The Spring-Damping Regularization Method and the Lie-Group Shooting Method for Inverse Cauchy Problems

Chein-Shan Liu[1,2], Chung-Lun Kuo[3] and Dongjie Liu[4]

Abstract: The inverse Cauchy problems for elliptic equations, such as the Laplace equation, the Poisson equation, the Helmholtz equation and the modified Helmholtz equation, defined in annular domains are investigated. The outer boundary of the annulus is imposed by overspecified boundary data, and we seek unknown data on the inner boundary through the numerical solution by a spring-damping regularization method and its Lie-group shooting method (LGSM). Several numerical examples are examined to show that the LGSM can overcome the ill-posed behavior of inverse Cauchy problem against the disturbance from random noise, and the computational cost is very cheap.

Keywords: Lie-group shooting method, Elliptic equations, Inverse Cauchy problem, Ill-posed problem, Spring-damping regularization method

1   Introduction

In a number of applications, such as vibration of structure, nondestructive testing technique, electro-cardiology, electro-magnetic scattering, the detection of corrosion inside a pipe, and so on, one needs to solve the inverse Cauchy problem for elliptic equation. Unfortunately, the inverse Cauchy problem is highly ill-posed, i.e., the solution does not depend continuously on the Cauchy data, and thus a small disturbance of the given data may destroy the numerical solution. In this paper, we consider the inverse Cauchy problem for elliptic equation in an annular domain by recovering the boundary data on the inaccessible inner boundary from the overspecified Cauchy data on the accessible outer boundary. Owing to its severely ill-posed nature, we have to tackle this type inverse Cauchy problem with a feasible numerical algorithm. The inverse Cauchy problems associated with the elliptic

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[1] Department of Civil Engineering, National Taiwan University, Taipei, Taiwan

[2] Corresponding author. E-mail: liucs@ntu.edu.tw

[3] Department of Systems Engineering and Naval Architecture, National Taiwan Ocean University, Keelung, Taiwan

[4] Department of Mathematics, College of Sciences, Shanghai University, Shanghai, P. R. China

equations have been extensively studied by using different numerical methods, such as, the boundary element method [Lesnic, Elliott and Ingham (1997); Mera, Elliott, Ingham and Lesnic (2000); Chi, Yeih and Liu (2009)], the modified collocation Trefftz method [Liu (2008a, 2008b, 2008c)], the method of fundamental solutions [Marin, Elliott, Heggs, Ingham, Lesnic and Wen (2005); Jin and Zheng (2006); Lin, Chen and Wang (2011)], the finite element method [Chakib and Nachaouli (2006)], the boundary particle method [Chen and Fu (2009)], iteration schemes [Jourhmane and Nachaouli (1999, 2002); Essaouini, Nachaouli and Hajji (2004); Jourhmane, Lesnic and Mera (2004)], the Fourier regularization method [Fu, Li, Qian and Xiong (2008); Fu, Feng and Qian (2009)], and analytical method [Liu (2011a)].

It is rare of this situation that one can directly apply a numerical integration method to solve the inverse Cauchy problem (i.e., initial value problem of elliptic equation). Liu and Atluri (2010) firstly applied the group-preserving scheme [Liu (2001)] to an elliptic equation in a small strip. Abbasbandy and Hashemi (2011) have extended the group-preserving scheme [Liu (2001)] to solve the inverse Cauchy problem of Laplace equation in a larger strip; however, they did not take the noisy data into account, because the numerical integration scheme is unstable and is easily vulnerable to the disturbance of noise. Recently, in order to overcome the numerical instability of the GPS, Liu and Chang (2012) and Liu and Kuo (2011) provided a numerical computation of the inverse Cauchy problem based on the mixed group-preserving scheme (MGPS), which is a combination of the forward group-preserving scheme [Liu (2001)] and the backward group-preserving scheme [Liu, Chang and Chang (2006)], and directly integrated the inverse Cauchy problem as an initial value problem after effectively regularizing the original equation into a stable one by a novel spring-damping regularization method.

In this paper, we take advantage of the Lie-group shooting method (LGSM) and the spring-damping regularization method (SDRM) to solve the inverse Cauchy problem. It is worth mentioning that the LGSM is based on a semi-discretization and the group-preserving scheme (GPS) developed by Liu (2001). In the construction of the LGSM, Liu (2006, 2008d, 2009) has introduced the idea of one-step GPS by utilizing the closure property and structure of the Lie-group. Hence, the Lie-group shooting method (LGSM) was developed. In the literature, there does not yet have a similar approach as being a combination of the LGSM and SDRM to the inverse Cauchy problem. This paper has such a novelty by applying the above two methods to this highly ill-posed inverse Cauchy problem, which is organized as follows. Section 2 is devoted to the mathematical formulation of the inverse Cauchy problem and the derivation of the LGSM. In Section 3, the numerical examples are given to illustrate the stability and accuracy of the LGSM. Finally, the conclusions

are given in Section 4.

2 Inverse Cauchy problem and the Lie-group shooting method

We consider the following inverse Cauchy problem of a linear elliptic equation:

$$\Delta u + pu = f(x,y), \quad (x,y) \in \Omega, \tag{1}$$

$$u = \varphi(x,y), \quad (x,y) \in \Gamma_1, \tag{2}$$

$$\frac{\partial u}{\partial v} = \phi(x,y), \quad (x,y) \in \Gamma_1, \tag{3}$$

where $\Omega = \{(x,y)|0 < R_0^2 < x^2 + y^2 < R^2\}$, $\Gamma_1 = \{(x,y)|x^2 + y^2 = R^2\}$ and $\Gamma_2 = \{(x,y)|x^2 + y^2 = R_0^2\}$. p is a real constant, f, φ and ϕ are given functions, and v is an outward unit normal. If the boundary value of $u|_{\Gamma_2}$ is available, then the data are completed in the whole boundary, and the solution of elliptic equation can be obtained.

