible variable parameters, the uncertainty inverse problems (UIPs) should be investigated.

Various inverse strategies and methods [Fonseca, Friswell and Mottershead (2005); Jiang, Liu and Han (2008); Li and Moor (2002); Pradlwarter and Schueller (2008); Nakagiri and Suzuki (1999); Rus, Lee and Gallego (2005); Turco(2005); Worden, Manson, Lord and Friswell (2005)] have been developed to deal with such problems. Nevertheless, there is very little literature concerned with UIPs with the insufficiency and imprecision in the input and/or output parameters. It is very common that the input and/or output parameters are available from experiments, and they always not be sufficient and exact in the practical engineering. Liu etc [Liu, Xu and Wu (2001)] proposed the concept of total solution to deal with UIPs with insufficiency in the input and/or output parameters. In their work, however, the imprecision of the partly known parameters are not considered. That is obviously improper to deal with the practical engineering problem. If we consider these parameters to be imprecise measurements that include random errors, then we are faced with the problem of finding the solution which is best from a statistical point of view. In addition, to incorporate uncertainty in structure analysis, design and evaluation, knowledge of the uncertainty in the obtained solutions is required. Therefore, uncertainty propagation from the known parameters to the unknown ones should be quantified. Apparently, this class of inverse problems is unable to be solved by traditional inverse methods. It is valuable to develop a corresponding inverse method.

Two methods, Sensitive matrix method (SMM) and maximum likelihood method (MLM), can be used to solve this class of UIPs. SMM is suitable for solving the inverse problems with insufficiency in the input and/or output parameters. The most important advantage of SMM is that it can be easily used to transform the partly known parameters to a resolvable formulation in the forward model. In the SMM, the sought parameters are always expressed in an explicit form, so that identification of mean values of these parameters can be simply performed using matrix inversion. However, it is difficult to cope with the imprecision of the partly known parameters. Additionally, it is unable to answer questions that how many errors present in the known imprecise parameters are transferred to the solution? and what is the confidence intervals of this obtained solutions? On the other hand, MLM holds complementary promises with respect to the SMM to deal with these problems. The MLM's advantages are the capability to estimate the maximum likelihood solution of the mean values of the unknown parameters. Then, confidence intervals of identified solutions can be calculated based on linearization and Monte Carlo methods. It is natural to expect that a combination of a SMM and MLM may provide an ideal performance for the inverse procedure needed for this class of inverse problems.

In this paper, a method is proposed for inverse problems with insufficiency and imprecision in the input and/or output parameters, by combining the advantages of SMM and MLM in the inverse procedure. The method is based on the assumption that a forward model is always available, and the partly known parameters in the input and/or output are random variables. Finite element method (FEM) is employed as the forward solver to calculate the responses corresponding to the given parameters. Firstly, SMM is used to transform the insufficient formulation in the input and/or output to a resolvable one, and then the mean values of the unknown parameters in the input and/or output can be identified by MLM. Finally, quantification of the uncertainty propagation from the known parameters to unknown parameters by linearization and Monte Carlo methods is also studied. Two numerical examples have been carried out to demonstrate the feasibility and validity of this combined method as well as the implementing techniques.

2 Statement of the problem

For a general deterministic engineering inverse problem, the forward model can be expressed by the equation:

$$\mathbf{Y} = \mathbf{T}(\mathbf{X}) \tag{1}$$

where $\mathbf{Y} = [y_1, y_2, \cdots, y_m]^T$ is a vector of output parameters representing the desired response quantities, for example displacements, stresses and natural properties, etc; $\mathbf{X} = [x_1, x_2, \cdots, x_n]^T$ is a vector that collects all input parameters representing the external loads, material properties and boundary conditions, etc; $\mathbf{T}$ is a is the system matrix of functions of vectors parameters representing the translation process from input to output. If the outputs of the system can be obtained by means of measurement, the deterministic inverse problem can be formulated as a simple optimization problem which minimizing the errors of the predicated outputs based on the forward model and an assumed $\mathbf{X}^*$ with respect to the measured outputs $\mathbf{Y}^m$. Then, various optimization strategies can be adopted to solve this problem.

For an uncertainty inverse problem (UIP), some input and/or output parameters are insufficient and imprecise, and Eq. (1) can be changed as:

$$(\mathbf{Y}_u, \mathbf{Y}_n) = \mathbf{T}(\mathbf{X}_u, \mathbf{X}_n) \tag{2}$$

where the vectors with subscript u are the unknown part and those with subscript n are the known parts in the corresponding random parameter vectors.

In order to state precisely what is meant by UIP with insufficiency and imprecision in the input and/or output parameters, it is helpful to look at Fig.1. The input vector $\mathbf{X}$ is subdivided into two parts. The first one $\mathbf{X}_n$ is the known parameters. They obtained from the experimental measures, and obey probability distribution with a certain mean value and variance. Conversely, the second one $\mathbf{X}_u$ is the unknown parameters to be determined. The same situation is occurred in the output vector$\mathbf{Y}$. Because of the uncertainty propagation, it is clear that the unknown parameters in the input $\mathbf{X}_u$ and/or output $\mathbf{Y}_u$ will also follow a certain probability distribution.

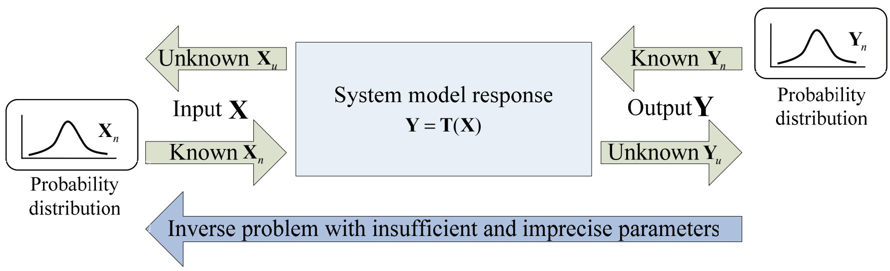


Figure 1: Schematic illustration of the inverse problem with insufficient and imprecise parameters

The main objective of this paper is to discuss method related to the determination of mean values of the unknown parameters $\mathbf{X}_u, \mathbf{Y}_u$ from the known probability distribution of $\mathbf{X}_n, \mathbf{Y}_n$, and the confidence intervals of the obtained solutions. It can be mathematically represented as follows:

To estimate mean values and confidence intervals:

$$(\mathbf{Y}_u, \mathbf{X}_u) \tag{3a}$$

For given the probability distributions:

$$(\mathbf{Y}_n, \mathbf{X}_n) \tag{3b}$$

Subject to:

$$(\mathbf{Y}_u, \mathbf{Y}_n) = \mathbf{T}(\mathbf{X}_u, \mathbf{X}_n) \tag{3c}$$

The above UIP is more complex and difficult to solve than the deterministic inverse problems, and the traditional inverse methods are incapable of dealing with this problem.

3 A combined method

3.1 Basic ideas

The first step, SMM [Liu, Xu and Wu (2001)] is used to cope with insufficiency in the input and/or output parameters. The main purpose is to divide the parameter into the known and unknown parts, and transform the complex engineering problem into a simple and solvable explicit form. The second step, MLM [Fonseca, Friswell and Mottershead (2005)] is applied to deal with imprecision in the input and/or output parameters. The main aim is to estimate mean values of those unknown parameters. Then, confidence intervals of the obtained parameters for a given confidence level can be calculated by using the linearization and Monte Carlo methods.

