Modeling of Effective Properties of Multiphase Magnetoelectroelastic Heterogeneous Materials

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Abstract: In this paper an N-phase Incremental Self Consistent model is developed for magnetoelectroelastic composites as well as the N-phase Mori-Tanaka and classical Self Consistent. Our aim here is to circumvent the limitation of the Self Consistent predictions for some coupling effective properties at certain inclusion volume fractions. The anomalies of the SC estimates are more drastic when the void inclusions are considered. The mathematical modeling is based on the heterogeneous inclusion problem of Eshelby which leads to an expression for the strain-electric-magnetic field related by integral equations. The effective N-phase magnetoelectroelastic moduli are expressed as a function of magnetoelectroelastic concentration tensors based on the considered micromechanical models. The effective properties are obtained for various types, shapes and volume fractions of inclusions and compared with the existing results.

Keywords: magnetoeletric, piezomagnetic, piezoelectric, composite, void, modeling, inclusion, effective moduli, micromechanical, Incremental Self Consistent, Mori-Tanaka, concentration tensor.

1   Introduction

The concept of new multifunctional materials constitutes now a scientific challenge especially for smart composites which include magnetoelectroelastic composites. Efforts are currently under way to develop materials that have superior properties to those currently existing. This has resulted in the development of composite materials that exhibit remarkable properties, which are created by the interaction between the constituent phases. There are many advantages to using composite materials more than traditional materials, such as the possibility of weight or volume reduction in a structure while maintaining a comparable or improved performance level.

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Composite materials consisting of a piezoelectric phase and a piezomagnetic phase show a remarkably large magnetoeletric coefficients and large coupling coefficients between elastic, electric and magnetic fields, which do not exist in either constituent. The magnetoeletric coupling in the composite is created through the interaction between the piezoelectric and the piezomagnetic phases. The product property of composites offers great opportunities to design new materials that are capable of responding in a desired way to the internal or environment changes, which may not be achieved by traditional materials. The coupling effects in magnetoelectroelastic composites materials have many uses in many engineering fields such aeronautics, automotives and medical imagery. The double coupling effects in piezoelectric materials and the triple ones in magnetoelectroelastic are very useful for sensors and actuators.

The effective properties of piezoelectric composites materials have been investigated by many researchers. Dunn and Wienecke (1996, 1997) have given the closed-form expressions for the infinite-body Green's functions for a transversally isotropic piezoelectric medium and the four Eshelby tensors for spheroid inclusions in transversally isotropic solids. Dunn and Taya (1993) predicted the effective properties using the Dilute, Self Consistent, Mori-Tanaka, and Differential micromechanical models. Fakri, Azrar and El Bakkali (2003) predicted the behavior of piezoelectric composite materials and presented the numerical results for the effective electroelastic properties in term of phase properties, orientation angles, volume fraction and shapes of inclusions. Odegard (2004) proposed a new modeling approach to predict the bulk electromechanical properties of piezoelectric composites and compared the obtained results with those obtained by the Mori-Tanaka approach and the finite element method. Li (2004) applied the Self Consistent approach to predict the effective pyroelectric and thermal expansion coefficients of ferroelectric ceramics taking into account the texture change due to domain switching during poling.

For magnetoelectroelastic composites, Li and Dunn (1998) investigated the magnetoelectroelastic coupling effects using the mean field Mori-Tanaka method and presented numerical results for fibrous and laminated composites. Wu and Huang (2000) investigated the magnetoeletric coupling effect in a fibrous composites with piezoelectric and piezomagnetic phases. Based on the eigenstrain formulation and Mori-Tanaka approach, the magnetoelectroelastic Eshelby tensors and the effective material properties of the composite are obtained explicitly. Li (2000) studied the average magnetoelectroelastic field in a multi-inclusion or inhomogeneities embedded in an infinite matrix. Feng, Fang and Hwang (2004) investigated the effective properties of composite consisting of piezomagnetic inhomogeneities embedded in a non-piezomagnetic matrix by using a unified energy method and the Mori-

Tanaka and Dilute approaches. Zhang and Soh (2005) extended the micromechanical Self Consistent, Mori-Tanaka and Dilute to study the coupled magnetoelectroelastic composite materials. Srinivas and Li (2005) developed a Self Consistent approach to calculate the macroscopic magnetoelectric coefficients by emphasizing the effects of shape, volume fraction and orientation distribution of particles of both phases. Lee, Boyd and Lagoudas (2005) developed a finite element analysis and micromechanics based averaging of a representative volume element to determine the effective dielectric, magnetic, mechanical, and coupled-field properties of an elastic matrix reinforced with piezoelectric and piezomagnetic fibers. A special emphasis on the poling directions of the piezoelectric and piezomagnetic fibers is done. Srinivas, Li, Zhou and Soh, (2006) developed a mean field Mori-Tanaka model to calculate the effective magnetoelectroelastic moduli of matrix-based multiferroic composites by emphasizing the effects of shape and orientation distribution of second phase particles composites. More recently, Fakri and Azrar (2010) developed the Incremental Self Consistent method to thermoelectroelastic materials to predict the electro elastic and thermal response of piezocomposites with and without voids.

The classical Self Consistent model, which is widely used, overestimates the predictions of some magnetoelectroelastic composites effective properties for moderate and high concentrations of reinforcements and diverges for some coefficients. For magnetoelectroelastic composites with void inclusions the predictions are limited for very low void concentrations and are erroneous for volume fraction greater than 10%. The aim of this paper is on one hand to develop an N-phase Incremental Self Consistent model for magnetoelectroelastic materials. On the other hand to present an accurate model based on the Self Consistent procedure for N-phases coupled materials

In this work, a micromechanical modeling is used to predict the behavior of multi-phase magnetoelectroelastic composites. The nine interaction tensors which are used to predict the effective moduli of multi-phase magnetoelectroelastic composites based on various micro mechanical approaches such Self Consistent, Mori-Tanaka, Dilute and Incremental Self Consistent schemes are derived. Numerical results are obtained for various shapes of inclusions and compared with the existing ones. A mathematical modeling based on the Incremental Self Consistent model is developed for multi-phase magnetoelectroelastic composites. It is clearly demonstrated in this work that the Incremental Self Consistent model gives more accurate results than the classical Self Consistent model.