Therefore, we face the following **inverse Cauchy problem**: to seek an unknown boundary value $u|_{\Gamma_2}$ under Eqs. (1)-(3).

2.1 Spring-damping regularization method

Here, it is convenient to consider the problem in the polar coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad r \in [R_0, R], \quad \theta \in [0, 2\pi], \tag{4}$$

and write

$$u(r,\theta) := u(x = r\cos\theta, y = r\sin\theta) \tag{5}$$

for saving notation.

In the present paper we develop a novel Lie-group shooting method (LGSM) to directly solve the above inverse Cauchy problem as an initial value problem by recovering the missing initial data at the inner circle. The integration direction will be in the r-axis from R_0 to R, and thus we consider the following variable transformation from u to U:

$$u(r,\theta) = e^{\alpha r} U(r,\theta). \tag{6}$$

Hence, we can transform the problem (1)-(3) into the following problem:

$$\frac{\partial^2 U}{\partial r^2} + \left[2\alpha + \frac{1}{r}\right]\frac{\partial U}{\partial r} + \left[\alpha^2 + \frac{\alpha}{r} + p\right]U + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = e^{-\alpha r}f(r,\theta), \tag{7}$$

$$U(R,\theta) = e^{-\alpha R}\varphi(\theta), \quad \theta \in [0, 2\pi], \tag{8}$$

$$U_r(R,\theta) = e^{-\alpha R}[\phi(\theta) - \alpha\varphi(\theta)], \quad \theta \in [0, 2\pi], \tag{9}$$

while the inverse Cauchy problem (1)-(3) is equivalent to the determination of the boundary value $U(R_0, \theta)$ from Eqs. (7)-(9).

One may appreciate the effect of α below, which plays a regularization parameter by introducing two new terms: a damping term $2\alpha U_r$ and a spring term $[\alpha^2 + \alpha/r]U$ into the governing equation. This regularization technique was named as the spring-damping regularization method (SDRM) [Liu and Chang (2012); Liu and Kuo (2011)]. In the mechanical vibration system, it is known that suitable spring and damping coefficients can enhance the stability of motion.

2.2 Semi-discretization

Eq. (7) can be transformed into the following equations:

$$U_r(r,\theta) = S(r,\theta), \tag{10}$$

$$S_r(r,\theta) = -\left[2\alpha + \frac{1}{r}\right]S(r,\theta) - \frac{1}{r^2}U_{\theta\theta} - \left[\alpha^2 + \frac{\alpha}{r} + p\right]U + e^{-\alpha r}f(r,\theta). \tag{11}$$

Then, by using a semi-discrete method to discretize the quantities of $U(r,\theta)$ and $S(r,\theta)$ in the polar angle domain, we can obtain a system of ODEs for U and S with r as an independent variable:

$$U_r^i(r) = S^i(r), \quad i = 1,\ldots,n, \tag{12}$$

$$
\begin{aligned}
S_r^1(r) &= -\left[2\alpha + \frac{1}{r}\right]S^1(r) - \frac{1}{r^2}\frac{U^2(r) - 2U^1(r) + U^n(r)}{(\Delta\theta)^2} \\
&\quad - \left[\alpha^2 + \frac{\alpha}{r} + p\right]U^1(r) + e^{-\alpha r}f^1(r), \\
S_r^i(r) &= -\left[2\alpha + \frac{1}{r}\right]S^i(r) - \frac{1}{r^2}\frac{U^{i+1}(r) - 2U^i(r) + U^{i-1}(r)}{(\Delta\theta)^2} \\
&\quad - \left[\alpha^2 + \frac{\alpha}{r} + p\right]U^i(r) + e^{-\alpha r}f^i(r), \quad i = 2,\ldots,n-1, \\
S_r^n(r) &= -\left[2\alpha + \frac{1}{r}\right]S^n(r) - \frac{1}{r^2}\frac{U^1(r) - 2U^n(r) + U^{n-1}(r)}{(\Delta\theta)^2} \\
&\quad - \left[\alpha^2 + \frac{\alpha}{r} + p\right]U^n(r) + e^{-\alpha r}f^n(r),
\end{aligned} \tag{13}
$$

where $\Delta\theta = 2\pi/n$ is a uniform increment of polar angle, and $\theta_i = (i-1)\Delta\theta$. $U^i(r) = U(r,\theta_i)$, $S^i(r) = S(r,\theta_i)$, $f^i(r) = f(r,\theta_i)$ are the discretized quantities at the nodal point θ_i. The Lie-group shooting method developed by Liu (2008d, 2009) is extended and applied to the above discretized equations.