3.2 Determination of the mean values

SMM, besides being useful for solving complex engineering problems, makes the formulation more readable and so easier to understand. We can see that Eq. (2) relates the cause vector $\mathbf{X}$ to the effect vector $\mathbf{Y}$ by the transformation matrix $\mathbf{T}$. However, the sensitivity matrix $\mathbf{S}$ is applied to relate a finite change between a compound vector of parameter $\mathbf{Q} = \overline{\mathbf{X}}$ and the effect vector $\overline{\mathbf{Y}}$ as follows:

$$\overline{\mathbf{Y}} = \mathbf{SQ} \tag{4}$$

where

$$\overline{\mathbf{Y}} = [\Delta y_i, i = 1, \cdots, m], \quad \mathbf{Q} = [\Delta x_j, j = 1, \cdots, n],$$

$$\mathbf{S} = [\Delta y_j / \Delta x_j, i = 1, \cdots, m; j = 1, \cdots, n].$$

The sensitivity matrix $\mathbf{S}$ can be calculated by numerical computation [Yech (1986)].

By dividing the parameter vectors into the known and unknown parts, the Eq. (4) can be rewritten as follows

$$\begin{bmatrix} \overline{\mathbf{Y}}_u \\ \overline{\mathbf{Y}}_n \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_u \\ \mathbf{Q}_n \end{bmatrix} \tag{5}$$

Then, all the known parts in the vectors $\overline{\mathbf{Y}}$,$\mathbf{Q}$ are moved to the right-hand-side of Eq. (5) and denoted as a new vector $\mathbf{d}$, while all the unknown parts are moved to the left-hand-side and denoted as a new vector $\mathbf{m}$. Eq. (5) is thus transferred as

$$\mathbf{Gm} = \mathbf{d} \tag{6}$$

where $\mathbf{d}=\begin{bmatrix}\mathbf{S}_{12}\mathbf{Q}_n\\ \overline{\mathbf{Y}}_n-\mathbf{S}_{22}\mathbf{Q}_n\end{bmatrix}$, $\mathbf{G}=\begin{bmatrix}\mathbf{I} & -\mathbf{S}_{11}\\ 0 & \mathbf{S}_{21}\end{bmatrix}$, $\mathbf{m}=\begin{bmatrix}\overline{\mathbf{Y}}_u & \mathbf{Q}_u\end{bmatrix}^T$. Thus, the insufficient parameters in the input and/or output in Eq. (2) can be transformed to a resolvable formulation by sensitive matrix method.

In many engineering problem, there is sufficient evidence to suspect the known imprecise parameter $\mathbf{X}_n$ and $\mathbf{Y}_n$ are generated by a normal distribution and independent:

$$x_n^i \sim N(\widehat{x_n^i},(\sigma_n^{xi})^2) \tag{7a}$$

$$y_n^i \sim N(\widehat{y_n^i},(\sigma_n^{yi})^2) \tag{7b}$$

where N is called the normal distribution. $\widehat{x_n^i}$ and $\widehat{y_n^i}$ denote the ith mean value, and σ_n^{xi} and σ_n^{yi} represent the ith deviation in $\mathbf{X}_n$ and $\mathbf{Y}_n$, respectively.

Therefore, maximum likelihood method can be applied to UIP where a joint probability density function can be assigned to the known parameters. The essential problem is to find the most likely parameter, and to estimate the confidence intervals of the obtained parameters in Eq. (6). The known parameters are independent so that we can use the product form of the joint probability density function:

$$f(\mathbf{d}|\mathbf{m}) = f_1(d_1|m)\cdot f_2(d_2|m)\cdots f_m(d_m|m) \tag{8}$$

where $f_i(d_i|m)$ is probability density function for each of the $i$parameter.

In practice, based on Eq. (7), the likelihood function is:

$$\begin{aligned} L(\mathbf{d}|\mathbf{m}) &= f(\mathbf{d}|\mathbf{m}) = f_1(d_1|m)\cdot f_2(d_2|m)\cdots f_m(d_m|m)\\ &= \frac{1}{(2\pi)^{m/2}\prod_{i=1}^m \sigma_i}\prod\nolimits_{i=1}^m e^{-(d_i-(\mathbf{Gm})_i)^2/2\sigma_i^2} \end{aligned} \tag{9}$$

According to the maximum likelihood principle we can determine the mean values of the unknown parameters that maximize the likelihood function. The constant factor does not affect the maximization of Eq. (9), so we can solve:

$$\max\prod\nolimits_{i=1}^m e^{-(d_i-(\mathbf{Gm})_i)^2/2\sigma_i} \tag{10}$$

The logarithm is a monotonically increasing function, so we can equivalently solve:

$$\max(\log\prod\nolimits_{i=1}^m e^{-(d_i-(\mathbf{Gm})_i)^2/2\sigma_i}) = \max(-\sum_{i=1}^m (d_i-(\mathbf{Gm})_i)^2/2\sigma_i^2) \tag{11}$$

Finally, the problem becomes:

$$\min \sum_{i=1}^{m} (d_i - (\mathbf{Gm})_i)^2 / \sigma_i^2 \tag{12}$$

From Eq. (12), we can find that aside from the distinct $1/\sigma_i^2$ factors in each term, this is identical to the least squares problem. As above discussed, we can use SMM to solve the mean values of vector $\mathbf{m}$ in Eq. (12). To incorporate the variance into a solution in each of iteration step, Eq. (12) is changed as:

$$\mathbf{G}_w \mathbf{m} = \mathbf{d}_w \tag{13}$$

where $\mathbf{G}_w = \mathbf{WG}$, $\mathbf{d}_w = \mathbf{Wd}$, $\mathbf{W} = diag(1/\overline{\sigma}_1, 1/\overline{\sigma}_2, \cdots, 1/\overline{\sigma}_m)$, and $\overline{\sigma}^2$ can be obtained from derivation by probability theory:

$$\overline{\sigma}^2 = \{\overline{\sigma}_1^2, \overline{\sigma}_2^2\} \tag{14}$$

where $\overline{\sigma}_1^2, \overline{\sigma}_2^2$ are the variance to $\mathbf{d}$ in Eq. (7) . The element of vector $\mathbf{d}$ $d^i \sim \begin{bmatrix} N(\overline{\mu}_1, \overline{\sigma}_1^2) \\ N(\overline{\mu}_2, \overline{\sigma}_2^2) \end{bmatrix}$, and $\overline{\mu}_1$, $\overline{\mu}_2$ and $\overline{\sigma}_1^2$, $\overline{\sigma}_2^2$ denote the mean value and variance of d_i, respectively:

$$\overline{\mu}_1 = s_{12}(\widehat{x_n^i} - x_n^{hi}) \tag{15a}$$

$$\overline{\mu}_2 = (\widehat{y_n^i} - y_n^{hi}) - s_{22}(\widehat{x_n^i} - x_n^{hi}) \tag{15b}$$

$$\overline{\sigma}_1^2 = s_{12}^2 (\sigma_n^{xi})^2 \tag{15c}$$

$$\overline{\sigma}_2^2 = (\sigma_n^{yi})^2 + s_{22}^2 (\sigma_n^{xi})^2 \tag{15d}$$

where $\widehat{x_n^i} - x_n^{hi}$ and $\widehat{y_n^i} - y_n^{hi}$represent the small change of the ith element in vectors $\mathbf{X}_n$ and $\mathbf{Y}_n$, respectively. In other word, they denote the element of vectors $\mathbf{Q}_n$ and $\overline{\mathbf{Y}}_n$ in Eq. (6), respectively. From Eq. (15c) and (15d), we can conclude that the vectors $\mathbf{X}_n$ and $\mathbf{Y}_n$ has limited change, while its variance is not changed.