2   Basic equations

Let us consider the linear magnetoelectroelastic effect, where the magnetic, electric and elastic fields are coupled through the following constitutive equations:

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - e_{lij}E_l - h_{lij}H_l$$
$$D_i = e_{ikl}\varepsilon_{kl} + \kappa_{il}E_l + \alpha_{il}H_l \tag{1}$$
$$B_i = h_{ikl}\varepsilon_{kl} + \alpha_{il}E_l + \mu_{il}H_l$$

where the elastic strain ε_{kl}, electric fields E_l, and magnetic fields H_l are independent variables related to stresses σ_{ij}, electric displacements D_i and magnetic inductions B_i. The tensors c_{ijkl}, e_{lij}, h_{lij}, α_{il}, κ_{il} and μ_{il} are the elastic, piezoelectric, piezomagnetic, magnetoeletric, dielectric and magnetic permeability constants respectively. Let us note that $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{jilk}$, $e_{lij} = e_{lji}$ and $h_{lij} = h_{lji}$. In the constitutive equations we use -E_l and -H_l rather than E_l and H_l as they will permit the construction of a symmetric matrix of constitutive moduli. The following gradient expressions are used:

$$\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}) \quad E_l = -\varphi_{,l}^e \quad H_l = -\varphi_{,l}^m \tag{2}$$

where u_k, φ^e, φ^m are the elastic displacements, electric and magnetic potentials, respectively.

The equilibrium equations, in the absence of body forces, electric charge and electric current densities, are as follows:

$$\sigma_{ij,i} = 0 \quad D_{i,i} = 0 \quad B_{i,i} = 0 \tag{3}$$

In order to make easy the manipulation of these equations, the condensed notations are used. These notations are identical to those using the conventional subscripts except that the lower case subscripts assume the range of 1-3, while the capital subscripts take the range of 1-5, and the repeated capital subscripts are summed over 1-5. With these notations, the generalized strain field denoted by Z_{Kl} can be expressed as

$$Z_{Kl} = \begin{cases} \varepsilon_{kl} & (K = k = 1,2,3) \\ -E_l & (K = 4) \\ -H_l & (K = 5) \end{cases} \tag{4}$$

Note that Z_{Kl} can be derived from the generalized potential field U_K given by

$$U_K = \begin{cases} u_k & (K = k = 1,2,3) \\ \varphi^e & (K = 4) \\ \varphi^m & (K = 5) \end{cases} \tag{5}$$

Similarly, the generalized stress field Σ_{iJ} is given by

$$\Sigma_{iJ} = \begin{cases} \sigma_{ij} & (J = j = 1,2,3) \\ D_i & (J = 4) \\ B_i & (J = 5) \end{cases} \tag{6}$$

The magnetoelectroelastic constants can then be represented as follows:

$$E_{iJKl} = \begin{cases} c_{ijkl} & (J,K = 1,2,3) \\ e_{lij} & (J = 1,2,3; \ K = 4) \\ h_{lij} & (J = 1,2,3; \ K = 5) \\ e_{ikl} & (J = 4; \ K = 1,2,3) \\ h_{ikl} & (J = 5; \ K = 1,2,3) \\ -\kappa_{il} & (J = 4; \ K = 4) \\ -\alpha_{il} & (J = 4; \ K = 5 \, or \, J = 5; \ K = 4) \\ -\mu_{il} & (J = 5; \ K = 5) \end{cases} \tag{7}$$

The symmetry of E_{iJKl} can be obtained from those of c_{ijkl}, e_{lij}, h_{lij}, κ_{il}, α_{il} and μ_{il}. By using these shorthand notations, eqs. (1) can be rewritten as a single equation as follows:

$$\Sigma_{iJ} = E_{iJKl} Z_{Kl} \tag{8a}$$

With

$$Z_{Kl} = U_{K,l} \tag{8b}$$

When standard notation matrix for tensors is adopted the constitutive equation can be written as follow:

$$\Sigma = EZ \tag{8c}$$

where

$$E = \begin{bmatrix} c & e^t & h^t \\ e & -\kappa & -\alpha \\ h & -\alpha & -\mu \end{bmatrix}, \quad Z = \begin{bmatrix} \varepsilon \\ -E \\ -H \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma \\ D \\ B \end{bmatrix}$$

The subscript t is used to denote matrix transpose. E is a (12x12) matrix. Similarly, Z, Σ are (12x1) matrices.

Thus, in order to express the equilibrium equations, each individual tensor must be transformed by the well known law of tensor transformations. The resulting tensors can then be reunified into the form of Eqs.(4) to (7). Substituting Eqs. (2) into Eqs.(1) and considering matrix symmetry, one obtains:

$$\sum_{iJ} = E_{iJKl}U_{K,l} \tag{9}$$

Introducing Equation (9) in the equilibrium equation (3), the following partial differential equation is obtained:

$$(E_{iJKl}U_{K,l})_{,i} = 0 \tag{10}$$

3   Integral equation formulation

Let us consider a homogeneous fictitious media called "reference media" which has the magnetoelectroelastic moduli E^0_{iJMn}. The expression of the local magneto-electroelastic moduli is given as follow:

$$E_{iJMn}(r) = E^0_{iJMn} + \delta E_{iJMn}(r) \tag{11}$$

where "r" is the position vector in the media considered and δE is the deviation part. The introduction of this expression into (10) leads to

$$E^0_{iJMn}U_{M,ni}(r) + (\delta E_{iJMn}(r)U_{M,n}(r))_{,i} = 0 \tag{12}$$

Now, let us introduce the magnetoelectroelastic Green's tensors, denoted by $G_{MJ}(r - r')$, of the reference media corresponding to the response at the position r due to a unit point force or charge at r'. These tensors satisfy the following partial differential equation:

$$E^0_{iJMn}G_{MK,in}(r - r') + \delta_{JK}\delta(r - r') = 0 \tag{13}$$

This partial differential equation, satisfied by the magnetoelectroelastic Green's tensors, condensed nine partial differential equations. Based on (13), and after some mathematical manipulations and the consideration of the boundary conditions, the expression of the local generalized field $U_M(r)$ is derived:

$$U_K(r) = U^0_K(r) + \int_V G_{JK}(r - r')(\delta E_{iJMn}(r')U_{M,n}(r'))_{,i'}dV' \tag{14}$$

Using the fact that $Z_{Kl} = U_{K,l}$ and considering the condition that the local generalized strain field vanishes at the boundaries the expression of the local generalized strain field can be written as:

$$Z_{Kl}(r) = Z^0_{Kl}(r) - \int_V \Gamma_{iJKl}(r - r')(\delta E_{iJMn}(r')Z_{Mn}(r'))dV' \tag{15}$$

where $\Gamma_{iJKl}(r - r') = -G_{JK,li}(r - r')$ is a condensed notation of nine tensors.

This equation is an integral formulation of the generalized strain field $Z_{Kl}(r)$. To solve this equation the equivalent inclusion approach will be used.