Define the following two vectors:

$$\mathbf{y} := \begin{pmatrix} \mathbf{U} \\ \mathbf{S} \end{pmatrix}, \quad \mathbf{f} := \begin{pmatrix} \mathbf{S} \\ \mathbf{h}(r,\mathbf{U},\mathbf{S}) \end{pmatrix}. \tag{14}$$

where

$$\mathbf{U} = (U^1(r),\ldots,U^n(r))^T, \quad \mathbf{S} = (S^1(r),\ldots,S^n(r))^T,$$

$$\mathbf{h}(r,\mathbf{U},\mathbf{S}) = -\left[2\alpha + \frac{1}{r}\right]\mathbf{S} - \frac{1}{r^2}\mathbf{H} - \left[\alpha^2 + \frac{\alpha}{r} + p\right]\mathbf{U} + \mathbf{L},$$

$$\mathbf{L} = e^{-\alpha r}(f^1(r),\ldots,f^n(r))^T,$$

$$\mathbf{H} = \begin{pmatrix} \frac{U^2(r)-2U^1(r)+U^n(r)}{(\Delta\theta)^2} \\ \frac{U^3(r)-2U^2(r)+U^1(r)}{(\Delta\theta)^2} \\ \vdots \\ \frac{U^n(r)-2U^{n-1}(r)+U^{n-2}(r)}{(\Delta\theta)^2} \\ \frac{U^1(r)-2U^n(r)+U^{n-1}(r)}{(\Delta\theta)^2} \end{pmatrix}. \tag{15}$$

Then, Eqs. (12) and (13) are expressed in a vector-form:

$$\mathbf{y}' = \mathbf{f}(r,\mathbf{y}), \tag{16}$$

where the prime denotes the differential with respect to r.

In order to utilize the LGSM to deal with the inverse Cauchy problem (7)-(9), i.e., to determine $\mathbf{U}(R_0)$, let us first briefly describe the group-preserving scheme (GPS) for ODEs. The details can refer Liu (2001).

2.3 The group-preserving scheme

First, we combine the vector $\mathbf{y}$ and its magnitude $\|\mathbf{y}\| = \sqrt{\mathbf{y}^T\mathbf{y}} = \sqrt{\mathbf{y}\cdot\mathbf{y}}$ into a single augmented vector

$$\mathbf{X} = \begin{pmatrix} \mathbf{y} \\ \|\mathbf{y}\| \end{pmatrix}, \tag{17}$$

and then, Eq. (16) can be embedded into the following augmented differential equations system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X}, \tag{18}$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{0}_{2n \times 2n} & \frac{\mathbf{f}(r,\mathbf{y})}{\|\mathbf{y}\|} \\ \frac{\mathbf{f}^T(r,\mathbf{y})}{\|\mathbf{y}\|} & 0 \end{pmatrix} \tag{19}$$

is an element of the Lie-algebra $so(2n,1)$ satisfying

$$\mathbf{A}^T\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}, \tag{20}$$

and

$$\mathbf{g} = \begin{pmatrix} \mathbf{I}_{2n} & \mathbf{0}_{2n \times 1} \\ \mathbf{0}_{1 \times 2n} & -1 \end{pmatrix} \tag{21}$$

is a Minkowski metric. Here, $\mathbf{I}_{2n}$ is the identity matrix of order $2n$, and the superscript T denotes the transpose. The augmented vector $\mathbf{X}$ satisfies the cone condition:

$$\mathbf{X}^T\mathbf{g}\mathbf{X} = \mathbf{y} \cdot \mathbf{y} - \|\mathbf{y}\|^2 = 0. \tag{22}$$

Hence, Liu (2001) has developed the group-preserving scheme (GPS) to assure that each $\mathbf{X}_k$ can automatically locate on the cone:

$$\mathbf{X}_{k+1} = \mathbf{G}_k\mathbf{X}_k, \tag{23}$$

where $\mathbf{X}_k$ represents the approximate value of $\mathbf{X}$ at the discrete points $r_k = R_0 + (k-1)(R-R_0)/m$, $k = 1, \cdots, m$, and $\mathbf{G}_k \in SO_o(2n,1)$, which is an $2n+1$-dimensional Lorentz group, satisfies

$$\mathbf{G}^T\mathbf{g}\mathbf{G} = \mathbf{g}, \ \ \det\mathbf{G} = 1, \ \ \mathbf{G}_0^0 > 0, \tag{24}$$

where $\mathbf{G}_0^0$ is the 00-th component of $\mathbf{G}$.

Step-by-step applying the scheme (23) to Eq. (18) and starting with an inner-boundary condition $\mathbf{X}(R_0) = \mathbf{X}^{R_0}$, we can calculate $\mathbf{X}(r)$ by the GPS. Assuming that the stepsize employed in the GPS is $\Delta r = (R - R_0)/m$ and starting from an augmented inner-boundary condition $\mathbf{X}^{R_0} = ((\mathbf{y}^{R_0})^T, \|\mathbf{y}^{R_0}\|)^T \neq \mathbf{0}$, we can calculate the value $\mathbf{X}^R = ((\mathbf{y}^R)^T, \|\mathbf{y}^R\|)^T$ at the outer-boundary $r = R$ by

$$\mathbf{X}^R = \mathbf{G}_m(\Delta r) \cdots \mathbf{G}_1(\Delta r)\mathbf{X}^{R_0}. \tag{25}$$

Because each $\mathbf{G}_i$, $i = 1, \ldots, m$, is an element of the Lie-group $SO_o(2n, 1)$, and by the closure property of the Lie-group, $\mathbf{G}_m(\Delta r) \cdots \mathbf{G}_1(\Delta r)$ is also an element of the Lie-group $SO_o(2n, 1)$, denoted by $\mathbf{G}$. Thus, we can achieve the following important result:

$$\mathbf{X}^R = \mathbf{G}\mathbf{X}^{R_0}. \tag{26}$$

This is a one-step Lie-group transformation (or one-step GPS) from $\mathbf{X}^{R_0}$ to $\mathbf{X}^R$. The above result is very important, which has been used as an elementarily mathematical tool to solve the inverse problem of parabolic type PDE, demonstrating that the new approach has high accuracy and efficiency [Liu (2008d, 2009)].