We can easily found that the least squares solution to Eq. (13) is the maximum likelihood solution. In other word, mean values of the unknown parameters can be determined from solving Eq. (13) by using SMM in the statistical point. Therefore, the simplified matrix inversion can be used to determine mean values of the unknown parameter vector $\mathbf{m}$ from the known parameter vector $\mathbf{d}_w$ in each of iteration step. As we known, the least squares solution of the unknown parameters can be calculated as:

$$\mathbf{m} = (\mathbf{G}_w^T \mathbf{G}_w)^{-1} \mathbf{G}_w^T \mathbf{d}_w \tag{16}$$

It is noted that $\mathbf{G}_w^{-1}$ obtainable only if matrix $\mathbf{G}_w$ is square and non-singular. In most cases, we have to use singular value decomposition [Golub, van Van Loan (1996)] to calculate **m**.

3.3 Determination of the confidence intervals

A sample formulation $\overline{\sigma}^2 = \{\overline{\sigma}_1^2, \overline{\sigma}_2^2\}$ obtained from Eq. (14) is adopted to measure the uncertainty propagation from the partly known parameters to the identification action. Consider the linear relationship between **m** and $\mathbf{d}_w$ in Eq. (16) in each of iteration step, we can use Linearization method to calculate the appropriate covariance:

$$cov(\mathbf{A}\mathbf{d}_w) = \mathbf{A}cov(\mathbf{d}_w)\mathbf{A}^T \tag{17}$$

Eq. (17) has $\mathbf{A} = (\mathbf{G}_w^T\mathbf{G}_w)^{-1}\mathbf{G}_w^T$. Therefore, the appropriate covariance for the unknown parameters can be computed by:

$$cov\mathbf{m} = (\mathbf{G}_w^T\mathbf{G}_w)^{-1}\mathbf{G}_w^T\mathbf{I}_m((\mathbf{G}_w^T\mathbf{G}_w)^{-1}\mathbf{G}_w^T)^T = (\mathbf{G}_w^T\mathbf{G}_w)^{-1} \tag{18}$$

where $\mathbf{I}_m$ is m by m identity matrix. In the case of independent and identically normal distribution, the covariance matrix $cov(\mathbf{d}_w)$ is simply the variance $\overline{\sigma}^2$ times $\mathbf{I}_m$, and Eq. (18) simplifies to:

$$cov\mathbf{m} = \overline{\sigma}^2(\mathbf{G}^T\mathbf{G})^{-1} \tag{19}$$

Then, we can calculate 95% confidence intervals for individual obtained parameters with the mean values which are identified by SMM and variance $cov(\mathbf{m})$ in each of iteration step. The 95% confidence intervals are given by [Aster, Borchers and Thurber (2005)]:

$$\mathbf{m} \pm 1.96 \cdot \sqrt{\text{diag}(\text{cov}(\mathbf{m}))} \tag{20}$$

where the 1.96 factor arises from $\frac{1}{\sigma\sqrt{2\pi}}\int_{1.96}^{1.96} e^{-\frac{x^2}{2\sigma^2}}dx = 0.95$.

Monte Carlo method is also used to estimate the confidence intervals of the obtained solutions, since it copes with linear and nonlinear mapping of known probability distribution into unknown probability distribution very well. We simulate a collection of known parameter vectors and then examine the statistics of the corresponding unknown parameter. This means that we resolve the least squares problem many times for system model corresponding to independent known parameter, computing a suite of solutions to $\mathbf{Gm} = \mathbf{d} = \mu + \overline{\sigma}$. Then, an empirical estimate of the covariance matrix can be calculated as:

$$cov(\mathbf{m}) = \mathbf{B}^T\mathbf{B}/N \tag{21}$$

where N denotes the number of simulation, $\mathbf{B} = \mathbf{m}^T - \overline{\mathbf{m}}^T$ is a $N \times m$ matrix that the ith row contains the difference between the ith parameter estimate $\mathbf{m}$ and the average parameter $\overline{\mathbf{m}}$. After this, the confidence intervals of the obtained parameters similarly calculated by Eq. (20). A great difference of Monte Carlo method from linearization method is that Monte Carlo method needs a lot of numerical computations, and it is very suit for linear and nonlinear problems.

3.4 Solution procedure

Based on the above discussion, the solution procedure of this type of UIP can be described as follows:

1. Construct a forward model Eq. (2) for the problem under consideration. The model parameters are simultaneously determined. Specify the known and unknown parameters in input and output for the practical problem. The known parameters are assumed to follow a normal distribution Eq. (7) from engineering experience or previous research results.

2. Finite element method (FEM) is adopted as forward solver to calculate the response from the given parameters. Assume initial mean values of the unknown parameters $\mathbf{X}_u^0$, combing them together with the known ones $\mathbf{X}_n$ into the forward solver to calculate the corresponding output $(\mathbf{Y}_u^0, \mathbf{Y}_n^0)$.

3. Compute the sensitivity matrix $\mathbf{S}^i$ using the calculated mean value vectors $\mathbf{X}_u^i$, $\mathbf{X}_n$, $\mathbf{Y}_u^i$ and $\mathbf{Y}_n^i$. Calculate matrix $\mathbf{G}^i$ in Eq. (6) from$\mathbf{S}^i$. According to the difference between the known mean $\mathbf{Y}_n$ and the calculated $\mathbf{Y}_n^i$, construct the matrix $\mathbf{d}^i$.

4. Calculate the parameter increment vector in Eq. (16) by SMM. Obtain a new set of parameters by adding the increment into the solution in the next iteration. We use linearization method Eq. (19) to compute variance for the unknown parameters, and adopt Eq. (20) to estimate the confidence intervals of the obtained solutions.

5. Calculate the output $(\mathbf{Y}_u^{i+1}, \mathbf{Y}_u^{i+1})$ again from the forward solver by the newly obtained mean values of $\mathbf{X}_u^{i+1}$ and known $\mathbf{X}_n$, $\mathbf{Y}_n$. Stopping criterion is created as

$$\left\|\mathbf{Y}_u^{i+1} - \mathbf{Y}_u^i\right\|_2 \leq \varepsilon_{yu} \tag{22a}$$

$$\left\|\mathbf{Y}_n^{i+1} - \mathbf{Y}_n\right\|_2 \leq \varepsilon_{yn} \tag{22b}$$

$$\left\|\mathbf{X}_u^{i+1} - \mathbf{X}_u^i\right\|_2 \leq \varepsilon_{x} \tag{22c}$$

where ε_{yu}, ε_{yn} and ε_x are a small constant.

6. If Eq. (22) is satisfied, this set of parameters is considered to be mean values for the problem. Otherwise, go back to step (2). Combining these parameters, we use Eq. (21) to calculate the variance for the unknown parameters by Monte Carlo

method. Confidence intervals of the obtained solutions can also be determined by Eq. (20), and the solution procedure stops.

For detail, this inverse procedure is illustrated in the flow chart of Fig. 2. “confidence interval 1” means that he confidence interval is calculated by linearization method, and “confidence interval 2” denotes that the one is obtained from Monte Carlo method.