4 Averaged field

Consider an infinite media with magnetoelectroelastic moduli E^0_{iJMn} which contains a single inclusion "I" of volume V^I and magnetoelectroelastic moduli E^I_{iJMn} assumed to be constant inside the volume V^I. The inhomogeneity can be simulated by an "equivalent inclusion". Based on these assumptions, as done by Eshelby (1957) in the elastic case and by Deeg (1980) in the electroelastic case, one obtains

$$\delta E_{iJMn} = (E^I_{iJMn} - E^0_{iJMn})\theta^I(r)$$

Or

$$\delta E_{iJMn} = \Delta E^I_{iJMn}\theta^I(r) \tag{16}$$

where $\theta^I(r)$ is the characteristic function of V^I ($\theta^I(r)$ equals 1 inside the volume V^I and 0 outside of V^I). Based on Eq. (15), the average generalized strain field Z^I_{Kl} in the considered inclusion is given by the following expression:

$$Z^I_{Kl} = Z^0_{Kl} - \frac{1}{V^I}\int_{V^I}\int_V \Gamma_{iJKl}(r - r')\Delta E^I_{iJMn}\theta^I(r')Z_{Mn}(r')dV'dV \tag{17}$$

The exact solution of the above integral equation is difficult to be obtained. An approximation is then made by replacing $Z_{Mn}(r')$ by its average value Z^I_{Kl} in the considered inclusion as follows:

$$Z^I_{Kl} = Z^0_{Kl} - \frac{1}{V^I}\int_{V^I}\int_{V^I} \Gamma_{iJKl}(r - r')\Delta E^I_{iJMn}Z^I_{Mn}dV'dV \tag{18}$$

This equation can be reformulated in the following form:

$$Z^I_{Kl} = Z^0_{Kl} - \frac{1}{V^I}T^{II}_{iJKl}\Delta E^I_{iJMn}Z^I_{Mn} \tag{19}$$

where $T^{II}_{iJKl} = \int_{V^I}\int_{V^I}\Gamma_{iJKl}(r - r')dV'dV$ represents the condensed notation of the nine interaction tensors. These tensors are computed numerically for various shapes of inclusions using the Gaussian quadrature integration for the considered inclusion shape.

4.1 Spherical inclusion

A spherical inclusion with radius "**q**" is considered. In spherical system attached at the inclusion, the vector $\vec{q}$ becomes

$$q_p = q\chi_p \quad p = 1,2,3 \tag{20}$$

where $\vec{\chi} = \begin{cases} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{cases}$

q, θ and φ are the spherical coordinates of the vector $\vec{q}$ defined in the following domains: **q**$\in$**[0, +∞[**, $\theta \in$**[0, π]** and $\varphi \in$**[0, 2π]**.

Application of the Fourier transform to Eq. (13) leads to the following expression

$$E^0_{iJMn}\tilde{G}_{JK}(q)q_nq_i = \delta_{MK} \tag{21}$$

The introduction of the equation (20) into (21) leads to the algebraic problem

$$E^0_{iJMn}\chi_n\chi_i(q^2\tilde{G}_{JK}(q)) = \delta_{MK} \tag{22}$$

Let us introduce a matrix M defined by

$$M_{JM} = E^0_{iJMn}\chi_n\chi_i \tag{23a}$$

The inverse of M is given by

$$M^{-1}_{JK} = q^2\tilde{G}_{JK}(q) \tag{23b}$$

The explicit expression of the matrix M is given by

$$M = \begin{bmatrix} A & B \\ B^t & C \end{bmatrix} \tag{23c}$$

Expressions of the matrices A, B and C are derived for transversely isotropic magnetoelectroelastic composites with x_3 the axis of symmetry. The used matrices to derive A, B and C are given in the appendix. With spherical coordinate, these matrices are expressed as:

$$A = \begin{bmatrix} c_{11}x_1^2 + c_{66}x_2^2 + c_{44}x_3^2 & (c_{12}+c_{66})x_1x_2 & (c_{13}+c_{44})x_1x_3 \\ (c_{12}+c_{66})x_1x_2 & c_{66}x_1^2 + c_{11}x_2^2 + c_{44}x_3^2 & (c_{13}+c_{44})x_2x_3 \\ (c_{13}+c_{44})x_1x_3 & (c_{13}+c_{44})x_2x_3 & c_{44}x_1^2 + c_{44}x_2^2 + c_{33}x_3^2 \end{bmatrix}$$

$$B = \begin{bmatrix} (e_{31}+e_{15})x_3x_1 & (h_{31}+h_{15})x_3x_1 \\ (e_{31}+e_{15})x_2x_3 & (h_{31}+h_{15})x_2x_3 \\ e_{15}x_1^2+e_{15}x_2^2+e_{33}x_3^2 & h_{15}x_1^2+h_{15}x_2^2+h_{33}x_3^2 \end{bmatrix}$$

$$C = - \begin{bmatrix} \kappa_{11}x_1^2 + \kappa_{11}x_2^2 + \kappa_{33}x_3^2 & \alpha_{11}x_1^2 + \alpha_{11}x_2^2 + \alpha_{33}x_3^2 \\ \alpha_{11}x_1^2 + \alpha_{11}x_2^2 + \alpha_{33}x_3^2 & \mu_{11}x_1^2 + \mu_{11}x_2^2 + \mu_{33}x_3^2 \end{bmatrix}$$

where $x_1 = \sin\theta\cos\varphi$, $x_2 = \sin\theta\sin\varphi$, and $x_3 = \cos\theta$.

The expression of T_{iJKl}^{II} in spherical coordinates system is then given by:

$$T_{iJKl}^{II} = \frac{a^3}{6} \int_0^\pi \sin\theta \left[\int_0^{2\pi} (\chi_i\chi_l q^2 \tilde{G}_{JK} d\varphi) + \int_0^{2\pi} (\chi_i\chi_K q^2 \tilde{G}_{Jl} d\varphi) \right] d\theta$$

$$K = 1,2,3$$

$$T_{iJ4l}^{II} = \frac{a^3}{3} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} (\chi_i\chi_l q^2 \tilde{G}_{J4} d\varphi)$$

$$T_{iJ5l}^{II} = \frac{a^3}{3} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} (\chi_i\chi_l q^2 \tilde{G}_{J5} d\varphi)$$

(24)

4.2 Ellipsoidal inclusion

An ellipsoidal inclusion with a, b, and c as half axes is considered. The used ellipsoidal coordinates are expressed in the principal system of the inclusion:

$$\vec{R} = \begin{cases} R_1 = r_1 \\ R_2 = \frac{a}{b}r_2 \\ R_3 = \frac{a}{c}r_3 \end{cases}$$

and

$$\vec{Q} = \begin{cases} Q_1 = r_1 q_1 \\ Q_2 = \frac{b}{a}q_2 \\ R_3 = \frac{c}{a}q_3 \end{cases}$$

The matrix relationship between $\vec{Q}$ and $\vec{q}$ is as follows:

$$q_i = \phi_{it} Q_t$$

(25)

With

$$\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{a}{b} & 0 \\ 0 & 0 & \frac{a}{c} \end{bmatrix}$$