Though it is impossible to erect the exact solution of $\mathbf{G}$, we can evaluate an appropriate value of $\mathbf{G}$ by employing a generalized mid-point rule, which is obtained from an exponential mapping of $\mathbf{A}$ by taking the values of the argument variables of $\mathbf{A}$ at a generalized mid-point. The Lie-group element generated from such an $\mathbf{A} \in so(2n, 1)$ by an exponential has a closed-form representation as follows [Liu (2008d)]:

$$\mathbf{G} = \begin{pmatrix} \mathbf{I}_{2n} + \frac{(a-1)}{\|\hat{\mathbf{f}}\|^2} \hat{\mathbf{f}}\hat{\mathbf{f}}^T & \frac{b\hat{\mathbf{f}}}{\|\hat{\mathbf{f}}\|} \\ \frac{b\hat{\mathbf{f}}^T}{\|\hat{\mathbf{f}}\|} & a \end{pmatrix}, \tag{27}$$

where

$$\hat{\mathbf{f}} = \mathbf{f}(\hat{r}, \hat{\mathbf{y}}) = \begin{pmatrix} \hat{\mathbf{S}} \\ \hat{\mathbf{h}} \end{pmatrix}, \tag{28}$$

$$\hat{\mathbf{y}} = \lambda \mathbf{y}^{R_0} + (1 - \lambda)\mathbf{y}^R, \tag{29}$$

$$a = \cosh\left[\frac{(R - R_0)\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{y}}\|}\right], \quad b = \sinh\left[\frac{(R - R_0)\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{y}}\|}\right]. \tag{30}$$

Here, we use the inner-boundary condition $\mathbf{y}^{R_0}$ and the outer-boundary condition $\mathbf{y}^R$ through a suitable weighting factor λ to evaluate $\mathbf{G}$, where $\lambda \in [0, 1]$ is a parameter and $\hat{r} = \lambda R_0 + (1 - \lambda)R$. The above approach is by applying a generalized mid-point rule to the computation of $\mathbf{G}$, and the result is a single-parameter Lie-group element denoted by $\mathbf{G}(\lambda)$, where a suitable value of λ is to be determined in Section 2.6 by matching the target equation specified by the outer boundary condition.

2.4 A Lie-group mapping between two points on the cone

Let us define a new vector:

$$\mathbf{F} := \frac{\hat{\mathbf{f}}}{\|\hat{\mathbf{y}}\|}, \tag{31}$$

such that Eqs. (27) and (30) can be also expressed as

$$\mathbf{G} = \begin{pmatrix} \mathbf{I}_{2n} + \frac{(a-1)}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^T & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^T}{\|\mathbf{F}\|} & a \end{pmatrix}, \tag{32}$$

$$a = \cosh[(R-R_0)\|\mathbf{F}\|], \quad b = \sinh[(R-R_0)\|\mathbf{F}\|]. \tag{33}$$

Then from Eqs. (17), (26) and (32) it follows that

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{y}^R - \mathbf{y}^{R_0}), \tag{34}$$

where

$$\eta = \frac{(R-R_0)\|\mathbf{y}^R - \mathbf{y}^{R_0}\|}{\ln Z}, \tag{35}$$

$$Z = \frac{(t-1)\|\mathbf{y}^{R_0}\|}{\|\mathbf{y}^{R_0}\|t + \|\mathbf{y}^R - \mathbf{y}^{R_0}\| - \|\mathbf{y}^R\|}, \tag{36}$$

$$t = \frac{(\mathbf{y}^R - \mathbf{y}^{R_0}) \cdot \mathbf{y}^{R_0}}{\|\mathbf{y}^R - \mathbf{y}^{R_0}\|\|\mathbf{y}^{R_0}\|}. \tag{37}$$

It can be seen that $\mathbf{F}$ is only dependent on $\mathbf{y}^{R_0}$ and $\mathbf{y}^R$. The reader can refer Liu (2011b) for a detailed derivation of the above three equations.

Thus, we come to an important result that between any two points $(\mathbf{y}^{R_0}, \|\mathbf{y}^{R_0}\|)$ and $(\mathbf{y}^R, \|\mathbf{y}^R\|)$ on the cone, there exists a Lie-group element $\mathbf{G} \in SO_o(2n,1)$ mapping $(\mathbf{y}^{R_0}, \|\mathbf{y}^{R_0}\|)$ onto $(\mathbf{y}^R, \|\mathbf{y}^R\|)$, which is given by

$$\begin{pmatrix} \mathbf{y}^R \\ \|\mathbf{y}^R\| \end{pmatrix} = \mathbf{G} \begin{pmatrix} \mathbf{y}^{R_0} \\ \|\mathbf{y}^{R_0}\| \end{pmatrix}, \tag{38}$$

where $\mathbf{G}$ is uniquely determined by $\mathbf{y}^{R_0}$ and $\mathbf{y}^R$ through Eqs. (32) and (34)-(37). We write it to be $\mathbf{G}(R-R_0)$. It is interesting that by letting $\mathbf{G}(\lambda)=\mathbf{G}(R-R_0)$, we can derive the following Lie-group shooting equation.

2.5   *Lie-group shooting method for solving the inverse Cauchy problem*

From Eqs. (12) and (13) it follows that

$$\mathbf{U}' = \mathbf{S}, \quad \mathbf{S}' = \mathbf{h}(r, \mathbf{U}, \mathbf{S}), \tag{39}$$

$$\mathbf{U}(R_0) = \mathbf{U}^{R_0}, \quad \mathbf{U}(R) = \mathbf{U}^R, \tag{40}$$

$$\mathbf{S}(R_0) = \mathbf{S}^{R_0}, \quad \mathbf{S}(R) = \mathbf{S}^R, \tag{41}$$

where $\mathbf{U}^R$ and $\mathbf{S}^R$ are known from Eqs. (8) and (9), but $\mathbf{U}^{R_0}$ and $\mathbf{S}^{R_0}$ are two unknown vectors. Next, we derive algebraic equations to solve them.

