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A 3D Constitutive Model for Magnetostrictive Materials

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Abstract: This paper is concerned with a 3-D general constitutive law of nonlinear magneto-thermo-elastic coupling for magnetostrictive materials. The model considered here is thermodynamically motivated and based on the Gibbs free energy function. A set of closed and analytical expressions of the constitutive relationships for the magnetostrictive materials are obtained, in which all parameters can be determined by those measurable experiments in mechanics and physics. Then the model can be simplified to two cases, i.e. magnetostrictive rods and films. It is found that the predictions from this model are in good accordance with the experimental data including both rods and films. In particular, the effects of pre-stress or in-plane residual stress and environment temperature on the magnetization or the magnetostriction are also discussed.

Keywords: Magnetostrictive; Constitutive relations; Magneto-thermo-elastic coupling; Nonlinear

1 Introduction

A huge strain up to the order of 10^{-3} arises when magneostrictive alloys such as Terfenol-D are applied in an external magnetic field, which results in a wide range of application in the high-performance smart structures and devices, especially in fields such as deformable surfaces, active vibration control, where large forces and small displacements are required [Olabi and Grunwald (2008)]. To efficiently control these intelligent devices and improve their designs, a study of constitutive behavior for magnetostrictive materials in various operating conditions needs to be carried out.

Experiments conducted by Butler [Butler (1988)], Clark et al [Clark, Teter and McMasters (1988)], Moffet et al [Moffet, Clark, Wun-Fogle, Linberg, Teter, and

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McLaughlin (1991)] show that Terfenol-D rods exhibit an inherent coupling between deformation and operating conditions such as magnetic excitation, pre-stress and environment temperature. The magnetization curves and magnetostrictive strain curves for Terfenol-D rods are nonlinear, and vary with the pre-stress and environment temperature. Furthermore, for magnetostrictive films, as presented by Schatz et al [Schatz, Hirscher, Schnell, Flik and Kronmüller (1994)], both internal stress and the direction of applied magnetic field also have an influence on their magnetic and mechanic behavior. Therefore, it is necessary to develop a general theoretical model of constitutive relationships which can predict those nonlinear coupled behavior of magnetostrictive materials, including both rods and films.

Piezomagnetic model, which is valid only within a narrow operating range, is still widely employed in engineering applications [Clark (1980)]. Since the piezomagnetic model is linear, it is inaccurate to describe the nonlinear coupled behavior of magnetostrictive materials in a broad range of magnetic field. To circumvent this disadvantage, some nonlinear constitutive models for magnetostrictive materials have been developed. A class of widespread constitutive models is thermodynamically motivated model, which is formulated in accordance with the principles of thermodynamics. The constitutive relationships are determined by defining a specific free energy function and truncating the polynomial expansion of the free energy function in certain ways, see e.g. Carman and Mitrovic [Carman and M. Mitrovic (1995)], Wan et al [Wan, Fang and Hwang (2003)], Duenas et al [Duenas, Hsu and Carmanin (1996)] and Zheng and Liu [Zheng and Liu (2005)]. Zheng and Liu's model is found to have wider applicability and higher precision in the prediction of magnetostriction of Terfenol-D rod under various pre-stress than those other models, which had been discussed in detail in [Zheng and Liu (2005)]. For magnetostrictive thin films, both Duenas et al's model and Zheng-Liu's model [Liu and Zheng (2005)] can be generalized to three-dimension problems to describe the magnetoelastic coupled behavior of films. Guerrero and Wetherhold [Guerrero and Wetherhold (2004)] calculated strain and stress in bulk magnetostrictive materials and thin films.

However, none of the above models can capture the magneto-thermo-elastic coupling characteristics of magnetostrictive materials. The influence of environment temperature on the magnetostrictive strain is treated as a thermal expansional term in both Duenas et al's model and Guerrero and Wetherhold's model, while the temperature dependence of magnetostriction has not be taken into consideration in Zheng and Liu's model. Since the magnetostrictive devices are unavoidably operated in an environment of changeable temperature in practice, and have the advantage of functioning under elevated temperatures [Clark (1992)], the study of environment temperature effect becomes significant and necessary for both the applications and theoretical reframe of the materials. Zheng and Sun [Zheng and Sun (2006)] proposed a model which can describe the temperature dependence for Terfenol-D rods. However, since it is a one-dimension model, it can not describe the magneto-thermo-elastic coupling characteristics of magnetostrictive films.

The purpose of this paper is the development of a general constitutive model for magnetostrictive materials, which can predict the nonlinear coupled behavior of both magnetostrictive rods and films subjected to magneto-mechanical loading and different environment temperature. In the following section, the derivation of the theoretical model is briefly displayed. Then, the simplified models for rods and films are shown in Sec. 3. Thirdly, the validation of the obtained theoretical models is conducted in Sec. 4 by comparing the predictions with those existing experimental results, wherein some theoretical predictions for magnetization and magnetostriction of magnetostrictive films under pre-stress or in-plane residual stress and environment temperature are also displayed. Finally, some conclusions are remarked in Sec. 5.

2 Theoretical model

The elastic Gibbs free energy $G(\sigma_{ij}, M_k, T)$ can be expressed as [Smith (2005)]

$$G(\sigma_{ij}, M_k, T) = U - TS - \sigma_{ij}\varepsilon_{ij}. \tag{1}$$

here, U is the internal energy, σ_{ij} is the stress, ε_{ij} is the strain, H_k is the magnetic field intensity, M_k is the magnetization, T is the temperature, S is the entropy density and μ_0 is the vacuum permeability. The thermodynamic relations can be derived as following

$$\varepsilon_{ij} = -\frac{\partial G}{\partial \sigma_{ij}}, \quad \mu_0 H_k = \frac{\partial G}{\partial M_k} \tag{2}$$

and

$$S = -\frac{\partial G}{\partial T}$$

To obtain polynomial constitutive relationships, the elastic Gibbs free energy is expressed as Taylor's series expansion with the independent variables of stress, magnetization and temperature at reference point $(\sigma_{ij}, M_k, T) = (0, 0, T_0)$. Since the nominal temperature of the magnetostrictive device is usually the room temperature, the reference temperature T_0 can be chosen as 20^oC.