The expression of $\vec{Q}$ in this coordinate system is then

$$Q_t = Q\chi_t \quad t = 1,2,3$$

The final expressions of T_{iJKl}^{II} are similarly given by:

$$T_{iJKl}^{II} = \frac{abc}{6} \int_0^\pi \sin\theta \left[\int_0^{2\pi} (\phi_{lt}\chi_t\phi_{ill}\chi_{ll}Q^2\tilde{G}_{JK}d\varphi + \int_0^{2\pi} (\chi_{Kt}\chi_t\phi_{ill}\chi_{ll}Q^2\tilde{G}_{JI}d\varphi) \right] d\theta$$

$$K = 1,2,3$$

$$T_{iJ4I}^{II} = \frac{abc}{3} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} (\phi_{lt}\chi_t\phi_{ill}\chi_{ll}Q^2\tilde{G}_{J4})d\varphi$$

$$T_{iJ5I}^{II} = \frac{abc}{3} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} (\phi_{lt}\chi_t\phi_{ill}\chi_{ll}Q^2\tilde{G}_{J5})d\varphi$$

$$(26)$$

In this case the matrix **M** is given by
$M_{JK} = E_{iJKl}^0 \phi_{it}\chi_t\phi_{lu}\chi_u$ and $M_{JK}^{-1} = Q^2\tilde{G}_{JK}(q)$
The explicit expression of the matrix **M** in ellipsoidal coordinates can be directly obtained by replacing $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ in (23-c) by:

$$x_1 = \sin\theta\cos\varphi, \quad x_2 = \frac{a}{b}\sin\theta\sin\varphi, \quad x_3 = \frac{a}{c}\cos\theta$$

5   Micromechanical models

5.1   *N-phase Self Consistent approach*

The Self Consistent model has been originally developed for estimating macroscopic moduli of polycrystalline metals (Hershey 1954, Kroner 1958). The Self Consistent model continues to be used by a great number of researchers for estimating homogenized moduli of heterogeneous materials including elastic, elastoplastic, viscoplasitic, piezoelectric materials, etc. In the one site Self Consistent approach the composite is considered as an inclusion embedded in a matrix which takes the properties of the whole composites E^{eff}. Based on the equivalent inclusion problem of Eshelby, the expression of the concentration tensor A^{SC} is given by Dunn et al and Fakri et al for piezoelectric composites.

$$A_{MnKl}^{SC} = (I_{KlMn} + \frac{1}{V^I}T_{iJKl}^{II}\Delta E_{iJMn}^I)^{-1}$$

$$(27)$$

where $\Delta E_{iJMn}^I = E_{iJMn}^I - E_{iJMn}^{eff}$.

For magnetoelectroelastic composite materials A_{MnKl}^{SC} is the shorthand notation of nine concentration tensors defined as functions of the tensor T^{II}. This tensor has to be computed first and the numerical procedure can be found in Hershey 1954.

For an N-phase medium, the effective magnetoelectroelastic moduli E^{eff}, predicted by the Self Consistent model, is expressed as:

$$E_{iJKl}^{eff} = \sum_{I=1}^{N} f^I E_{iJMn}^I A_{MnKl}^{SC} \tag{28}$$

where $f^I = \frac{V^I}{V}$.

is the concentration of the inclusions I. If the first phase (N=1) is taken as the matrix (symbol 'm'), the last expression becomes

$$E_{iJKl}^{eff} = E_{iJKl}^m + \sum_{I=2}^{N} f^I (E_{iJMn}^I - E_{iJMn}^m) A_{MnKl}^{SC} \tag{29}$$

Let us recall that $\sum_{I=1}^{n} f^I A_{MnKl}^{SC} = I_{MnKl}$, where I_{MnKl} is the shorthand notation of the four identity tensors, E_{iJMn}^m corresponds to the magnetoelectroelastic matrix moduli and E_{iJMn}^I corresponds to the magnetoelectroelastic inclusions moduli. These formulations permit one to predict the effective magnetoelectroelastic moduli for the N-phase composites. For a two phase composites, the expression of E_{iJKl}^{eff} becomes

$$E_{iJKl}^{eff} = E_{iJKl}^m + f^I (E_{iJMn}^I - E_{iJMn}^m) A_{MnKl}^{SC} \tag{30}$$

Note that equations (28) and (30) give coupled and implicit expression of the effective magnetoelectroelastic moduli of the magnetoelectroelastic material. The concentration tensors A^{SC} are functions of E^{eff}. This kind of equations is generally solved by iterative methods. A detailed algorithm for numerical computation is given in [6].

5.2 N-phase Incremental Self Consistent scheme

The development of the N-phase Incremental Self Consistent approach for the magnetoelectroelastic heterogeneous materials is one the main theoretical and numerical results of this paper. This is due to the fact that the Self Consistent method gives erroneous predictions of effective coefficients of composites materials at high concentration of reinforcements. An improvement of SCM, by an incremental way has been developed for piezo composite materials by Fakri and Azrar for two phases. In this paper, an extension of the ISC scheme to magneto-electro-elastic effective

properties and its development for N-phase magnetoelectroelastic composites are done.

For N-phase materials, the resulting composite must be characterized by concentrations "f^J" of phases (reinforcements) *$0 < f^J < 1; J = 1, N$*.

$\Delta f_J = \frac{f^J}{S}$ is considered as partial concentration of the phase J and S is the number of steps. At the ith step, the volume fraction of the phase J is $f_i^J = i \Delta f_J$. The concept of the volume preservation must be used for computing the finite increment of the total volume fraction of reinforcements which will be added at the ith step. This volume preservation can be expressed by means of magnetoelectroelastic behaviors of each phase in the following manner:

After (i-1) steps, the magnetoelectroelastic coefficients of composite can be expressed by means of magnetoelectroelastic coefficients of each phase as:

$$\sum_{J=1}^{N} (i-1) \Delta f_J E^J + \left[1 - \sum_{J=1}^{N} (i-1) \Delta f_J \right] E^M = E_{i-1}^C \tag{31}$$

where E^J and E^M are the magnetoelectroelastic coefficients of the phase J and the matrix respectively. E_{i-1}^C represents the composite magnetoelectroelastic coefficients for the step (i-1).

At the i-th step in the Self Consistent scheme, the next increment of phase J is Δf_i^J. It must be introduced in an equivalent matrix which has the behavior of the built composite in the last steps. So, one can write:

$$\sum_{J=1}^{N} \Delta f_i^J E^J + \left[1 - \sum_{J=1}^{N} \Delta f_i^J \right] E_{i-1}^C = \sum_{J=1}^{N} i \Delta f_J E^J + \left[1 - \sum_{J=1}^{N} i \Delta f_J \right] E^M \tag{32}$$

where Δf_i^J is the increment that must be added at the ith step into the equivalent matrix.