4 Numerical examples and discussion

4.1 Identification of material properties of a beam

A stepped section beam with a centralized force $P = 1000$N on its end is shown schematically in Fig. 3. This frame has different material and geometrical properties in the spans 1-2 and 2-3, respectively. The span 1-2 has length $l_1 = 600$mm, and the span 2-3 has length $l_2 = 250$ mm. This frame is fixed at node 1, simply supported at node 2 and free at node 3. This example can be performed only in the xy plane for simplicity. The input parameters include Young’s modulus (E_1,E_2) and section areas (A_1,A_2) of the beam. And the translational and rotational displacements of the beam at the node 2 (x_2,θ_2) and the node 3 (x_3,y_3,θ_3) can be known as the output parameters.

The objective of material identification for this problem is to determine the parameters E_1,E_2 and estimate the confidence intervals of the obtained solutions. For testing the presented combined method, we intentionally assume that the output x_2, θ_2 and θ_3 are unknown, but the transnational displacements x_3,y_3 and the section areas A_1,A_2 are experimentally obtained. Consider imprecise measurements that include random errors, these known parameters follow a normal distribution. Take 3% deviation level of measurement for example, the normal distribution of these parameters can be described as $A_1 \sim N(600, 18^2)mm^2$, $A_2 \sim N(448, 13.44^2)mm^2$, $x_3 \sim N(0.011, 0.0033^2)mm$ and $y_3 \sim N(1.487, 0.0446^2)mm$. In other word, this problem has both partly known parameters in the input and output, and these known parameters are described as a random distribution. This situation is most likely to happen in practical engineering. In this case, apparently, identification of mean values and confidence intervals of the material parameters would be very difficult when using the traditional inverse algorithms only based on the partly known output distribution of x_3, y_3. However, the presented combined method is available for this problem. It is able to make use of the other known input parameters of A_1 and A_2.

According to the solution procedure described in Section 3, the parameter vectors for this problem are thus set up as follows: $\mathbf{X} = \{\mathbf{X}_u, \mathbf{X}_n\}$, $\mathbf{X}_u = \{E_1, E_2\}$, $\mathbf{X}_n = \{A_1, A_2\}$, $\mathbf{Y} = \{\mathbf{Y}_u, \mathbf{Y}_n\}$, $\mathbf{Y}_u = \{x_2, \theta_2, \theta_3\}$, $\mathbf{Y}_n = \{x_3, y_3\}$, and the initial mean

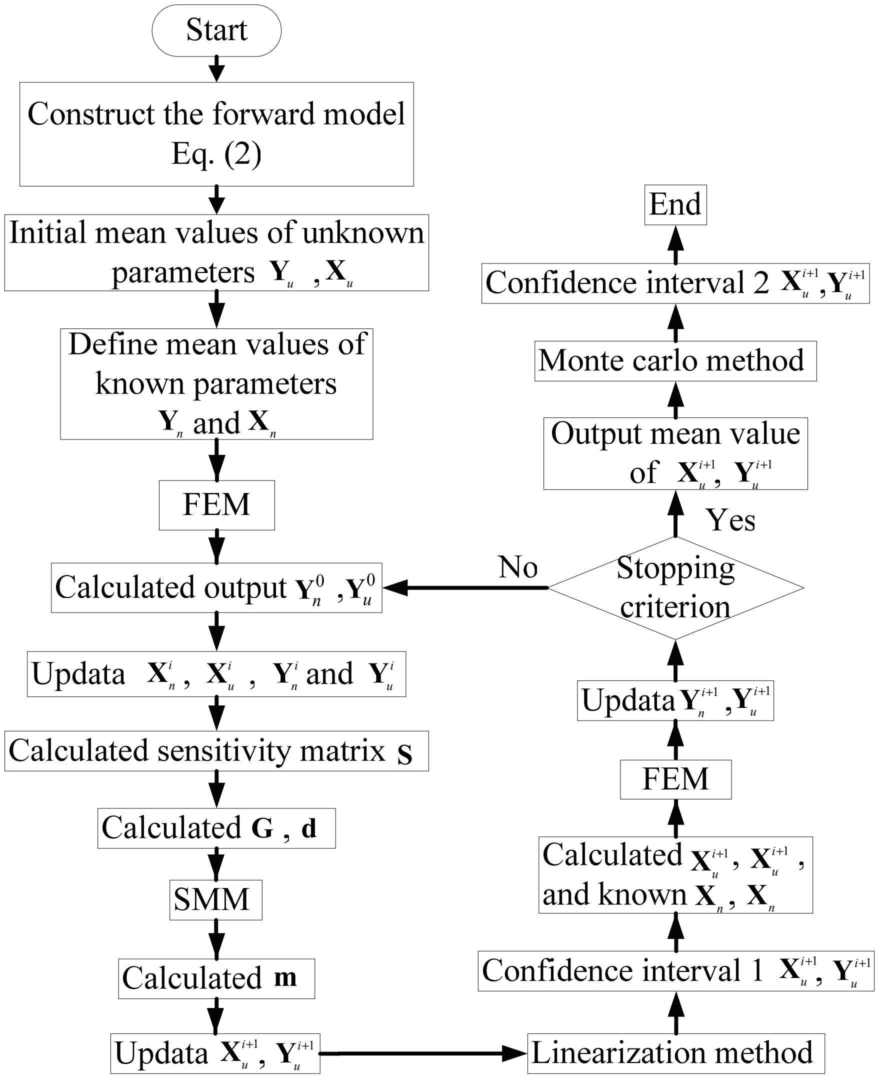


Figure 2: Solution procedure for inverse problem with insufficient and imprecise parameters

values of E_1 and E_2 are assumed 1.8×10^5 and 2.5×10^5, respectively. Combing these assumed mean values of E_1 and E_2 and the known A_1 of 600, A_2 of 448 into the forward solver FEM, we obtained the corresponding first output parameters x_2, θ_2, x_3, y_3 and θ_3 to be 4.8×10^{-3}mm, 2.3×10^{-3}, 4.5×10^{-3} mm, 6.7×10^{-3}mm