By using Eq. (14) for $\mathbf{y}$ we have

$$\mathbf{y}^{R_0} = \begin{pmatrix} \mathbf{U}^{R_0} \\ \mathbf{S}^{R_0} \end{pmatrix}, \quad \mathbf{y}^R = \begin{pmatrix} \mathbf{U}^R \\ \mathbf{S}^R \end{pmatrix}, \tag{42}$$

which as being inserted into Eq. (34) yields

$$\mathbf{F} = \frac{1}{\eta} \begin{pmatrix} \mathbf{U}^R - \mathbf{U}^{R_0} \\ \mathbf{S}^R - \mathbf{S}^R \end{pmatrix}. \tag{43}$$

On the other hand, from Eqs. (31) and (28) it follows that

$$\mathbf{F} = \frac{1}{\|\hat{\mathbf{y}}\|} \begin{pmatrix} \hat{\mathbf{S}} \\ \hat{\mathbf{h}} \end{pmatrix}, \tag{44}$$

Comparing Eq. (43) with Eq. (44), we can derive the following nonlinear algebraic equations with regard to $\mathbf{U}^{R_0}$ and $\mathbf{S}^{R_0}$ for a given λ:

$$\mathbf{U}^{R_0} = \mathbf{U}^R - \frac{\eta}{\|\hat{\mathbf{y}}\|} \hat{\mathbf{S}}, \tag{45}$$

$$\mathbf{S}^{R_0} = \mathbf{S}^R - \frac{\eta}{\|\hat{\mathbf{y}}\|} \hat{\mathbf{h}}, \tag{46}$$

where $\|\hat{\mathbf{y}}\|$ and η are calculated, respectively, from Eqs. (29) and (35) by inserting Eq. (42), and $\hat{\mathbf{h}}$ is given by

$$\hat{\mathbf{h}} = \mathbf{h}(\hat{r}, \hat{\mathbf{U}}, \hat{\mathbf{S}}) = -\left[2\alpha + \frac{1}{\hat{r}}\right]\hat{\mathbf{S}} - \frac{1}{\hat{r}^2}\hat{\mathbf{H}} - \left[\alpha^2 + \frac{\alpha}{\hat{r}} + p\right]\hat{\mathbf{U}} + \hat{\mathbf{L}}, \tag{47}$$

in which

$$\hat{\mathbf{U}} = \lambda \mathbf{U}^{R_0} + (1-\lambda)\mathbf{U}^R, \quad \hat{\mathbf{S}} = \lambda \mathbf{S}^{R_0} + (1-\lambda)\mathbf{S}^R,$$

$$\hat{\mathbf{H}} = \begin{pmatrix} \frac{\hat{U}^2(r) - 2\hat{U}^1(r) + \hat{U}^n(r)}{(\Delta\theta)^2} \\ \frac{\hat{U}^3(r) - 2\hat{U}^2(r) + \hat{U}^1(r)}{(\Delta\theta)^2} \\ \vdots \\ \frac{\hat{U}^n(r) - 2\hat{U}^{n-1}(r) + \hat{U}^{n-2}(r)}{(\Delta\theta)^2} \\ \frac{\hat{U}^1(r) - 2\hat{U}^n(r) + \hat{U}^{n-1}(r)}{(\Delta\theta)^2} \end{pmatrix},$$

$$\hat{\mathbf{L}} = e^{-\alpha\hat{r}}(\hat{f}^1, \dots, \hat{f}^n)^T, \tag{48}$$

$\hat{U}^i(r) = \lambda U(R_0, \theta_i) + (1 - \lambda)U(R, \theta_i)$ and $\hat{f}^i = f(\hat{r}, \theta_i)$ with $\hat{r} = \lambda R_0 + (1 - \lambda)R$, $i = 1, \ldots, n$.

Upon considering Eq. (47) and the first equation in Eq. (48), Eq. (46) can be rearranged to

$$\mathbf{S}^{R_0} = \frac{1}{\hat{A}}[\hat{B}\mathbf{S}^R + \hat{\mathbf{C}}], \tag{49}$$

where

$$
\begin{aligned}
\hat{A} &:= 1 - \frac{\eta \lambda}{\|\hat{\mathbf{y}}\|}\left[2\alpha + \frac{1}{\hat{r}}\right], \\
\hat{B} &:= 1 + \frac{\eta(1 - \lambda)}{\|\hat{\mathbf{y}}\|}\left[2\alpha + \frac{1}{\hat{r}}\right], \\
\hat{\mathbf{C}} &:= \frac{\eta}{\|\hat{\mathbf{y}}\|}\left(\frac{1}{\hat{r}^2}\hat{\mathbf{H}} + \left[\alpha^2 + \frac{\alpha}{\hat{r}} + p\right]\hat{\mathbf{U}} - \hat{\mathbf{L}}\right).
\end{aligned}
\tag{50}
$$

2.6   Numerical procedures for solving the inverse Cauchy problem

In the following we use an iteration procedure to calculate $\mathbf{U}^{R_0}$, i.e., the boundary value $U(R_0, \theta_i)$.

Step 1: For each given $\lambda \in [0, 1]$, choose an initial value of $(\mathbf{U}_0^{R_0}, \mathbf{S}_0^{R_0})$, and set $k = 0$.