The substitution of (31) into (32) leads to the following equation:

$$\sum_{J=1}^{N} \left[\Delta f_i^J + \left[1 - \sum_{J=1}^{N} \Delta f_i^J \right] (i-1) \Delta f_J \right] E^J + \left[1 - \sum_{J=1}^{N} \Delta f_i^J \right] \left[1 - \sum_{J=1}^{N} (i-1) \Delta f_J \right] E^M$$

$$= \sum_{J=1}^{N} i \Delta f_J E^J + \left[1 - \sum_{J=1}^{N} i \Delta f_J \right] E^M \tag{33}$$

From this equation, the following formulations are derived:

$$\Delta f_i^J + \left[1 - \sum_{J=1}^{N} \Delta f_i^J \right] (i-1) \Delta f_J = i \Delta f_J \tag{34}$$

$$\left[1 - \sum_{J=1}^{N} \Delta f_i^J\right]\left[1 - \sum_{J=1}^{N} (i-1)\Delta f_J\right] = 1 - \sum_{J=1}^{N} i\Delta f_J \tag{35}$$

From (34) and (35) the general expression of the volume fraction Δf_i^J to be injected at the step 'i' into the phase J is given by:

$$\Delta f_i^J = \frac{\Delta f_J}{1 - \sum\limits_{J=1}^{N} (i-1)\Delta f_J} \tag{36}$$

Expression (36) shows that the incremental volume fraction of reinforcements Δf_i^J continuously increases as a function of the step number 'i'. It is important to point out that the overall properties of the equivalent homogeneous material obtained by this procedure depends on the number of steps S.

$$E_{iJKl}^{eff(i)} = E_{iJKl}^{eff(i-1)} + \sum_{J=1}^{N} \Delta f_i^J (E_{iJMn}^I - E_{iJMn}^{eff(i-1)}) A_{MnKl}^{SC} \tag{37}$$

with $E^{eff(0)} = E^M$.

Note that the Incremental Self Consistent scheme does not affect the expression of the concentration tensors A on which the method is based. So, the formulations used in this study and in the traditional Self Consistent method are the same. The two methods differ only in the manner of introducing the reinforcements' concentration. In order to compare the effectiveness of the presented approach the Mori Tanaka as well as the dilute approach is presented for N-phase composites.

5.3   N-phase Mori-Tanaka approach

The Mori-Tanaka model has been and continues to be the most widely used approach in the micro mechanics dilute heterogeneous materials with ellipsoidal inclusions. The Mori-Tanaka mean field approach takes into account the effect of other inhomogeneities by considering a finite concentration of inclusions embedded in an infinite matrix of magnetoelectroelastic moduli E_{iJKl}^I and E_{iJKl}^m, and gives a straightforward explicit expression of the effective moduli. The corresponding concentration tensor A^{MT} is then given by the solution for a single inclusion embedded in an infinite matrix in the same manner as the heterogeneous inclusion problem of Eshelby.

For N phases, the Mori-Tanaka concentration tensor A^{MT} is given as follows:

$$A_{iJKl}^{MT} = A_{iJMn}^{Dil}(f^m I_{KlMn} + \sum_{I=1}^{N} f^I A_{KlMn}^{Dil})^{-1} \tag{38}$$

To apply this to N phase composites, it is necessary to find A^{Dil} and A^{MT} for each phase [2]. Similarly to the Self Consistent approach, the effective behavior of N phase composites can be obtained by

$$E^{eff}_{iJKl} = E^m_{iJKl} + \sum_{I=2}^{N} f^I(E^I_{iJMn} - E^m_{iJMn})A^{MT}_{MnKl} \tag{39}$$

Note that the matrix phase is explicitly taken into account but only in an average sense.

5.4   N-phase Dilute approach

This approach has an equivalent scheme than the above approaches but does not consider any interaction between the inhomogeneities. The expression of strain-electro-magnetic fields Z^I_{Kl} of inclusion can be then derived from that obtained in the Self Consistent approach with the difference that in this case, the infinite matrix has magnetoelectroelastic moduli E^m as equivalent behavior. The concentration tensor A^{Dil} is given [2]

$$A^{Dil}_{MnKl} = (I_{KlMn} + \frac{1}{V^I}T^{II}_{iJKl}\Delta E^I_{iJMn})^{-1} \tag{40}$$

where, $\Delta E^I_{iJMn} = E^I_{iJMn} - E^m_{iJMn}$.

The effective behavior prediction of N phase composites, in this case, is expressed as

$$E^{eff}_{iJKl} = E^m_{iJKl} + \sum_{I=2}^{N} f^I(E^I_{iJMn} - E^m_{iJMn})A^{Dil}_{MnKl} \tag{41}$$

The effective magnetoelectroelastic formulations (29, 37, 39 and 41) are applicable to a wide range of inclusion types, shapes and volume fractions. The coupling elastic-electric-magnetic effective behaviors can be investigated and optimized with respect to the volume fraction, shape and type of inclusions which may be elastic, piezoelectric or magnetoelectroelastic.

6   Numerical results

6.1   Two phase composites

The Micromechanical models presented in this paper are used to predict the effective magnetoelectroelastic coefficients. These models permit to take into account the effect of phase number and concentrations, shape inclusions, as well as

its polling orientation. Before investigating the three phase composites effective behaviors, the numerical results of the two phase composites is first considered.

Consider a magnetoelectroelastic composite in which the matrix is piezomagnetic ($CoFe_2O_4$) and the elliptic inclusions are piezoelectric ($BaTiO_3$) having half axes a, b and c. The global coordinate system for the matrix is (x_1, x_2, x_3) and the third half axis c is on the polling direction x_3. The material properties of both phases are transversely isotropic with x_3 the axis of symmetry. The magnetoelectroelastic characteristics of the two materials, used in this paper, are both listed in table 1 and are obtained from [20].

Note that in the two considered phases the magnetoeletric effect does exist neither in the matrix nor in the inclusion. This coupling effect will be induced in magneto-electroelastic composite through the interaction between phases.

Numerical results of effective properties for different inclusions shapes based on the Mori-Tanaka, Dilute, Self Consistent and Incremental Self Consistent approaches are obtained using the presented concentration tensors and the obtained numerical results are well compared with available numerical ones [11, 12, 20].

Figure 1 shows the magnetoeletric coefficient α_{33} for fibrous composite (c/a=1000, b=a) with respect to the volume fraction predicted by the Mori-Tanaka and Self consistent models. The two models predict the same results and α_{33} is maximized at 45% of inclusion concentration. The same results are already obtained by Zhang and Soh [20].

Table 1: Material properties of $BaTiO_3/CoFe_2O_4$

	C_{11}	C_{12}	C_{13}	C_{33}	C_{44}
$BaTiO_3$	166	77	78	162	43
$CoFe_2O_4$	286	173	170	269.5	45.3

	e_{15}	e_{31}	e_{33}	κ_{11}	κ_{33}
$BaTiO_3$	11.5	-4.4	18.6	11.2×10^{-9}	12.6×10^{-9}
$CoFe_2O_4$	0	0	0	0.08×10^{-9}	0.093×10^{-9}

	h_{15}	h_{31}	h_{33}	μ_{11}	μ_{33}
$BaTiO_3$	0	0	0	5×10^{-6}	10×10^{-6}
$CoFe_2O_4$	550	580.3	699.7	-590×10^{-6}	157×10^{-6}

Units: elastic constant GPa; dielectric constants C^2/Nm^2; magnetic constants Ns^2/C^2, piezoelectric constants C/m^2; piezomagnetic constants N/Am; magnetoeletric coefficients Ns/VC.