$$\begin{aligned}
G(\sigma_{ij}, M_k, T) =& \frac{1}{2}\frac{\partial^2 G}{\partial\sigma_{ij}\partial\sigma_{kl}}\sigma_{ij}\sigma_{kl} + \frac{1}{3!}\frac{\partial^3 G}{\partial\sigma_{ij}\partial\sigma_{kl}\partial\sigma_{mn}}\sigma_{ij}\sigma_{kl}\sigma_{mn} + \cdots \\
&+ \frac{1}{2}\frac{\partial^2 G}{\partial M_k\partial M_l}M_k M_l + \frac{1}{4!}\frac{\partial^4 G}{\partial M_k\partial M_l\partial M_i\partial M_j}M_k M_l M_i M_j + \cdots \\
&+ \frac{\partial G}{\partial T}\Delta T + \frac{1}{2}\frac{\partial^2 G}{\partial T^2}\Delta T^2 + \frac{1}{3!}\frac{\partial^3 G}{\partial T^3}\Delta T^3 + \frac{1}{4!}\frac{\partial^4 G}{\partial T^4}\Delta T^4 + \cdots \\
&+ \frac{3}{3!}\frac{\partial^3 G}{\partial\sigma_{ij}\partial M_k\partial M_l}\sigma_{ij}M_k M_l + \frac{6}{4!}\frac{\partial^4 G}{\partial\sigma_{ij}\partial\sigma_{mn}\partial M_k\partial M_l}\sigma_{ij}\sigma_{mn}M_k M_l + \cdots \\
&+ \frac{3}{3!}\frac{\partial^3 G}{\partial M_k\partial M_l\partial T}M_k M_l\Delta T + \frac{6}{4!}\frac{\partial^4 G}{\partial M_k\partial M_l\partial^2 T}M_k M_l\Delta T^2 + \ldots \\
&+ \frac{\partial^2 G}{\partial\sigma_{ij}\partial T}\sigma_{ij}\Delta T + \frac{3}{3!}\frac{\partial^3 G}{\partial\sigma_{ij}\partial\sigma_{kl}\partial T}\sigma_{ij}\sigma_{kl}\Delta T + \cdots \\
&+ \frac{12}{4!}\frac{\partial^4 G}{\partial\sigma_{ij}\partial M_k\partial M_l\partial T}\sigma_{ij}M_k M_l\Delta T + \cdots
\end{aligned} \tag{3}$$

The symmetric properties about the magnetization variable are taken into account in the above expansions [Zheng and Liu (2005)], and $\Delta T = T - T_0$. Substitution of Eq. (3) into the thermodynamic relations Eq. (2), we can obtain the polynomial forms of constitutive relation as follows

$$\begin{aligned}
\varepsilon_{ij} =& -\frac{\partial^2 G}{\partial\sigma_{ij}\partial\sigma_{kl}}\sigma_{kl} - \frac{1}{2}\frac{\partial^3 G}{\partial\sigma_{ij}\partial\sigma_{kl}\partial\sigma_{mn}}\sigma_{kl}\sigma_{mn} + \cdots \\
&- \frac{1}{2}\left(\frac{\partial^3 G}{\partial\sigma_{ij}\partial M_k\partial M_l} + \frac{\partial^4 G}{\partial\sigma_{ij}\partial\sigma_{mn}\partial M_k\partial M_l}\sigma_{mn} + \cdots\right)M_k M_l \\
&- \frac{\partial^2 G}{\partial T\partial\sigma_{ij}}\Delta T + \cdots - \frac{1}{2}\frac{\partial^4 G}{\partial\sigma_{ij}\partial M_k\partial M_l\partial T}M_k M_l\Delta T + \cdots
\end{aligned} \tag{4a}$$

$$\begin{aligned}
\mu_0 H_k =& \frac{\partial^2 G}{\partial M_k\partial M_l}M_l + \frac{1}{3!}\frac{\partial^4 G}{\partial M_k\partial M_l\partial M_i\partial M_j}M_l M_i M_j + \cdots \\
&+ \left(\frac{\partial^3 G}{\partial\sigma_{ij}\partial M_k\partial M_l}\sigma_{ij} + \frac{1}{2}\frac{\partial^4 G}{\partial\sigma_{ij}\partial\sigma_{mn}\partial M_k\partial M_l}\sigma_{ij}\sigma_{mn} + \cdots\right)M_l \\
&+ \frac{\partial^3 G}{\partial T\partial M_k\partial M_l}\Delta T M_l + \ldots + \frac{\partial^4 G}{\partial T\partial\sigma_{ij}\partial M_k\partial M_l}\Delta T\sigma_{ij}M_l
\end{aligned} \tag{4b}$$

In order to give a more concise expression, a detailed discussion on the terms in

Eqs. (4a) and (4b) will be given. First of all, the items of elastic strain and magnetostrictive strain in Eq. (4a) can be expressed as [Liu and Zheng (2005)]:

$$-\frac{\partial^2 G}{\partial\sigma_{ij}\partial\sigma_{kl}}\sigma_{kl}-\frac{1}{2}\frac{\partial^3 G}{\partial\sigma_{ij}\partial\sigma_{kl}\partial\sigma_{mn}}\sigma_{kl}\sigma_{mn}+\cdots=S_{ijkl}\sigma_{kl}+\lambda_{0ij}(\sigma_{mn}), \tag{5}$$

$$-\frac{1}{2}\left(\frac{\partial^3 G}{\partial\sigma_{ij}\partial M_k\partial M_l}+\frac{\partial^4 G}{\partial\sigma_{ij}\partial\sigma_{mn}\partial M_k\partial M_l}\sigma_{mn}+\cdots\right)M_kM_l=\left[m_{ijkl}-\frac{\lambda_{0ij}(\sigma_{mn})}{M_s^2}\delta_{kl}\right]M_kM_l, \tag{6}$$

where, S_{ijkl} is intrinsic (or saturation) flexibility tensor, the nonlinear term $\lambda_{0ij}(\sigma_{mn})$ represents the elastic strain dependent on the domain rotation or movement, the tensor m_{ijkl} is introduced to describe the magnetostrictive strain without a pre-stress ($\sigma_{mn}=0$), and δ_{kl} is the Kronecker delta. The term of thermal expansion strain can be expressed as:

$$-\frac{\partial^2 G}{\partial T\partial\sigma_{ij}}\Delta T=b_{ij}\Delta T, \tag{7}$$

where, b_{ij} is the coefficient of thermal expansion. The experimental results show that the saturated magnetostrictive strain decreases with increasing temperature above the spin reorientation temperature by a nearly linear change [Clark and Crowder (1985)]. Thus the strain item concerning magneto-thermal coupling in Eq. (4a) can be expressed as:

$$-\frac{1}{2}\frac{\partial^4 G}{\partial\sigma_{ij}\partial M_k\partial M_l\partial T}M_kM_l\Delta T=\frac{\beta_{ij}}{M_s^2}M_kM_l\Delta T\,\delta_{kl}, \tag{8}$$

where β_{ij} is the slope of magnetostriction versus increment temperature at the saturation magnetization. Next, let us discuss Eq. (4b). The terms independent of stress represent the nonlinear magnetization with a saturation trend at the free state $\sigma_{mn}=0$ and the terms related to stress represent inverse magnetostrictive effect can be obtained, respectively, as follows [Liu and Zheng (2005)]

$$\frac{\partial^2 G}{\partial M_k\partial M_l}M_l+\frac{1}{3!}\frac{\partial^4 G}{\partial M_k\partial M_l\partial M_i\partial M_j}M_iM_jM_l+\cdots+\frac{\partial^3 G}{\partial T\partial M_k\partial M_l}\Delta TM_l+\ldots$$
$$=\mu_0 f_k^{-1}(M_l,\Delta T), \tag{9}$$

$$\left(\frac{\partial^3 G}{\partial\sigma_{ij}\partial M_k\partial M_l}\sigma_{ij}+\frac{1}{2}\frac{\partial^4 G}{\partial\sigma_{ij}\partial\sigma_{mn}\partial M_k\partial M_l}\sigma_{ij}\sigma_{mn}+\cdots\right)M_l$$
$$=-2\left[m_{ijkl}\sigma_{ij}-\frac{\Lambda_0(\sigma_{mn})}{M_s^2}\delta_{kl}\right]M_l, \tag{10}$$

where $f_k^{-1}(M_l,\Delta T)$ is the inverse function of the stress-free nonlinear magnetization $M_l = M_s^T f_l(k(\Delta T)H_k)$, $\Lambda_0(\sigma_{mn}) = \int_0^{\sigma_{mn}} \lambda_{0ij}(\sigma_{mn})d\sigma_{ij}$ is the primary function of $\lambda_{0ij}(\sigma_{mn})$. For the terms concerning magneto-thermo-elastic coupling, we can write this part in the form

$$\frac{\partial^4 G}{\partial T\partial\sigma_{ij}\partial M_k\partial M_l}\Delta T\sigma_{ij}M_l = -\frac{2\beta_{ij}}{M_s^2}\Delta T\delta_{kl}M_l\sigma_{ij}. \tag{11}$$

Thus, Eq. (4) can be reduced to

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} + \lambda_{0ij}(\sigma_{mn}) + [m_{ijkl} - \frac{\lambda_{0ij}(\sigma_{mn})}{M_s^2}\delta_{kl}]M_kM_l + b_{ij}\Delta T + \frac{\beta_{ij}}{M_s^2}\Delta T\delta_{kl}M_kM_l, \tag{12a}$$

$$\mu_0 H_k = \mu_0 f_k^{-1}(M_l,\Delta T) - 2[m_{ijkl}\sigma_{ij} - \frac{\Lambda_0(\sigma_{mn})}{M_s^2}\delta_{kl}]M_l - \frac{2\beta_{ij}}{M_s^2}\Delta T\delta_{kl}M_l\sigma_{ij}. \tag{12b}$$

This is the general form of nonlinear constitutive model for magnetostrictive materials presented in this paper concerning magneto-thermo-elastic coupling. The specific form of nonlinear functions $\lambda_{0ij}(\sigma_{mn})$, $f_k^{-1}(M_l,\Delta T)$ and the parameters can be determined by experiments.

3 Simplified model

For the isotropic material, the constant tensors and nonlinear tensor functions in the Eqs. (12a) and (12b) can be further simplified. The elastic strain $S_{ijkl}\sigma_{kl}$ and magnetostrictive strain $m_{ijkl}M_kM_l$ can be expressed as [Liu and Zheng (2005)]

$$S_{ijkl}\sigma_{kl} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}], \tag{13}$$

$$m_{ijkl}M_kM_l = \frac{\lambda_s}{M_s^2}(\frac{3}{2}M_iM_j - \frac{1}{2}M_kM_k\delta_{ij}), \tag{14}$$

where E and ν are respectively the Young's modulus and the Poisson's ratio; λ_s and M_s are respectively the maximum magnetostriction and the saturation magnetization. The coefficient of thermal expansion can be written as $b_{ij} = \alpha\delta_{ij}$ and the coefficient β_{ij} can be expressed as $\beta_{ij} = \bar{\beta}\delta_{ij}$. The isotropic vector function $f_k^{-1}(M_l,T)$ can be expressed by a scalar function

$$f_k^{-1}(M_l,T) = \frac{1}{k(T)M}f^{-1}\left(\frac{M}{M_s(T)}\right)\delta_{kl}M_l. \tag{15}$$

where $M=\sqrt{M_kM_k}$ and $H=\sqrt{H_kH_k}$ are the magnitudes of the magnetization vector and the magnetic field strength vector, respectively; $k(T)=3\chi_m/M_s(T)$ is the relaxation factor and χ_m is the susceptibility;the saturation magnetization $M_s(T)$ is temperature-related, which can be written as [Zheng and Sun (2006)]

$$M_s(T)=M_s(\frac{1-T/T_c}{1-T_0/T})^{1/2}, \tag{16}$$

where T_c is Curie temperature, and the temperature units in Eq.(16) are Kelvin. For the nonlinear funcion $f(x)$, several different forms have been used, such as hyperbolic tangent function *f(x)=tanh(x)*[Duenas, Hsu and Carmanin (1996)] and Langevin function *f(x)=coth(x)-1/x* [Jiles and Atherton (1984)]. Since Langevin function is based on the Botlzmann statistics and has a clear physical background [Dapino, Smith and Flatau (2000)], it will be adopted in the current model.