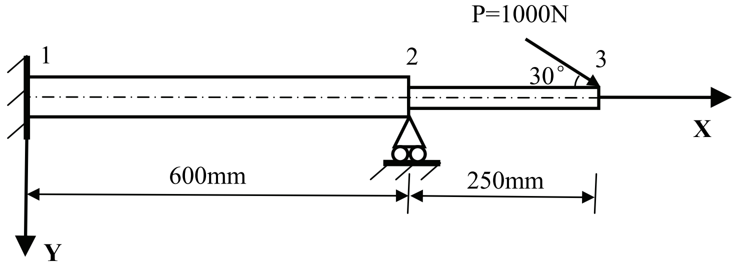


Figure 3: A beam with a centralized force

and 0.9346, respectively. Then, a sensitivity matrix **S** centering on this set of parameters is obtained. This is done by way of varying each input parameter among E_1 , E_2, A_1 and A_2, and then calculating the corresponding changes of output parameters x_2, θ_2, x_3, y_3 and θ_3 for each varied combination of input parameters. Subsequently the matrix **G** and the vector $\mathbf{d}_w$ in Eq. (16) are formed. Using the SVD algorithm to solve Eq. (16), the mean values of unknown input parametersE_1 , E_2 and output parameters x_2, θ_2, θ_3 are computed to be 3.59×10^4Mpa, 1.87×10^5 Mpa, 9.62×10^{-4}mm, 4.63×10^{-4} and 2.1×10^{-3}, respectively (as shown in the 2nd row in Table 1). Now, we use linearization method to estimate confident intervals of the unknown parameters. In this study, 1%, 3% and 5% deviation level of the known parameters is considered, respectively. In linearization method, the variance can be obtained by Eq. (19) at each of the iterations based on the variance $\overline{\sigma}^2$ and sensitively matrix **G**. Then, confident intervals of the obtained parameters can be computed by Eq. (20). Updating the originals of input parameters E_1 and E_2 with newly computed values 3.59×10^4 Mpa and 1.87×10^5 Mpa, and combining them together with the known input parameters A_1 and A_2 into the forward solver FEM once again. Then, a new set of output parameters x_2, θ_2, x_3, y_3 and θ_3 is calculated to be 0.024mm, 0.012, 0.015mm, 0.027mm and 3.38, respectively (as shown in the 3rd row in Table 1). The stopping criterion is examined to decide if the solution procedure ends. As the stopping criterion is not satisfied, the iteratively process is required to continue. After 6 times of such iterations, the stopping criterion is satisfied. The solution procedure for mean values and confidence interval of material parameters stops. For detail, the solution at each of the iterations is given in Table 1. In Table 1, data in bold italic are the value obtained at corresponding solving process.

Table 1: Solution procedure towards mean values of material parameters of the beam using SMM and MLM

No. of iteration	Method	Input information				Output information				
		$E_1(\times 10^5)$ Mpa	$A_1(\times 10^3)$ mm^2	$E_2(\times 10^5)$ Mpa	$A_2(\times 10^3)$ mm^2	$x_2(\times 10^{-3})$ mm	$\theta_2(\times 10^{-3})$	$\theta_3(\times 10^{-3})$	$x_3(\times 10^{-3})$ mm	y_3 mm
Initial	FEM	1.80	6.00	2.50	4.48	***4.80***	***2.30***	***4.50***	***6.70***	***0.93***
1	SMM	***0.36***	6.00	***1.87***	4.48	***0.96***	***0.46***	***2.10***	6.70	0.93
	FEM	0.36		1.87		***24.20***	***11.60***	***14.50***	***26.70***	***3.38***
2	SMM	***0.59***	6.00	***1.99***	4.48	***39.60***	***19.10***	***22.10***	26.70	3.38
	FEM	0.59		1.99		***14.70***	***7.10***	***9.80***	***17.10***	***2.22***
3	SMM	***0.83***	6.00	***2.00***	4.48	***20.80***	***10.00***	***12.70***	17.10	2.22
	FEM	0.83		2.00		***10.40***	***5.00***	***7.70***	***12.80***	***1.70***
4	SMM	***0.97***	6.00	***2.00***	4.48	***12.20***	***5.90***	***8.50***	12.80	1.70
	FEM	0.97		2.00		***8.90***	***4.30***	***7.00***	***11.30***	***1.52***
5	SMM	***0.99***	6.00	***2.00***	4.48	***9.20***	***4.40***	***7.10***	11.30	1.52
	FEM	0.99		2.00		***8.70***	***4.20***	***6.80***	***11.10***	***1.49***
6	SMM	***1.00***	6.00	***2.00***	4.48	***8.70***	***4.20***	***6.80***	11.10	1.49
	FEM	1.00		2.00		***8.70***	***4.20***	***6.80***	***11.10***	***1.49***
True mean		1.00	6.00	2.00	4.48	8.70	4.20	6.80	11.10	1.49

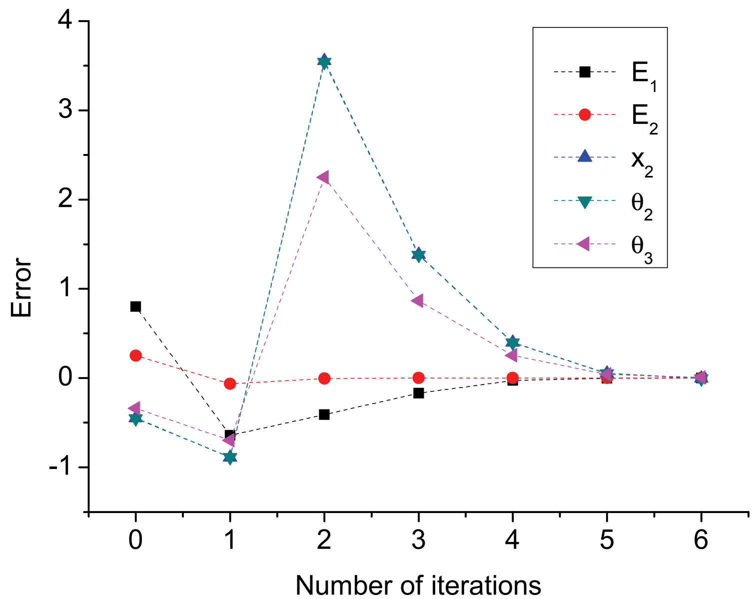


Figure 4: Convergence process of the errors

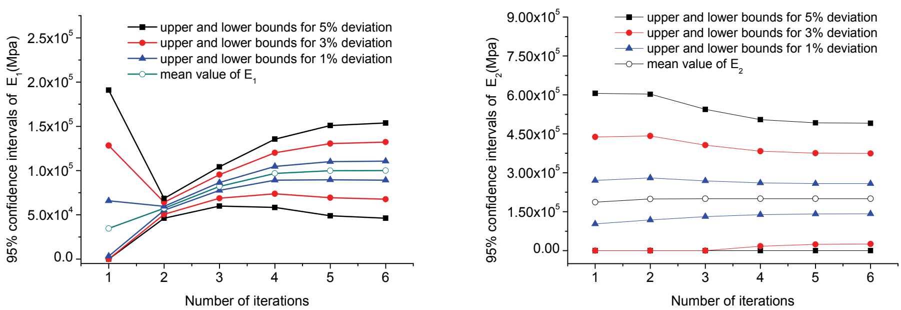
Fig. (4) shows the convergence process of errors ε_i

$$\varepsilon_i = (o_i^{j+1} - o_i^0)/o_i^{j+1} \text{ , i = } 1,\cdots,5 \tag{23}$$

where o_i^{j+1} denotes the output parameters computed from the forward solver FEM at the $j+1$ iteration, $o_i^0(i=1,2,3)$ are the output parameters x_2, θ_2 and θ_3 calculated from the SMM at the $j+1$ iteration, and $o_i^0(i=4,5)$ represent the known output parameters x_3, y_3. The maximal error between the actual and computed mean values of the sought parameters is only 0.012. It can be easy found that the required number of iterations to convergence of the SMM is very less, about six iterations in total required in the present method. Furthermore, this example demonstrates the high efficiency of the proposed method.