Step 2: From Eqs. (45) and (49), we can obtain a new value of $(\mathbf{U}_k^{R_0}, \mathbf{S}_k^{R_0})$, $k = 1, 2, \ldots$.

Step 3: Increase k by one and go to Eqs. (45) and (49) until the convergence is achieved under a given stopping criterion ε:

$$\sqrt{\|\mathbf{U}_{k+1}^{R_0} - \mathbf{U}_k^{R_0}\|^2 + \|\mathbf{S}_{k+1}^{R_0} - \mathbf{S}_k^{R_0}\|^2} \le \varepsilon. \tag{51}$$

Under the above new boundary values $\mathbf{U}^{R_0}$ and $\mathbf{S}^{R_0}$, we can return to Eqs. (12) and (13) and integrate them to obtain $\mathbf{U}^R$ by using the GPS. The above process can be done for all $\lambda \in [0, 1]$, among these solutions we can pick up the best λ, which lends to the smallest error of

$$\min_{\lambda \in [0,1]}\|\mathbf{U}^R - e^{-\alpha R}\mathbf{M}\|, \tag{52}$$

where

$$\mathbf{M} = (\varphi(\theta_1), \ldots, \varphi(\theta_n))^T, \tag{53}$$

such that the Cauchy data in Eq. (8) can be matched as best as possible.

When the process terminates, by inserting the best λ into Eq. (45) we can recover the boundary value $U(R_0, \theta)$, and then $u(R_0, \theta) = e^{\alpha R_0} U(R_0, \theta)$, at the discretized point θ_i.

3 Numerical examples

Before embarking the numerical tests we consider the influence of noise on the present algorithm, which is added on the data by

$$\hat{\varphi}_i = \varphi(\theta_i) + sR(i), \quad \hat{\phi}_i = \phi(\theta_i) + sR(i), \tag{54}$$

where $R(i)$ are random numbers in $[-1, 1]$.

3.1 Example 1

In this example we apply the LGSM to the following Laplace equation:

$$u_{xx} + u_{yy} = 0, \tag{55}$$

$$u(x, y) = \frac{x}{x^2 + y^2}, \quad u(r, \theta) = \frac{\cos \theta}{r}. \tag{56}$$

The required data can be obtained from the exact solution. We fix $R_0 = 1$ and $R = 1.5$, and intend to recover the data of u at $r = R_0$. We use the following parameters: $\Delta \theta = 2\pi/60$ and $\Delta r = 0.1$, and $\varepsilon = 10^{-4}$, and let $\lambda \in [0, 0.001]$. We consider two noised cases with $s = 0.01$ and $s = 0.05$. In the case of $s = 0.05$ we use $\alpha = -0.01$ while for $s = 0.01$ we use $\alpha = 0.1$. From Fig. 1(a) it can be seen that the LGSM converges very fast within two iterations for most λ. Due to its ill-posed nature of the inverse Cauchy problem, the error of mis-matching to target as shown in Fig. 1(a) by the solid line is quite large. Although in this situation, the present LGSM can still pick up the best λ. For the noisy cases with $s = 0.05$ and $s = 0.01$ we compare the numerical results obtained by the present LGSM with the exact one in Fig. 1(b), while the absolute errors are shown in Fig. 1(c). The maximum error for $s = 0.05$ is 0.0445, while the maximum error for $s = 0.01$ is 0.01. Very accurate solutions are obtained.

3.2 Example 2

We consider the following inverse Cauchy problem of the Poisson equation:
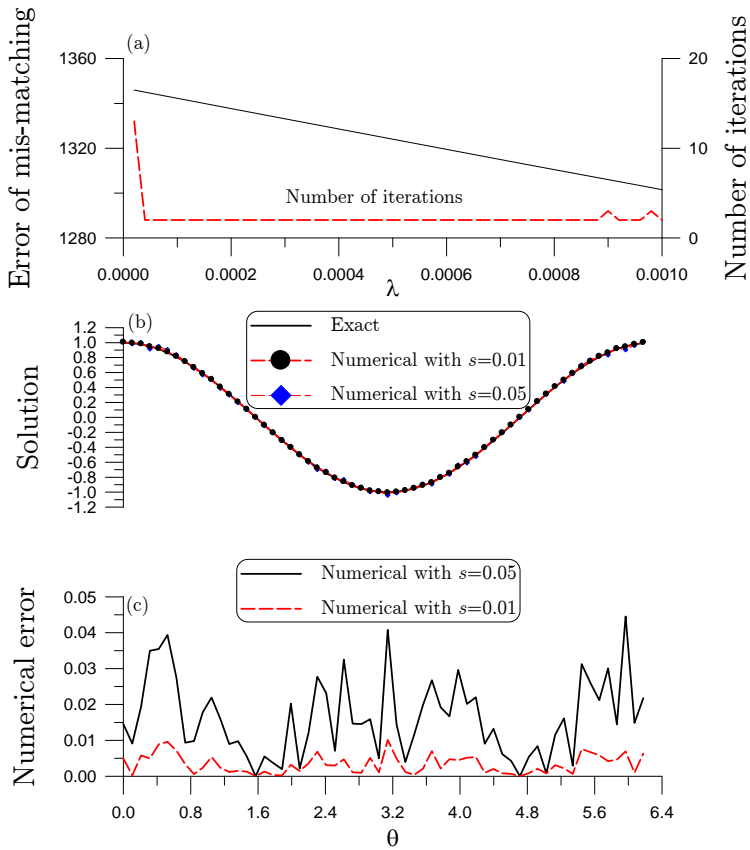
$$u_{xx} + u_{yy} = f(x, y), \tag{57}$$