The prediction based on the Incremental Self Consistent method is presented in figure 2 at different steps (2 to 100). The convergence of the procedure is demonstrated for e_{33} with respect to inclusion volume fraction. It is shown that the Incremental Self Consistent Model improves the prediction of the Self Consistent model which is usually criticized for its deficiency at high concentrations of inclusions. It is demonstrated that with 10 steps of increments, this method gives nearly the same results as with 20, 30, 50, and 100 steps until the concentration 50% of spherical BaTiO$_3$. For large concentrations, a good convergence is clearly seen with 50 steps, but 20 steps give very close results. For the numerical predictions in this paper, the Incremental Self Consistent model with 20 steps of the increment is used.

In figures 3 and 4 the electromagnetic coefficient α_{11} and the permeability coefficient μ_{11} are presented respectively for fibrous composites (a=b, c/a=1000) based on Incremental Self Consistent, Self Consistent, Mori-Tanaka and Dilute models. It is clearly shown that the predictions given by these models are in agreement with each other for low volume fractions of inclusions. The figure 3 demonstrates that the Self Consistent model is not able to conduct the predictions for moderate and higher concentrations and it diverges beyond 40% concentration of inclusions. This is the main raison why the Incremental Self Consistent is developed here. This figure shows also that the Incremental Self Consistent model improves the prediction of the classical Self Consistent one and gives closer results to Mori-Tanaka's predictions.

Note also that the effective moduli of the composite predicted by the Dilute model does not take the property of the inclusion when the volume faction is close to 1. This is expected because the Dilute model is only applicable when the volume fraction of the inclusions is very small.

In figure 5, the electromagnetic coefficient $-\alpha_{11}$ is presented for laminated magnetoelectroelastic composites (a=b and c/a=0.001). In this case the Mori-Tanaka, Self Consistent and ISC micromechanical models predict the same results. $-\alpha_{11}$ is maximized at 50% of inclusions concentration. These results are well compared with those obtained by [11, 12, 21].

The piezoelectric modulus e_{33} is presented in figure 6 for ellipsoidal inclusions. The results obtained by Self Consistent model and Mori-Tanaka and those obtained by ISC model are different and particularly in the vicinity of 50%. Experimental results are needed to test the accuracy of these predictions.

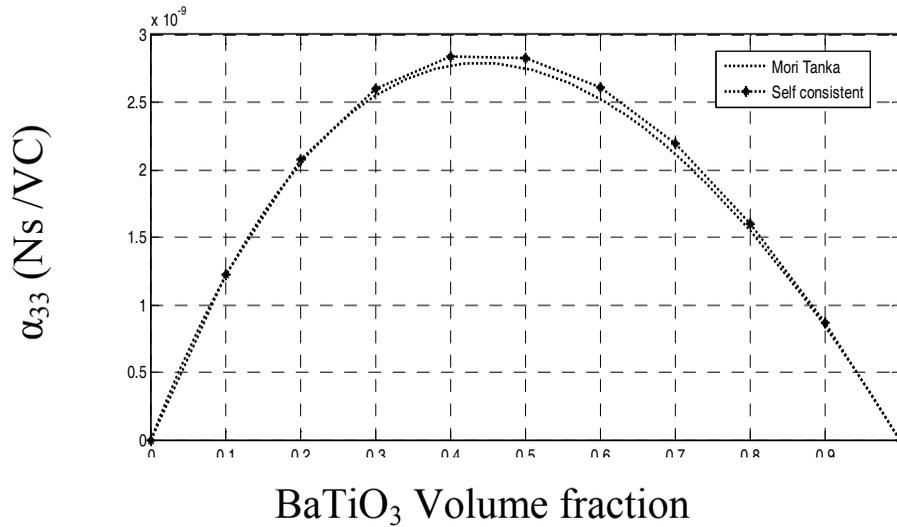In figures 7 and 8, the piezoelectric modulus e_{31} and the dielectric modulus κ_{33}

Figure 1: Effective electromagnetic modulus α_{33} for fibrous composite BaTiO$_3$/ CoFe$_2$O$_4$ predicted by Mori-Tanaka and Self Consistent models
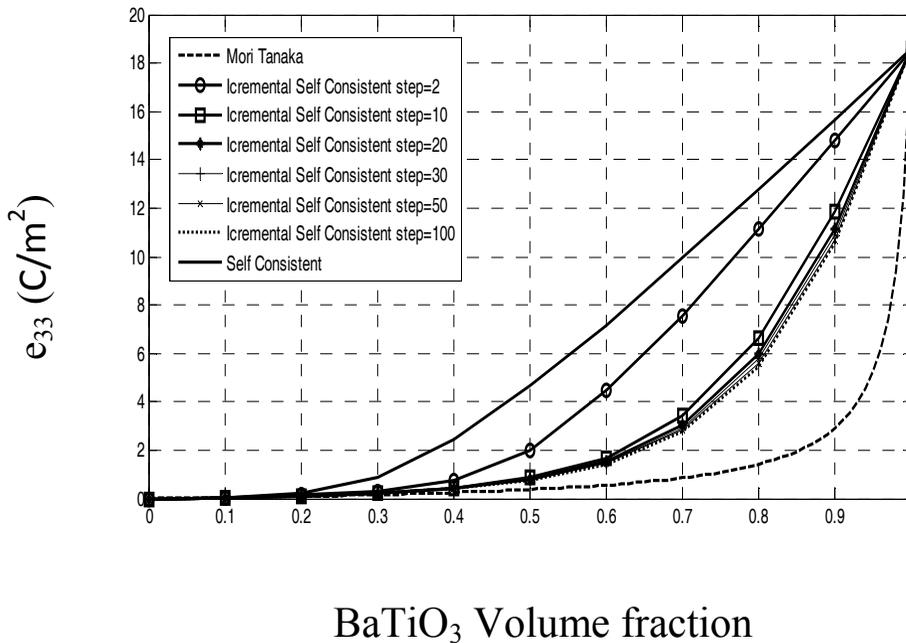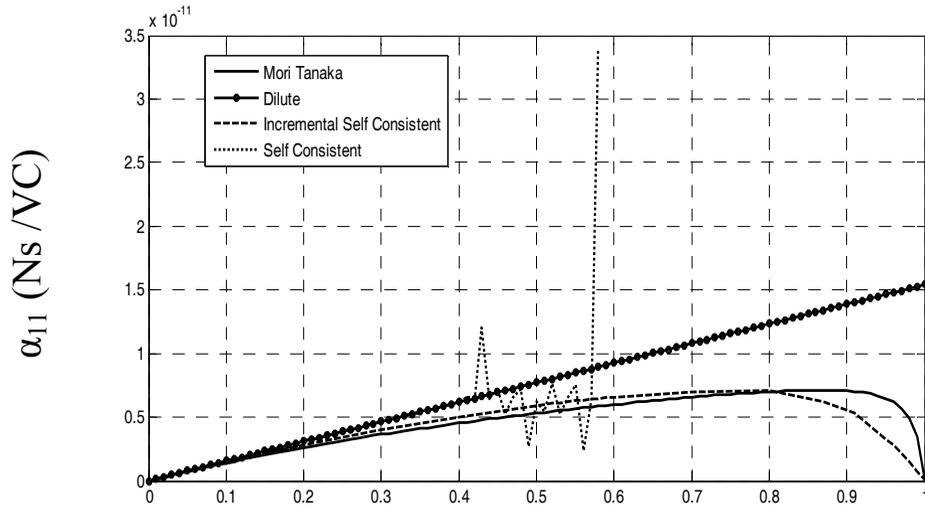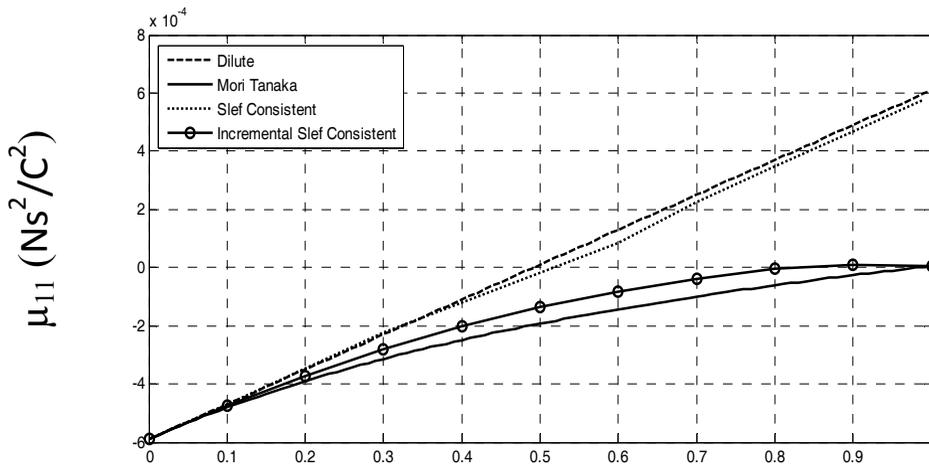