Using the mean values of x_2, θ_2, x_3, y_3 and θ_3 obtained by SMM, in Monte Carlo method, the variance can be calculated by Eq. (21). 95% confident intervals of them are then estimated by Eq. (20). The results are listed in Table 2. It has been find that variance of the identified parameters obtained by linearization and Monte Carlo Methods are agreed well (as shown in column 3 and 5 of Table 2). 95% confidence intervals estimated by two methods are shown in column 4 and 6 of Table 2, and its range also agreed well regardless of the presence of the deviation. Fig. 5 shows

95% confidence intervals of E_1 and E_2 obtained by using linearization method at each of iterations for different deviation levels. From this figure, it can be found that the confidence intervals of the obtained parameters enlarge as the deviation level increase. Fig. 6 depicts the histogram of E_1, E_2, x_2, θ_2 and θ_3 calculated by Monte carlo method using 1000 samples. It is easy to see that each of these five distributions of the obtained solutions is mostly symmetric about their mean values. Fig. 7 illustrates the comparison of the confidence interval of E_1 and E_2 between the present combined method and numerical simulation using Monte Carlo method. The abscissa of the figure is assigned for E_1, and the ordinate stands for E_2. The mark of cross "* " in the figure shows the relationship between E_1 and E_2. The thick and thin lines along the ordinate and abscissa indicate the 95% confidence intervals for 1% deviation of all known measurement given in row 2 and 3 of Table 2. These are within the projection regions of the relationship between E_1 and E_2. Consequently, the presented method can be judged to be valid for estimation of confidence interval in regard to material identification.



(a) E_1 (b) E_2

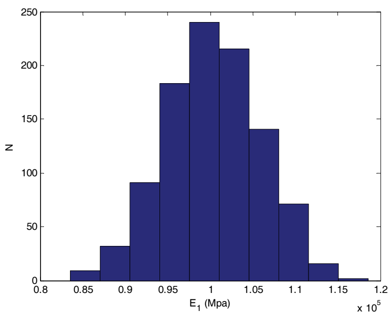
Figure 5: 95% confidence interval of E_1 and E_2 by linearization method

4.2 *Reconstruction of external Loads of a truss*

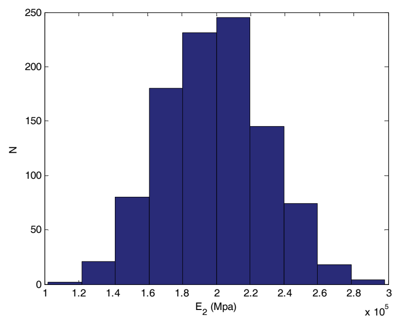
Fig. 8 illustrates the model used for loads reconstruction of a nine-bar truss made of steel. This truss is fixed at node 1 without in-plane rotation and translation. Joint 4 is subjected to a vertical load F_1, and joint 5 is subjected to a horizontal load F_2. All bars have a same cross-section area denoted by A. The truss element is used to create an FEM mesh. Each bar is an element and hence there are total of nine truss elements. In this case, the input parameters include external loads at node 4 and 5 (F_1,F_2), Young's module E, area of cross section A. And the output parameters

Table 2: Comparison of confidence intervals for identified parameters of the beam obtained from Linearization and Monte carlo methods

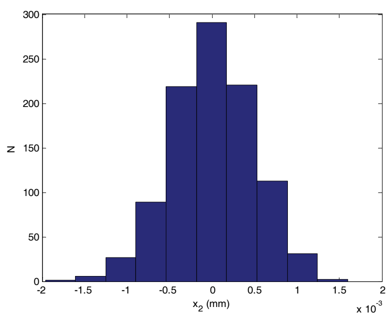
Parameters	Deviation of measurement	Linearization method		Monte carlo method	
		Covariance matrix $cov\mathbf{m} = \overline{\sigma}^2(\mathbf{G}^T\mathbf{G})^{-1}$	Confidence interval $\mathbf{m} \pm 1.96 \cdot \sqrt{\text{diag}(\text{cov}(\mathbf{m}))}$	Covariance matrix $cov(\mathbf{m}) = \mathbf{B}^T\mathbf{B}/N$	Confidence interval $\mathbf{m} \pm 1.96 \cdot \sqrt{\text{diag}(\text{cov}(\mathbf{m}))}$
$E_1(\times 10^5)$ Mpa	1%	0.05492	[0.8924, 1.1077]	0.05343	[0.89527, 1.1089]
	3%	0.16477	[0.67705, 1.323]	0.1642	[0.67817, 1.3218]
	5%	0.27462	[0.46175, 1.5383]	0.27491	[0.46119, 1.5127]
$E_2(\times 10^5)$ Mpa	1%	0.29675	[1.4184, 2.5816]	0.29056	[1.4305, 2.5925]
	3%	0.89024	[0.25513, 3.7449]	0.8837	[0.26794, 3.7321]
	5%	1.4837	[0, 4.9081]	1.4825	[0, 4.755]
$x_2(\times 10^{-3})$ mm	1%	0.47928	[7.7, 9.6]	0.46305	[7.8, 9.6]
	3%	1.4	[5.9, 11.5]	1.4	[5.9, 11.5]
	5%	2.4	[4.0, 13.4]	2.4	[4.0, 13.1]
$\theta_2(\times 10^{-3})$	1%	0.22687	[3.7, 4.6]	0.22074	[3.7, 4.6]
	3%	0.68062	[2.8, 5.5]	0.67827	[2.8, 5.5]
	5%	1.1	[1.9, 6.4]	1.1	[1.9, 6.3]
$\theta_3(\times 10^{-3})$	1%	0.17555	[6.5, 7.2]	0.17347	[6.5, 7.2]
	3%	0.52665	[5.8, 7.9]	0.52151	[5.8, 7.9]
	5%	0.87775	[5.1, 8.6]	0.87174	[5.1, 8.5]



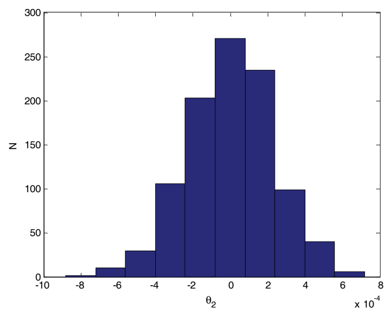
(a) E_1



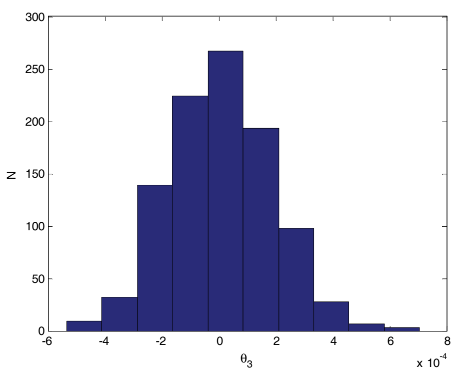
(b) E_2



(c) x_2



(d) θ_2



(e) θ_3

Figure 6: Histogram of E_1 ,E_2, x_2, θ_2 and θ_3 by Monte carlo method

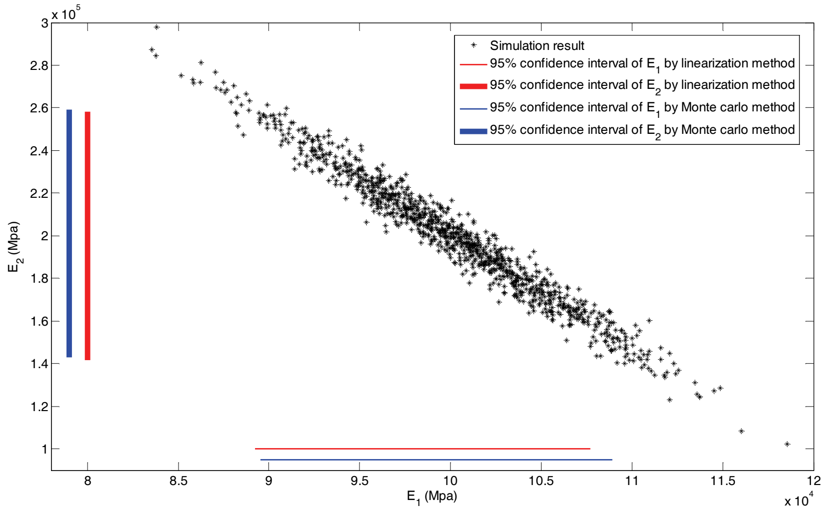


Figure 7: Comparison of four confidence interval with simulation result for 1% deviation level of measurements

include stresses of the bar 1 and 2 (σ_1, σ_2) and the translational displaces at node 2 (x_2) and node 3 (x_3, y_3).