3.1 For magnetostrictive Terfenol-D rods

Since Terfenol-D rods are usually subjected to an axial stress and axial applied magnetic field, the expressions of constitutive relations can be simplified to 1-D form

$$\varepsilon=\frac{\sigma}{E}+\lambda_0(\sigma)+\frac{\lambda_s-\lambda_0(\sigma)}{M_s^2}M^2+\alpha\Delta T+\frac{\bar{\beta}\Delta TM^2}{M_s^2}, \tag{17a}$$

$$H=f^{-1}(M,\Delta T)-\frac{2[\lambda_s\sigma-\Lambda_0(\sigma)]}{\mu_0M_s^2}M-\frac{2\bar{\beta}}{M_s^2}\Delta TM\sigma. \tag{17b}$$

Base on an abundance of experimental observations, the hyperbolic tangent function can be employed to approximate the nonlinear strain $\lambda_0(\sigma)$ for Terfenol-D rods [Zheng and Liu (2005)]. Thus, as a Terfenol-D rod subjected to the compressive pre-stresses, $\lambda_0(\sigma)$ and its primary function can be respectively expressed as

$$\lambda_0(\sigma)=\frac{\lambda_s}{2}\tanh(\frac{2\sigma}{\sigma_s}),\quad \Lambda_0(\sigma)=\frac{\lambda_s\sigma_s}{4}\ln(\cosh(\frac{2\sigma}{\sigma_s})), \tag{18}$$

here, σ_s reresents the axial pre-stress value, which is determined by the strain-stress curves. Substituting above Eq. (18) into Eqs. (17a) and (17b), the 1-D constitutive relations for Terfenol-D rod can be rewritten as

$$\varepsilon=\frac{\sigma}{E}+\frac{\lambda_s}{2}\tanh(\frac{2\sigma}{\sigma_s})+[1-\frac{1}{2}\tanh(\frac{2\sigma}{\sigma_s})]\frac{\lambda_sM^2}{M_s^2}+\alpha\Delta T+\frac{\bar{\beta}\Delta TM^2}{M_s^2}, \tag{19a}$$

$$H=f^{-1}(M,\Delta T)-\frac{2\lambda_sM}{\mu_0M_s^2}[\sigma-\frac{\sigma_s}{4}\ln(\cosh(\frac{2\sigma}{\sigma_s}))]-\frac{2\bar{\beta}\Delta TM\sigma}{M_s^2}. \tag{19b}$$

It should be noted that all the material parameters in Eqs. (19a) and (19b) can be easily measured in experiments. So, the model can be directly and conveniently used in practice.

3.2 For magnetostrictive films

Magnetostrictive amorphous thin films have an immeasurable applied prospect in the field of MEMS micro-actuator because of its excellent low-field performance. The elastic strain term dependent on the domain movement of magnetostrictive alloy films can be simplified as [Liu and Zheng (2005)]:

$$\lambda_{0ij}(\sigma_{mn}) = \frac{\lambda_s}{\sigma_s}\left(\frac{3}{2}\sigma_{ij} - \frac{1}{2}\sigma_{kk}\delta_{ij}\right) = \frac{\lambda_s}{\sigma_s}\tilde{\sigma}_{ij}, \tag{20}$$

here, $\tilde{\sigma}_{ij}$ is 3/2 times as much as the deviatoric component of stress tensor σ_{ij}. The terms representing the inverse magnetostrictive effect can be expressed as

$$-2[m_{ijkl}\sigma_{ij} - \frac{\Lambda_0(\sigma_{mn})}{M_s^2}\delta_{kl}]M_l = -\frac{\lambda_s}{M_s^2}[2\tilde{\sigma}_{kl} - (\mathrm{I}_\sigma^2 - 3\mathrm{II}_\sigma)\delta_{kl}/\sigma_s]M_l, \tag{21}$$

where $\mathrm{I}_\sigma = \sigma_{mm}$ and $\mathrm{II}_\sigma = (\sigma_{mm}\sigma_{nn} - \sigma_{mn}\sigma_{mn})/2$ are the first and second invariants of stress tensor, respectively. Thus, the constitutive relations Eqs. (12a) and (12b) can be simplified as

$$\begin{aligned}\varepsilon_{ij} = {} & \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}] + \frac{\lambda_s}{\sigma_s}\tilde{\sigma}_{ij} + \alpha\delta_{ij}\Delta T \\ & + \frac{1}{M_s^2}\left\{\frac{3}{2}\lambda_s M_i M_j - M_k M_k\left[\frac{\lambda_s}{2}\delta_{ij} - \bar{\beta}\delta_{ij}\Delta T + \frac{\lambda_s\tilde{\sigma}_{ij}}{\sigma_s}\right]\right\},\end{aligned} \tag{22a}$$

$$H_k = \left\{\frac{\delta_{kl}}{k(T)M}f^{-1}\left(\frac{M}{M_s(T)}\right) - \frac{\lambda_s}{\mu_0 M_s^2}[2\tilde{\sigma}_{kl} - (\mathrm{I}_\sigma^2 - 3\mathrm{II}_\sigma)\delta_{kl}/\sigma_s] - \frac{2\,\Delta T\bar{\beta}\sigma_{ij}\delta_{ij}\delta_{kl}}{\mu_0 M_s^2}\right\}M_l. \tag{22b}$$

It should be noticed that linearized function $\lambda_{0ij}(\sigma_{mn})$, shown in Eq. (20), is applicable within a proper region when the pre-stress is not very large. If the pre-stress is beyond the applicable region, the simplified model may produce enormous deviance and some physically impossible results. For an isotropic positive ($\lambda_s > 0$) or negative ($\lambda_s < 0$) magnetostrictive material, the applicable region can be expressed in the uniform dimensionless form $\min(\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3) \geq -1/2$[Liu and Zheng (2005)], where $\bar{\sigma}_i = \tilde{\sigma}_i/\sigma_s$ is the dimensionless quantity of the principal stress component of $\tilde{\sigma}_{ij}$.