Figure 1: For example 1: (a) showing the error of mis-matching and number of iterations, (b) comparing the numerical solutions with the exact one, and (c) displaying the numerical errors.

with the exact solution given by

$$u(x,y) = x^2 - y^2 + e^{x+y} + \frac{x+y}{x^2+y^2}, \quad u(r,\theta) = r^2 \cos 2\theta + \exp[r(\sin\theta + \cos\theta)]$$

$$+ \frac{\cos\theta + \sin\theta}{r}. \quad (58)$$

From it the data on the outer circle with a radius R are given by

$$\varphi(\theta) = R^2 \cos 2\theta + \exp[R(\sin\theta + \cos\theta)] + \frac{\cos\theta + \sin\theta}{R}, \quad (59)$$

$$\phi(\theta) = 2R\cos 2\theta + (\sin\theta + \cos\theta)\exp[R(\sin\theta + \cos\theta)] - \frac{\cos\theta + \sin\theta}{R^2}, \quad (60)$$
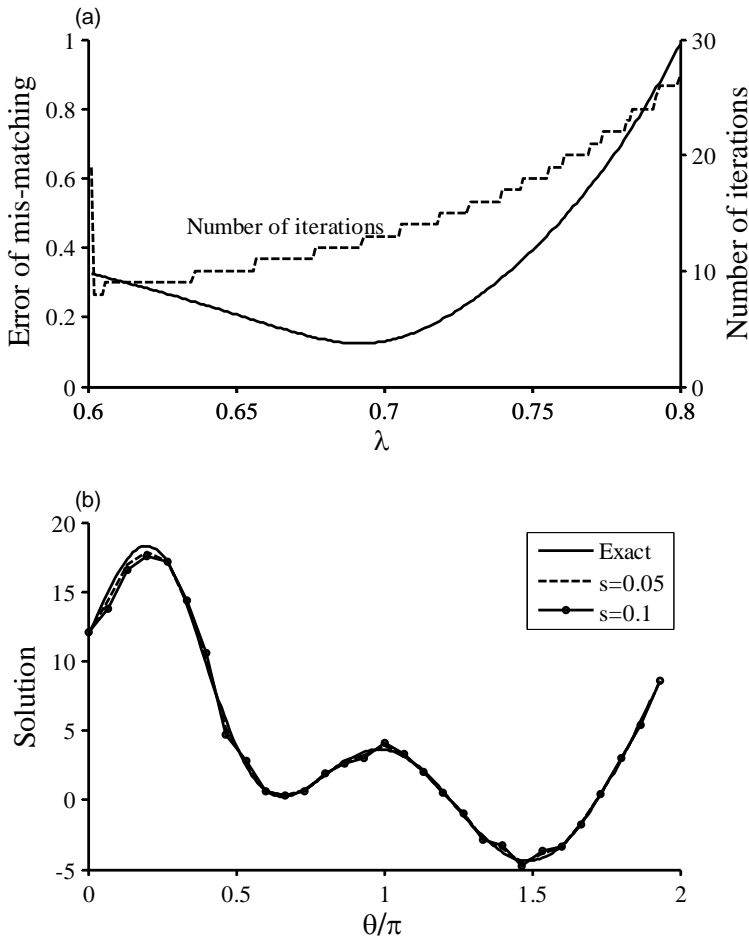
Figure 2: For example 2: (a) comparing the numerical solutions with the exact one, and (b) displaying the absolute errors

and the non-homogeneous term is given by

$$f(r,\theta) = 2\exp[r(\cos\theta + \sin\theta)]. \tag{61}$$

Here we fix $R_0 = 2$ and $R = 2.2$, and intend to recover the data of u at $r = R_0$. By employing the following parameters: $\Delta\theta = 2\pi/30$, $\Delta r = 0.1$, and $\alpha = 0.005$ for the noisy cases with noises $s = 0.1$ and $s = 0.05$, the mis-matching error and the number of iterations are plotted in Fig. 2(a) in the range of $\lambda \in [0.6, 0.8]$. We compare the numerical results with the exact one in Fig. 2(b). The accuracy as can
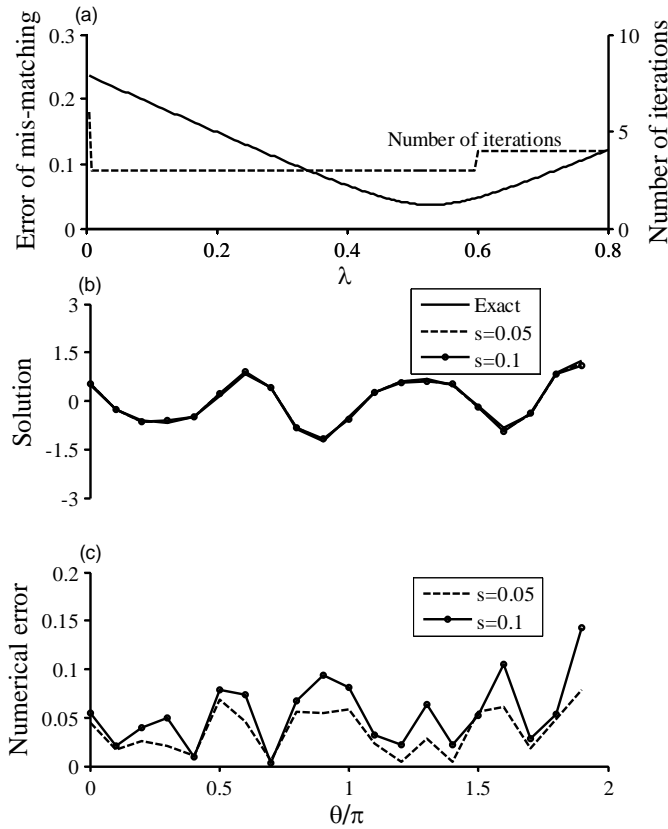
Figure 3: For example 3: (a) comparing the numerical solutions with the exact one, (b) showing the numerical errors under noises, and (c) displaying the relative errors under noises.

be seen from Fig. 2(b) is rather good.