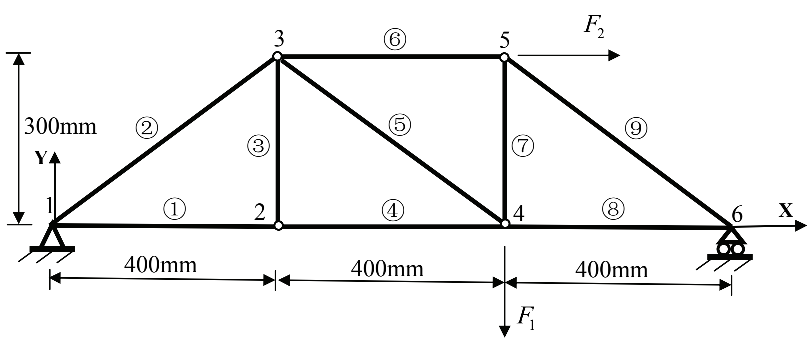


Figure 8: A nine bar truss with two point load

It is assumed that the partly known parameters in the input (E,A) and output (σ_1, σ_2) follow a normal distribution. Consider 5% deviation level of measurement, these parameters can be described as $E \sim N(2\times10^5,(1\times10^4)^2)$Mpa, $A \sim N(25,1.25^2)mm^2$, $\sigma_1 \sim N(71.1,3.6^2)$ Mpa and $\sigma_2 \sim N(-38.9,(-1.9)^2)$Mpa. This problem is also commonly happen in the real situations.

The problem dealt with in this section is to reconstruct unknown external loads (F_1, F_2)in the form of nodal forces and to estimate the confidence intervals of the identified nodal forces so as to correspond to the probability distribution of the partly known input and output (E,A,σ_1, σ_2). The presented method is available for this problem.

This case can be solved in the exactly same way as the material identification of the beam. Firstly, we assume F_1 and F_2 are 1.0×10^3N and 0.5×10^3N, respectively. Displacements x_2, x_3, y_3 and stressesσ_1, σ_2 are calculated to be -0.89 mm, -4.58 mm, 18.84 mm, -4.44 Mpa and 30.56 Mpa by forward solver, respectively. Then, the sensitivity matrix is obtained from these parameters. Subsequently, F_1, F_2, x_2, x_3 and y_3 are computed in the same way as that in the previous example (as shown in 2nd row in Table 3). Using the FEM model, with the calculated F_1 and F_2, the known E and A, a new set of parameters are computed to be 14.62 mm, 38.13 mm, -67.11 mm, 71.32 Mpa and -39.07 Mpa. We also use the Eq. (22) to assess whether the present solution is satisfactory in accuracy. This solving process experiences a total of three iterations. Fig. (9) shows the convergence process of the parameter errors. The maximal error of the solved parameters with respect to their actual values is 1.87%. The corresponding solutions at each of iteration are listed in Table 3.

The confidence intervals of the solved parameters are estimated as the same way in the previous example. We also considered three deviation levels 1%, 3% and 5%. The results are given in Table 4. We can find that 95% confidence intervals estimated by two methods are agreed well regardless of the presence of the deviation. Fig. 10 shows the histogram of F_1, F_2, x_2, θ_2 and $\theta_3$calculated by Monte carlo method using 1000 samples. Fig. 11 depicts the comparison between the confidence interval estimation of F_1 and F_2 obtained from the suggest method and that from numerical simulation. F_1 is denote the abscissa of the figure, and F_2 represents ordinate. The symbol "* " in the figure shows the relationship between F_1 and F_2. The colorful thick and thin lines along the ordinate and abscissa indicate the 95% confidence intervals for 5% deviation of all known measurement given in row 2 and 3 of Table 4. These are within the projection regions of the relationship between F_1 and F_2. Therefore, the propose method is feasibility for confidence interval estimation in regard to load reconstruction.

From the two examples, it can be clearly concluded that the present combined method is very efficient for solution of uncertainty inverse problem with insufficiency and imprecision in the input and/or output parameters. In practical appli-

Table 3: Solution procedure towards mean values of external loads of the truss using SMM and MLM

No. of iteration	Method	Input parameters				Output parameters				
		$F_1(\times 10^3)$ N	$E(\times 10^5)$ Mpa	$F_2(\times 10^3)$ N	A mm^2	$x_2(\times 10^{-2})$ mm	$x_3(\times 10^{-2})$ mm	$y_3(\times 10^{-2})$ mm	σ_1 Mpa	σ_2 Mpa
Initial	FEM	1.0	2.0	0.5	25.0	***-0. 89***	***-4.58***	***18.84***	***-4.44***	***30.56***
1	SMM	***-2.51***	2.0	***1.002***	25.0	***-16.04***	***-47.28***	***104.79***	-4.44	30.56
	FEM	-2.51		1.002		***14.26***	***38.13***	***-67.11***	***71.32***	***-39.07***
2	SMM	***-2.5***	2.0	***1.0***	25.0	***14.30***	***38.24***	***-67.35***	71.32	-39.07
	FEM	-2.5		1.0		***14.22***	***38.01***	***-66.88***	***71.11***	***-38.89***
3	SMM	***-2.5***	2.0	***0.99***	25.0	***14.22***	***38.01***	***-66.88***	71.11	-38.89
	FEM	-2.5		0.99		***14.22***	***38.01***	***-66.88***	***71.11***	***-38.89***
True mean		-2.5	2.0	1.0	25.0	14.22	38.01	-66.88	71.11	-38.89

Table 4: Comparison of confidence intervals for identified parameters of the truss obtained from Linearization and Monte carlo methods

Parameters	Deviation of measurement	Linearization method		Monte carlo method	
		Covariance matrix $cov\mathbf{m} = \overline{\sigma}^2(\mathbf{G}^T\mathbf{G})^{-1}$	Confidence interval $\mathbf{m} \pm 1.96 \cdot \sqrt{\text{diag}(\text{cov}(\mathbf{m}))}$	Covariance matrix $cov(\mathbf{m}) = \mathbf{B}^T\mathbf{B}/N$	Confidence interval $\mathbf{m} \pm 1.96 \cdot \sqrt{\text{diag}(\text{cov}(\mathbf{m}))}$
$F_1(\times 10^3)$ N	1%	0.0249	[-2.549, -2.451]	0.0239	[-2.547, -2.453]
	3%	0.0748	[-2.647, -2.353]	0.0758	[-2.649, -2.351]
	5%	0.1247	[-2.743, -2.256]	0.1262	[-2.747, -2.253]
$F_2(\times 10^3)$ N	1%	0.0273	[0.946, 1.054]	0.0274	[0.946, 1.054]
	3%	0.0819	[0.839, 1.161]	0.0839	[0.836, 1.164]
	5%	0.1365	[0.732, 1.268]	0.1378	[0.730, 1.271]
$x_2(\times 10^{-2})$ mm	1%	0.28	[13.67, 14.78]	0.28	[13.68, 14.76]
	3%	0.85	[12.56, 15.88]	0.86	[12.53, 15.91]
	5%	1.41	[11.46, 16.99]	1.40	[11.47, 16.97]
$x_3(\times 10^{-2})$ mm	1%	0.73	[36.59, 39.43]	0.74	[36.57, 39.45]
	3%	2.18	[33.74, 42.28]	2.22	[33.65, 42.37]
	5%	3.63	[30.89, 45.12]	3.53	[31.08, 44.94]
$y_3(\times 10^{-2})$ mm	1%	1.18	[-69.19, -64.57]	1.20	[-69.23, -64.54]
	3%	3.54	[-73.81, -59.95]	3.53	[-73.80, -59.96]
	5%	5.89	[-78.43, -55.33]	5.80	[-78.25, -55.52]