In order to describe the magneto-thermo-elastic coupling behavior of the amorphous thin films, a 2-D constitutive model is needed. In this case, all the off-plane stress components should be ignored, i.e. $\sigma_z = \tau_{zx} = \tau_{zy} = 0$, and the magnetic field is applied along a direction of in-plane (H_z=0) or transverse (H_x= H_y=0). Thus, the expressions of $\mathrm{I}_\sigma^2 - 3\mathrm{II}_\sigma$and $\tilde{\sigma}_{ij}$are$\mathrm{I}_\sigma^2 - 3\mathrm{II}_\sigma = \sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2$, $\tilde{\sigma}_x = \sigma_x - \frac{1}{2}\sigma_y$, $\tilde{\sigma}_y = \sigma_y - \frac{1}{2}\sigma_x$, $\tilde{\sigma}_z = -\frac{1}{2}(\sigma_x + \sigma_y)$, $\tilde{\tau}_{xy} = \frac{3}{2}\tau_{xy}$, $\tilde{\tau}_{yz} = \tilde{\tau}_{zx} = 0$ and the constitutive relations for an isotropic magnetostrictive film can be derived from Eqs. (22a) and (22b) in the following matrix forms (the engineering shear strain $\gamma_{xy} = 2\varepsilon_{xy}$ is used)

(1) In an in-plane magnetic field

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \begin{bmatrix} \frac{\lambda_s\tilde{\sigma}_x}{\sigma_s} + \alpha\Delta T \\ \frac{\lambda_s\tilde{\sigma}_y}{\sigma_s} + \alpha\Delta T \\ \frac{\lambda_s\tilde{\tau}_{xy}}{\sigma_s} \end{bmatrix}$$
$$+ \frac{1}{M_s^2} \begin{bmatrix} \lambda_s + \bar{\beta}\Delta T - \frac{\lambda_s\tilde{\sigma}_x}{\sigma_s} & -\frac{\lambda_s}{2} + \bar{\beta}\Delta T - \frac{\lambda_s\tilde{\sigma}_x}{\sigma_s} & 0 \\ -\frac{\lambda_s}{2} + \bar{\beta}\Delta T - \frac{\lambda_s\tilde{\sigma}_y}{\sigma_s} & \lambda_s + \bar{\beta}\Delta T - \frac{\lambda_s\tilde{\sigma}_y}{\sigma_s} & 0 \\ -2\tilde{\tau}_{xy}\lambda_s/\sigma_s & -2\tilde{\tau}_{xy}\lambda_s/\sigma_s & 3\lambda_s \end{bmatrix} \begin{Bmatrix} M_x^2 \\ M_y^2 \\ M_xM_y \end{Bmatrix}, \quad (23a)$$

$$\begin{Bmatrix} H_x \\ H_y \end{Bmatrix} = \left(\frac{1}{k(T)M} f^{-1}\left(\frac{M}{M_s^T}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right.$$
$$\left. - \frac{\lambda_s}{\mu_0 M_s^2} \begin{bmatrix} 2\tilde{\sigma}_x - (\mathrm{I}_\sigma^2 - 3\mathrm{II}_\sigma)/\sigma_s & 2\tilde{\tau}_{xy} \\ 2\tilde{\tau}_{xy} & 2\tilde{\sigma}_y - (\mathrm{I}_\sigma^2 - 3\mathrm{II}_\sigma)/\sigma_s \end{bmatrix} - \frac{2\bar{\beta}\Delta T}{\mu_0 M_s^2} I_\sigma \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$
$$\begin{Bmatrix} M_x \\ M_y \end{Bmatrix}. \quad (23b)$$

(2)In a transverse magnetic field

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$
$$+ \begin{bmatrix} \lambda_s\tilde{\sigma}_x/\sigma_s + \alpha\Delta T \\ \lambda_s\tilde{\sigma}_y/\sigma_s + \alpha\Delta T \\ \lambda_s\tilde{\tau}_{xy}/\sigma_s \end{bmatrix} + \frac{1}{M_s^2} \begin{Bmatrix} -\lambda_s/2 + \bar{\beta}\Delta T - \lambda_s\tilde{\sigma}_x/\sigma_s \\ -\lambda_s/2 + \bar{\beta}\Delta T - \lambda_s\tilde{\sigma}_y/\sigma_s \\ -2\tilde{\tau}_{xy}\lambda_s/\sigma_s \end{Bmatrix} M_z^2, \quad (24a)$$

$$H_z = \frac{1}{k(T)} f^{-1}\left(\frac{M_z}{M_s^T}\right) - \left[\frac{2\bar{\beta}\Delta T I_\sigma}{\mu_0 M_s^2} + \frac{\lambda_s[2\tilde{\sigma}_z - (\mathrm{I}_\sigma^2 - 3\mathrm{II}_\sigma)/\sigma_s]}{\mu_0 M_s^2} \right] M_z. \quad (24b)$$

Since films deposited on substrate often show in-plane residual stress, the biaxial residual stress $\sigma_x = \sigma_y = \sigma$ often exists. Thus, Eqs. (23) (24) can be further simplified as:

In an in-plane magnetic field (along the x-direction)

$$\varepsilon_x = \frac{1-\nu}{E}\sigma + \frac{\lambda_s \sigma}{2\sigma_s} + \alpha \Delta T + \frac{\lambda_s[1-\sigma/(2\sigma_s)] + \bar{\beta}\Delta T}{M_s^2} M_x^2, \tag{25a}$$

$$\varepsilon_y = \frac{1-\nu}{E}\sigma + \frac{\lambda_s \sigma}{2\sigma_s} + \alpha \Delta T - \frac{\lambda_s(1+\sigma/\sigma_s)/2 + \bar{\beta}\Delta T}{M_s^2} M_x^2, \tag{25b}$$

$$\gamma_{xy} = 0, \tag{25c}$$

$$H_x = \frac{1}{k(T)} f^{-1}\left(\frac{M_x}{M_s}\right) - \frac{\lambda_s \sigma (1-\sigma/\sigma_s)}{\mu_0 M_s^2} M_x - \frac{4\bar{\beta}\Delta T \sigma}{\mu_0 M_s^2} M_x, \tag{25d}$$

$$H_y = 0. \tag{25e}$$

In a transverse magnetic field ((along the z-direction)

$$\varepsilon_x = \varepsilon_y = \frac{1-\nu}{E}\sigma + \frac{\lambda_s \sigma}{2\sigma_s} + \alpha \Delta T + \frac{-\lambda_s(1+\sigma/\sigma_s)/2 + \bar{\beta}\Delta T}{M_s^2} M_z^2, \tag{26a}$$

$$\gamma_{xy} = 0, \tag{26b}$$

$$H_z = \frac{1}{k(T)} f^{-1}\left(\frac{M_z}{M_s}\right) + \left[\frac{\lambda_s \sigma (2+\sigma/\sigma_s)}{\mu_0 M_s^2} - \frac{4\bar{\beta}\Delta T \sigma}{\mu_0 M_s^2}\right] M_z, \tag{26c}$$

It is evident that the material parameters in above formulas of Eqs. (25) and (26) also can be easily determined by the measurable experiments in mechanics and physics.