BaTiO$_3$ Volume fraction

Figure 2: Effective piezoelectric modulus e$_{33}$ for spherical composite BaTiO$_3$/ CoFe$_2$O$_4$ predicted by Mori-Tanaka, Self Consistent and Incremental Self Consistent models.
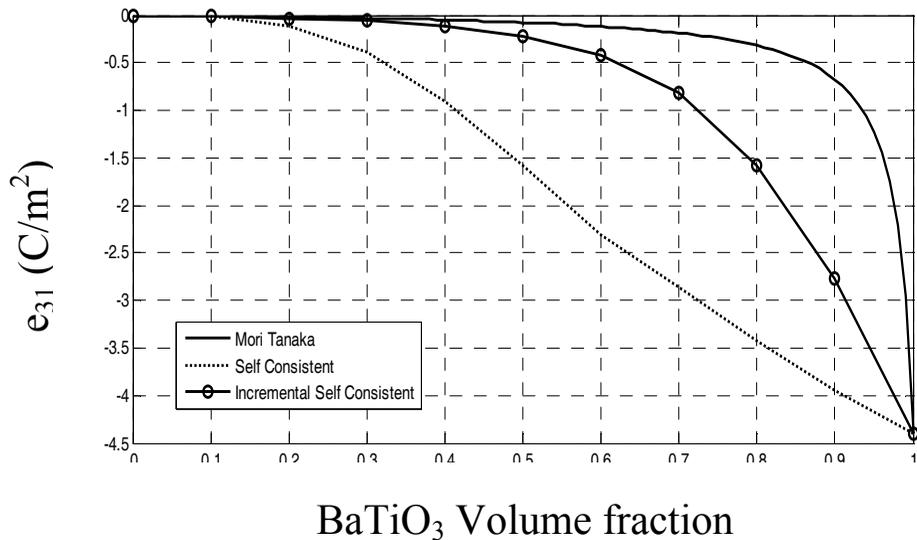
BaTiO$_3$ Volume fraction
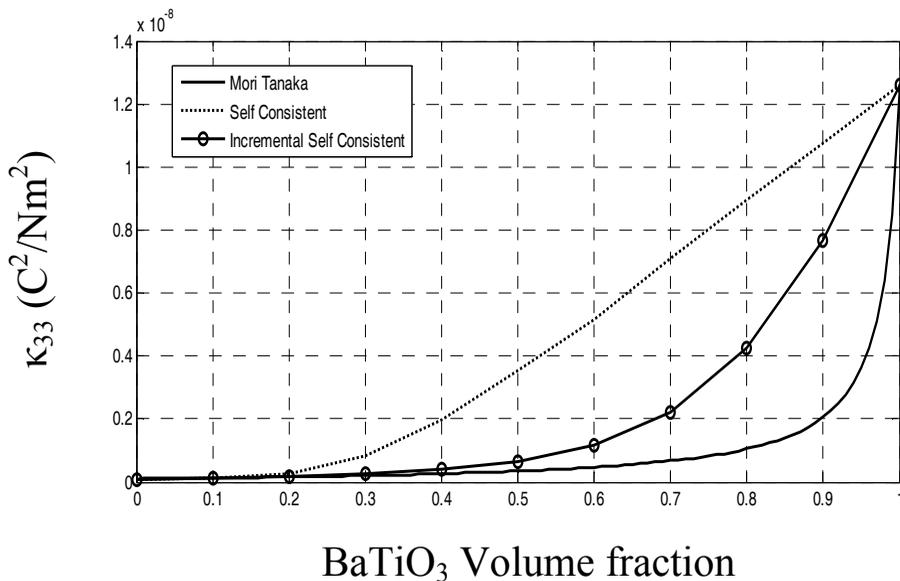
Figure 3: Effective electromagnetic modulus α_{11} for fibrous composite BaTiO$_3$/ CoFe$_2$O$_4$ predicted by Mori-Tanaka, Dilute, Self Consistent and Incremental Self Consistent models.