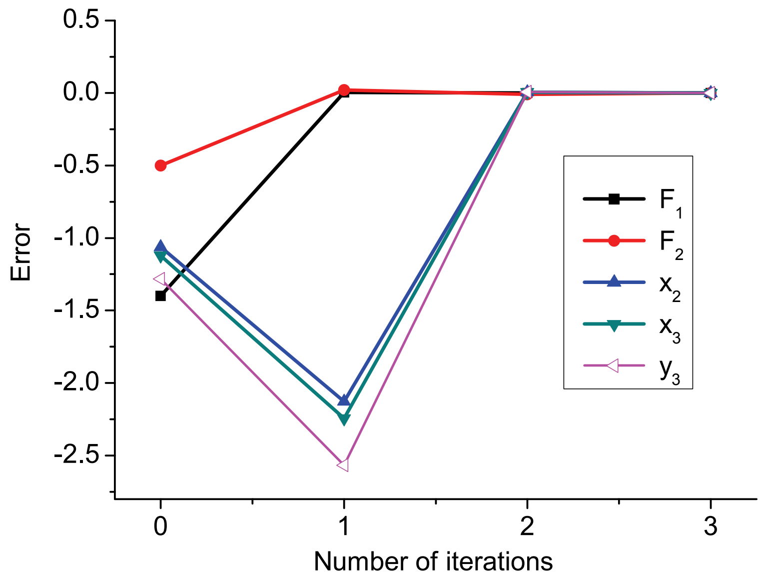
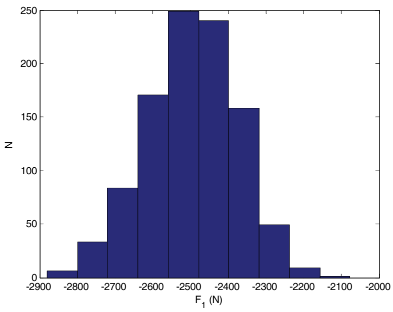


Figure 9: Convergence process of the unknown parameter errors

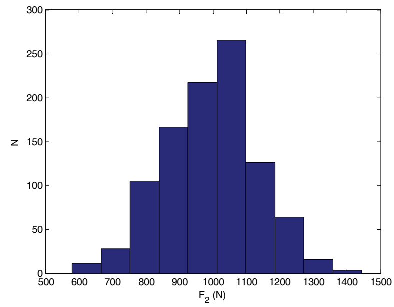
cation of the combined method, however, two points should be noticed. Firstly, as one of gradient based approaches, the SMM requires an initial estimation for mean values of the unknown parameters. This work should be performed by a rational combination of physical theory and engineering experience. Secondly, confidence intervals estimated by the linearization and Monte Carlo methods are agreed well regardless of the presence of the deviation in the linear problem. Additionally, the obtained confidence intervals should be having physical sense. For example, the Young's modulus should be greater than 0 in Fig. 5(a).

5 Conclusions

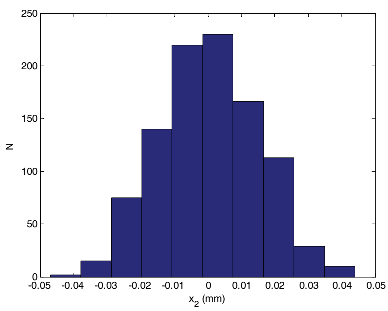
A method that combines sensitive matrix method and maximum likelihood estimation is proposed for UIPs with insufficiency and imprecision in the input and/or output parameters. This combined method has been used as the inverse operator to determine mean values of the unknown parameters and quantify confidence intervals for the identified solutions. This inverse operator takes both the advantage of easier to perform of SMM and the advantage of quantifying uncertainty propagation of MLM. We have demonstrated, though the use of the present combined method for solving two practical problems of material identification and load re-



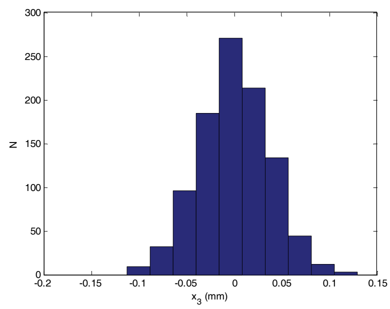
(a) F_1



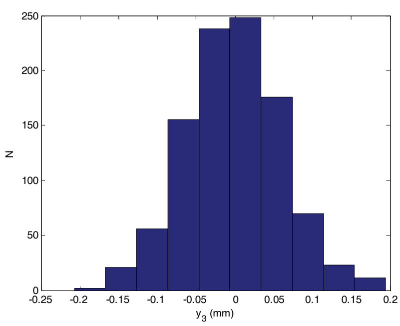
(b) F_2



(c) x_2



(d) x_3



(e) y_3

Figure 10: Histogram of F_1, F_2, x_2, θ_2 and θ_3 by Monte carlo method

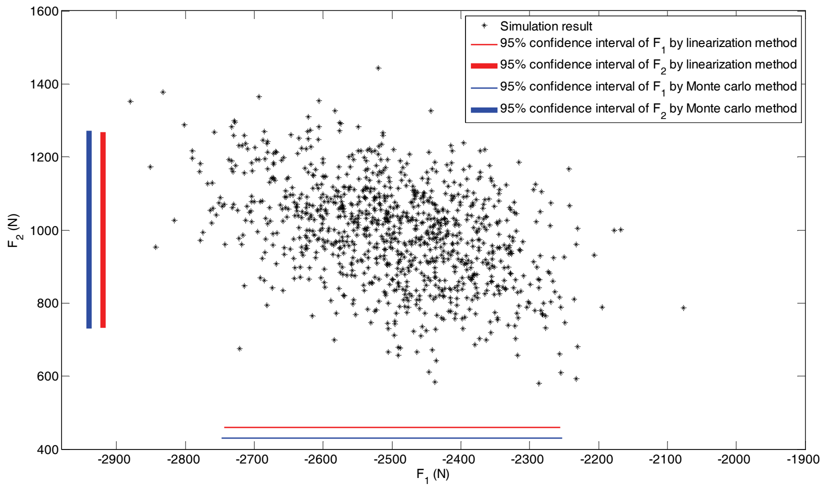


Figure 11: Comparison of four confidence interval with simulation result for 5% deviation level of measurements

construction, that this method has a high feasibility and validity for this class of UIP. Issues to be addressed include the extension of the proposed method to the case of more complex problems.

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