3.3 Example 3

We deliberate the Cauchy problem for the Helmholtz equation and apply the LGSM to

$$u_{xx} + u_{yy} + 2u = 0, \tag{62}$$

$$u(x,y) = \sin(x+y) + \frac{x}{x^2+y^2}, \quad u(r,\theta) = \sin[r(\cos\theta + \sin\theta)] + \frac{\cos\theta}{r}. \tag{63}$$
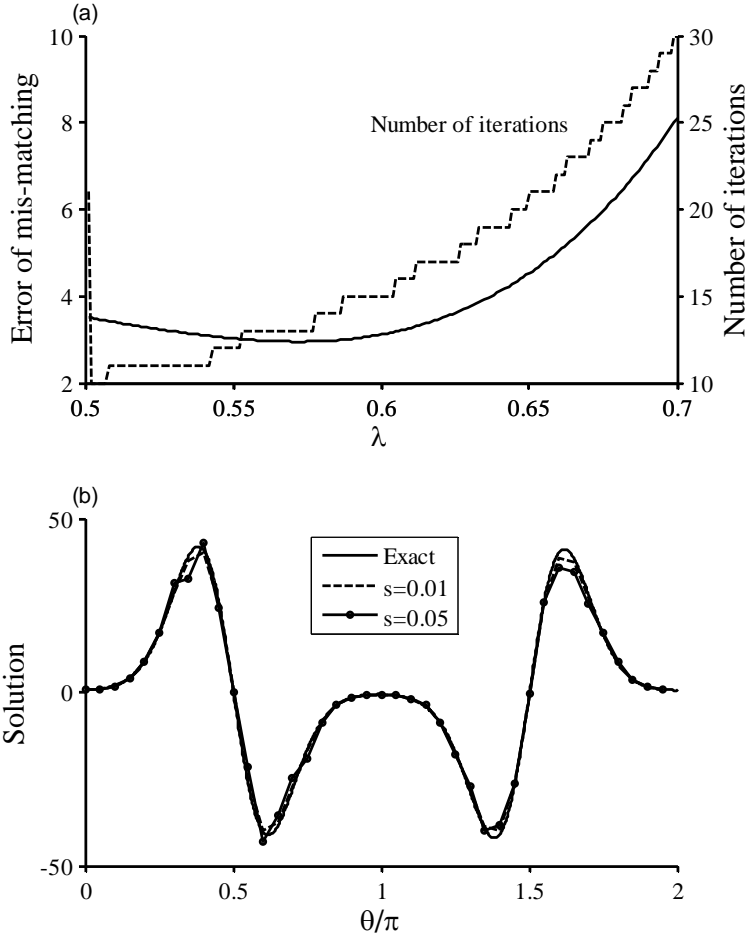
Figure 4: For example 4: (a) comparing the numerical solutions with the exact one, and (b) exhibiting the numerical errors.

For this computational experiment, we fix $R_0 = 3$ and $R = 3.3$. By employing the following parameters: $\Delta\theta = 2\pi/20$, $\Delta r = 0.01$, and $\alpha = 0.05$ for the noisy cases with noises $s = 0.1$ and $s = 0.05$, the mis-matching error and the number of iterations are plotted in Fig. 3(a) in the range of $\lambda \in [0, 0.8]$. We compare the numerical results with the exact one in Fig. 3(b), while the numerical errors are presented in Fig. 3(c). The maximum error for $s = 0.1$ is 0.143 and for $s = 0.05$ is 0.079. Even under large noises, the LGSM can still obtain very accurate result.

3.4 Example 4

We deliberate the Cauchy problem for the modified Helmholtz equation and apply the LGSM to

$$u_{xx} + u_{yy} - 3u + \frac{3y}{x^2 + y^2} = 0, \tag{64}$$

$$u(x,y) = \sin(x)\cosh(2y) + \frac{y}{x^2 + y^2},$$

$$u(r,\theta) = \sin(r\cos\theta)\cosh(2r\sin\theta) + \frac{\sin\theta}{r}.$$

Here we fix $R_0 = 2.5$ and $R = 2.7$ and employ the following parameters: $\Delta\theta = 2\pi/40$, $\Delta r = 0.05$, and $\alpha = 0.01$ for the noisy cases with noises $s = 0.1$ and $s = 0.05$. The mis-matching error and the number of iterations are plotted in Fig. 4(a) in the range of $\lambda \in [0.5, 0.7]$. We compare the numerical results with the exact one in Fig. 4(b).

4 Conclusions

By employing the spring-damping regularization method (SDRM) and the Lie-group shooting method (LGSM) we can recover the data on the inner circle very well. The LGSM could first attribute a stable numerical solution of the inverse Cauchy problem, and then the parameter α, which played both a spring and a damping constant, could further stabilize and adapt the numerical solution near to the exact one. Several numerical examples of the inverse Cauchy problems for linear elliptic equations were worked out, which showed that the present LGSM plus the SDRM were applicable to the inverse Cauchy problems, even for the severely ill-posed ones. The efficiency of the LGSM plus the SDRM was due to its easy numerical implementation and easy to treat different inverse Cauchy problems; moreover, the LGSM is remarkably robust against the noisy disturbance.

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