4 Verification and discussion

In order to evaluate the reliability of the constitutive relationships obtained here for magnetostrictive rods and films, we give some quantitative predictions to compare with experimental data. The material parameters appeared in the proposed model are taken as$\bar{\beta} = -1.0 \times 10^{-6}/^{\circ}C$,$c = 0.18$, $K = 2000A/m$, E_s=110Gpa, $\alpha = 1.2 \times 10^{-5}/^{o}C$, T_C=383.3^{o}C, and other parameters for Terfenol-D rods and TbDyFe amorphous thin films are shown in Table 1 respectively.

Firstly, the resulting magnetostrictive curves predicted by the proposed model and Duenas et al's model [Duenas, Hsu and Carmanin (1996)] under different pre-stress levels compared with Butler's experimental data [Butler (1988)] are depicted in Fig.1(a) and Fig.1(b), respectively. It can be seen that the predictions calculted by

Table 1: Parameters in the proposed model for Terfenol-D rods and TbDyFe amorphous thin films.

Parameters	λ_s(ppm)	$\mu_0 M_s$(T)	χ_{m}	σ_s(MPa)
Terfenol-D rods	1300	0.8	80	200
TbDyFe amorphous thin films	600	0.6	15	80

the proposed model are in agreement with the experimental data well, not only in the region of the low and moderate magnetic fields but also in the region of the high field. While Duenas et al's model can accurately predict magnetostrictive strain values in the region of the low magnetic fields for various pre-stress levels, it fails to simulate the phenomenon that the maximum magnetostrictive strain increased with the compressive pre-stress detected by expreiments. The magnetostrictive strains under different pre-stress levels predicted by the Duenas et al's model reach the same value in the region of the high field (see Fig.1(b)).

The temperature dependence of magnetostrictive strain for Terfenol-D rods is shown in Fig. 2(a) and 2(b). Here, the theoretical predictions by the proposed model and Duenas et al's model are respectively compared with Clark's observed results [Clark, Teter and McMasters (1988)]. It is noted that the magnetostrictive strain has a remarkable decrease with enhancing environment temperature for the applied magnetic field greater than 500 Oe, which is perfectly coincident with the experimental data. However, the magnetostrictive strain predicted by Duenas et al's model is nearly the same for the environment temperature changes from 0^oC to 80^oC, which means adopting a thermal expansional strain instead of magneto-thermo-elastic coupling term in the model may inaccurately capture the nonlinear coupled behavior of magnetostrictive materials.

The experimental data of TbDyFe amorphous thin films is also analyzed in terms of the proposed model. As shown in Fig. 3(a) and 3(b), the proposed model can predict the effect of in-plane stress on the magnetization, in the presence of excellent agreement with the experimental results [Schatz, Hirscher, Schnell, Flik and Kronmüller (1994)] of magnetostrictive films subjected to a applied magnetic field parallel or perpendicular to the film plane. In-plane tensile stress induces an easy direction of the magnetization parallel to the film plane (//), i.e. along the x-direction. In contrast, in-plane compressive stress makes the easy direction perpendicular to the film plane ($\perp$), i.e. along the z-direction. This is quite reasonable at the initial stage of magnetization process. In that case, according to the viewpoint of magnetic domains, an in-plane tensile stress tends to align the magnetic domains along the direction, and an in-plane compressive stress tends to align them along the direction perpendicular to the plane. In addition, as shown in

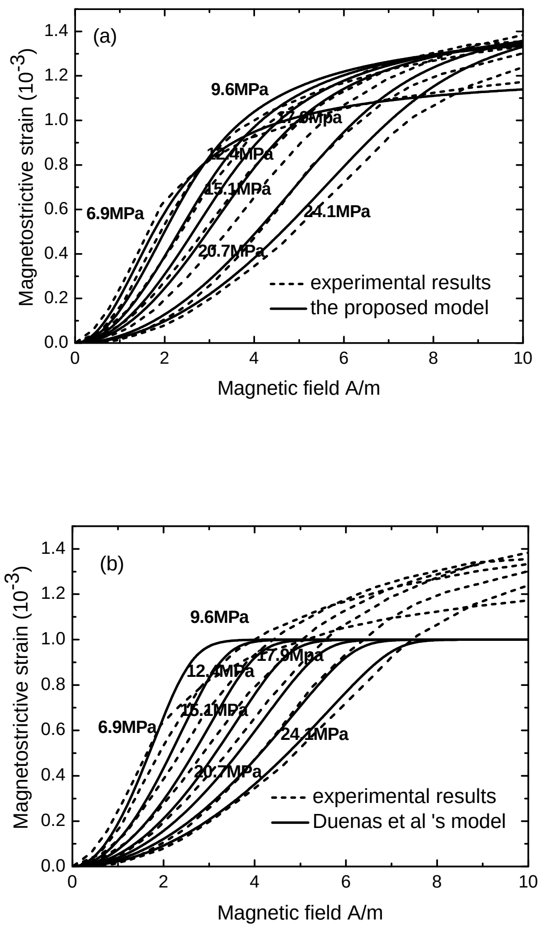

Figure 1: Comparison of the theoretical magnetostrictve strain curves calculated by the proposed model (a) and Duenas et al's model [Duenas, Hsu and Carmanin (1996)] (b) with the experimental data [Butler (1988)] for Terfenol-D rods under various compressive pre-stresses (Dashed lines: experimental data; Solid lines: theoretical results).

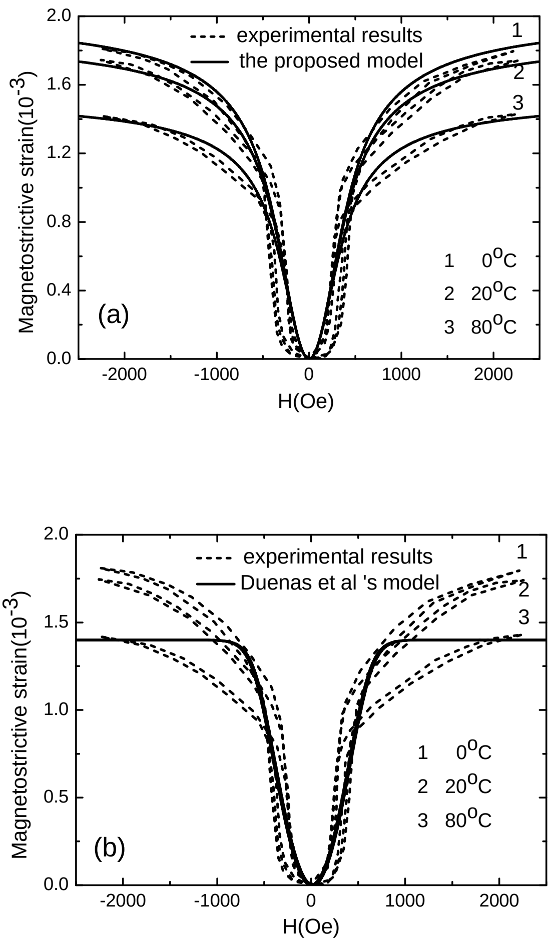

Figure 2: Comparison of the theoretical magnetostrictve strain curves calculated by the proposed model (a) and Duenas et al's model (b) with the experimental data [Clark, Teter and McMasters (1988)] for Terfenol-D rods under various environment temperature (Dashed lines: experimental data; Solid lines: theoretical results).

Fig. 4, when an in-plane magnetic field is applied along the x-direction of the film, the in-plane residual stress has different influences on the curve of magnetostriction curves in the various regions. A higher magnetostriction (550ppm) is achieved with compressive stress than the one (350ppm) with tensile stress for the magnetic field up to 1T. The predicted magnetostriction for the films with tensile or compressive stress subjected to an in-plane magnetic field is consistent with the experimental data.

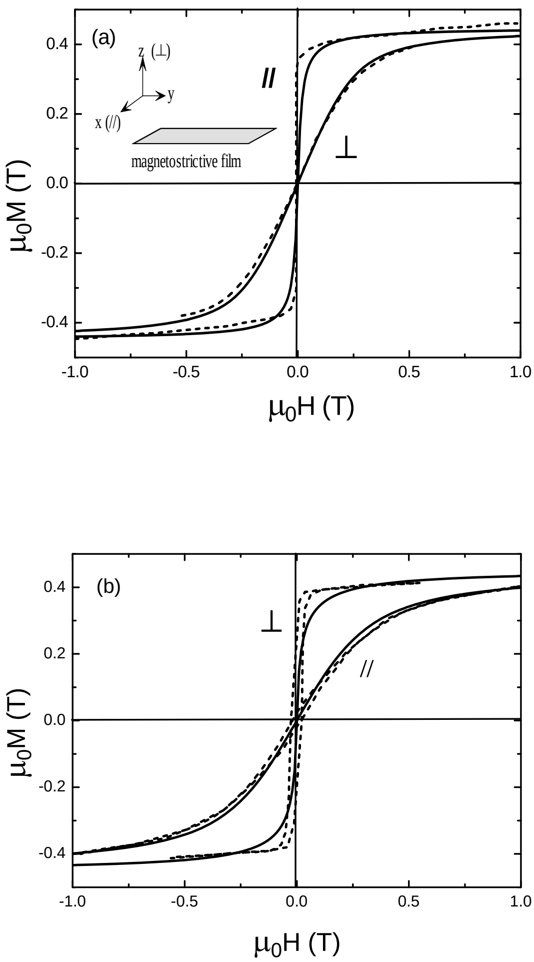

Figure 3: Comparison of the predicted magnetization curves with the experimental data for TbDyFe amorphous thin film under (a) tensive (b) compressive in-plane stress (Dashed lines: experimental data [Schatz, Hirscher, Schnell, Flik and Kronmüller (1994)]; Solid lines: predicted results.)

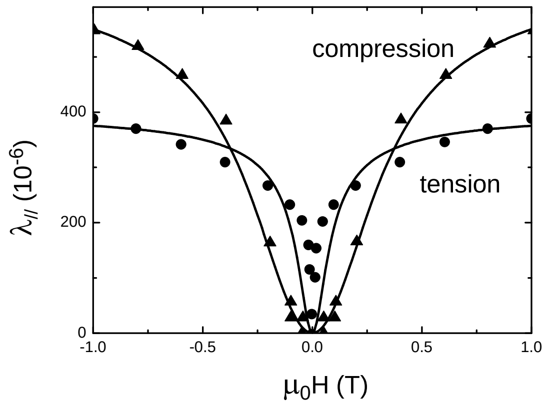

Figure 4: Comparison of the predicted magnetostriction curves with the experimental data for TbDyFe amorphous thin film under tensive and compressive in-plane stress (Scatters: experimental data [Schatz, Hirscher, Schnell, Flik and Kronmüller (1994)]; Lines: predicted results.)

Fig. 5 illustrates the magnetization curves of magnetostrictive thin films with different in-plane residual stresses for the environment temperature changes from 0^oC to 40^oC. It is evident that the saturation magnetization is independent on the in-plane residual stress in both the in-plane and transverse magnetic field, while it is highly sensitive to the environment temperature. The magnetization has a remarkably decrease with increasing temperature when the magnetization approaches to saturation whether the magnetic field is parallel or perpendicular to the film.

Then we give the curves of magnetostrictive strain for films with different in-plane residual stresses for the environment temperature varies from 0^oC to 40^oC in Fig. 6 and Fig. 7. As shown in Fig. 6(a), when an in-plane magnetic field is applied along the x-direction of the film, the magnetostriction component λ_x of the film with tensile or compressive stress has a dramatic decrease with increasing the environment temperature. That means the environment temperature also has a significant influence on the magnetic domain rotation. The high temperature gives rise to the thermal fluctuation of domain rotation, which hinders the domain rotating to the direction of magnetic field to some extent and decreases the saturation magnetization and magnetostriction correspondingly. These results suggest that the measurements on the magnetization and the magnetostriction should take the environment temperature into account. As for the component λ_y, which is negative, the effects of in-plane residual stresses and environment temperature are obviously different from the characteristics for the component λ_x. The values of magnetostriction component λ_y for the film with an in-plane tensile stress are always larger than the ones

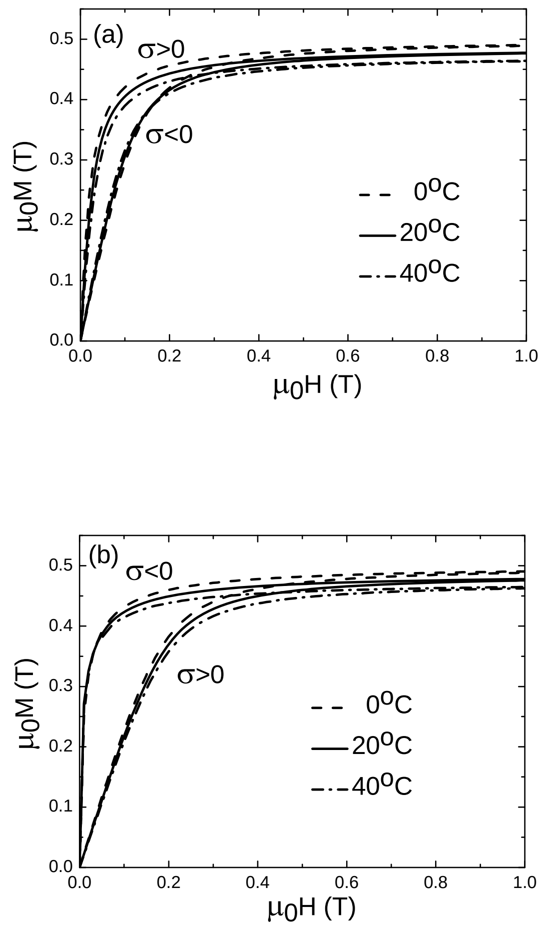

Figure 5: Predicted magnetization curves of a TbDyFe amorphous thin film under various in-plane stresses and environment temperature in an in-plane (a) or transverse (b) magnetic field.

with a compressive stress in the low, moderater and high magnetic field regions. In addition, the values of λ_y follow a slow increase with the environment temperature. When the applied magnetic field is perpendicular to the film, the components λ_x and λ_y, as shown in Fig.7, are always equal and negative. The values of λ_x or λ_y increase with environment temperature whether the residual stress is tensile or compressive. Furthermore, the in-plane residual stress has different influences on the magnetostriction in the various regions of applied magnetic field (see Fig.7).

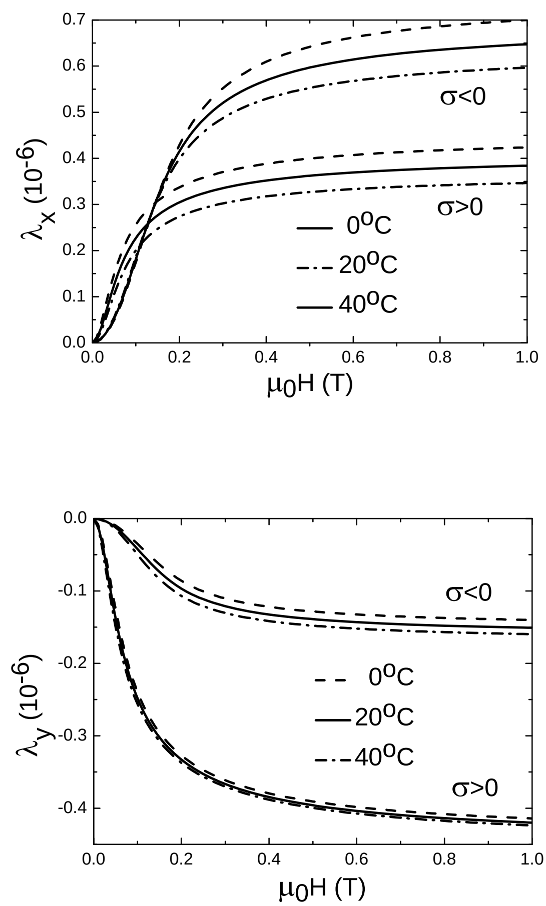

Figure 6: Predicted magnetostrictive strain curves of a TbDyFe amorphous thin film under various in-plane stresses and environment temperature in an in-plane magnetic field.

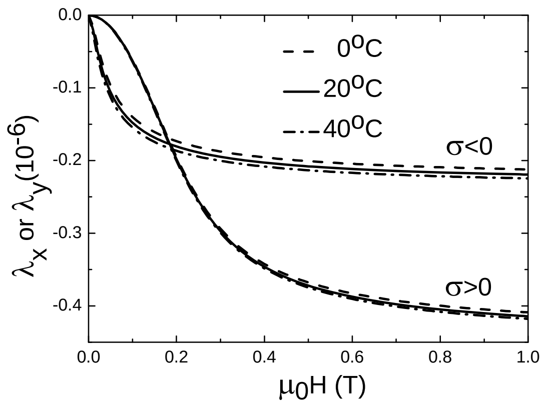

Figure 7: Predicted magnetostrictive strain curves of a TbDyFe amorphous thin film under various in-plane stresses and environment temperature in a transverse magnetic field.

5 Conclusions

A new 3D constitutive model of nonlinear magneto-thermo-elastic coupling for magnetostrictive materials is proposed in this paper. In the model, the constitutive relations are analytically formulated in closed forms, and all parameters referred to the model can be easily determined by experiments. The quantitative results demonstrate that this model can effectively predict the nonlinear coupled magnetostrictive behavior for magnetostrictive rods and films. Furthermore, the magneto-thermo-elastic coupling features for magnetostrictive films are dicussed in detail. The results show that magnetostrictive films are sensitive to the environment temperature and the in-plane residual stress. These essential and important investigations will be of benefit to both the theoretical researches and the applications of magnetostrictive materials in smart or intelligent structures and systems.

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