BaTiO$_3$ Volume fraction

Figure 4: Effective magnetic modulus μ_{11} for fibrous composite BaTiO$_3$/ CoFe$_2$O$_4$ predicted by Mori-Tanaka, Dilute, Self Consistent and Incremental Self Consistent models.
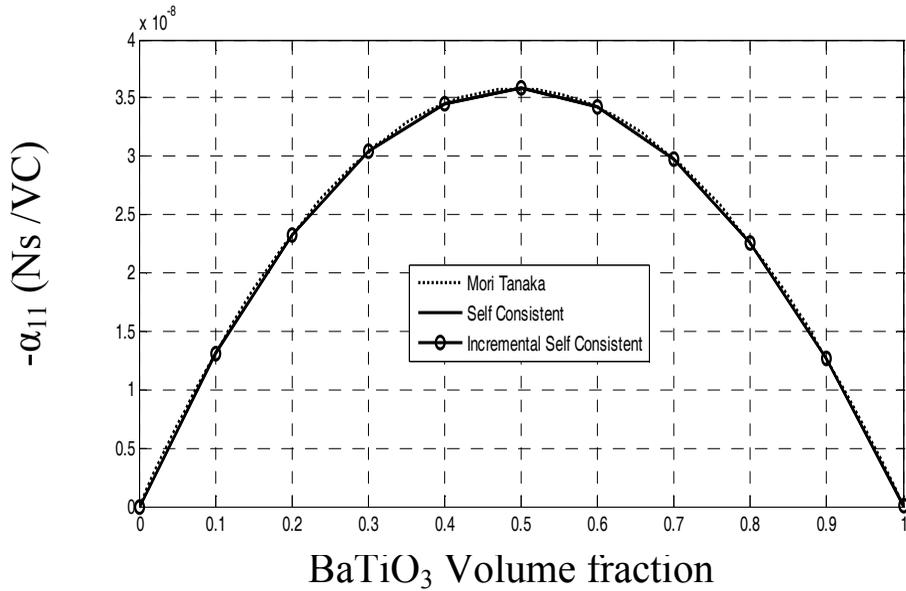
Figure 5: Effective electromagnetic modulus $-\alpha_{11}$ for laminated composite $BaTiO_3$/ $CoFe_2O_4$ predicted by Mori-Tanaka, Self Consistent and Incremental Self Consistent models.

BaTiO$_3$ Volume fraction
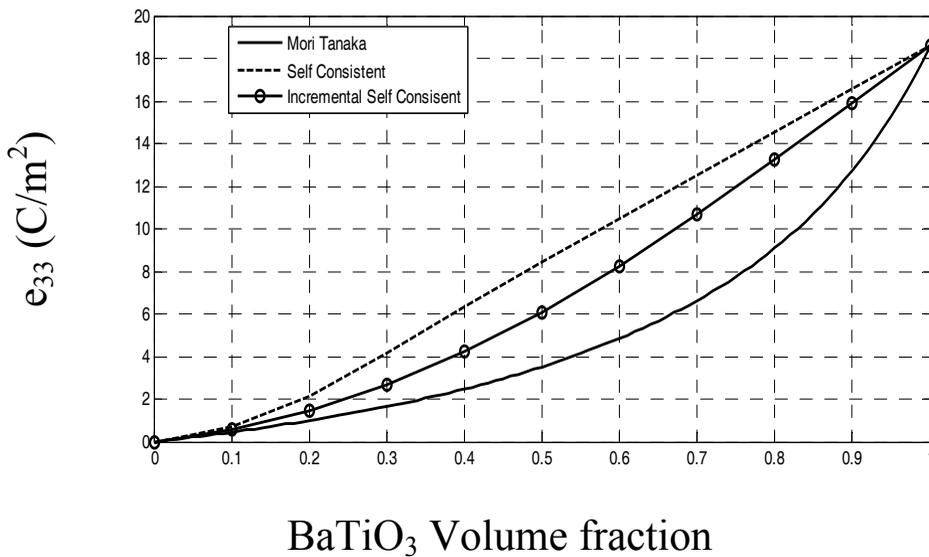
Figure 6: Effective piezoelectric modulus e_{33} for ellipsoidal composite (a=b, c/a=10) $BaTiO_3$/ $CoFe_2O_4$ predicted by Mori-Tanaka, Self Consistent and Incremental Self Consistent models.

Figure 7: Effective piezoelectric modulus e_{31} for spherical composite BaTiO$_3$/ CoFe$_2$O$_4$ predicted by Mori-Tanaka, Self Consistent and Incremental Self Consistent models.

Figure 8: Effective dielectric modulus κ_{33} for spherical two- phase composite BaTiO$_3$/ CoFe$_2$O$_4$ predicted by Mori-Tanaka, Self Consistent and Incremental Self Consistent models.

are presented respectively for magnetoelectroelastic spherical composites (a=b=c). Again the Self Consistent model shows an over estimation especially over 20% volume fraction of inclusions. The ISC model improves the prediction of the classical Self Consistent model for high volume fraction of inclusions.

The effective piezomagnetic coefficients h_{33} is presented in figure 9 for $CoFe_2O_4$ matrix with fibrous voids (a=b, c/a=1000). It is clearly shown that the prediction given by the classical Self Consistent approach is limited for very low void concentration and the model diverges at 12% of voids inclusions. On the other hand, it is seen that the Incremental Self Consistent approach improves the prediction of the classical Self Consistent approach and conducted far the prediction.
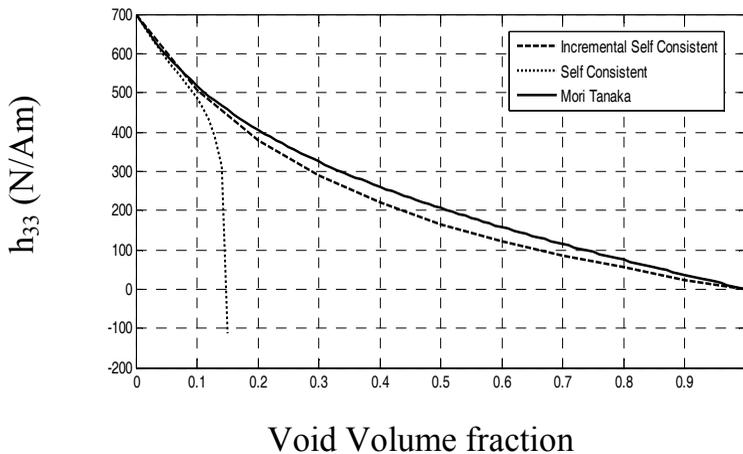

Figure 9: Effective piezomagnetic modulus h_{33} for fibrous composite $CoFe_2O_4$/ Void predicted by Mori-Tanaka, Self Consistent and Incremental Self Consistent models.

6.2   Three phase composites

In this subsection, the numerical results for three-phase composite materials are presented. Two kinds of three phase composites are investigated. One is consisting of a piezoelectric phase and a piezomagnetic phase surrounded by a matrix assumed to be Epoxy whose properties are listed in table 2. The other is consisting of piezoelectric phase and void phase surrounded by a piezomagnetic matrix. These voids are simulated as empty inclusions, which may have several forms. Here, Mori-Tanaka, Self Consistent and Incremental Self Consistent micro mechanical models are used to predict the behavior of the considered three-phase magnetoelectroelastic composites. Numerical results are presented for various shapes and types of

inclusions. In all presented results, the volume fraction of the matrix is fixed and the volume fractions of inclusions are varied.

Let us note that the Mori-Tanaka method has been already used by Lee, Boyd and Lagoudas [11] for three-phase magnetoelectroelastic composites materials. Thus, the numerical results presented in this section using the Mori-Tanaka method are the same as those of [11].

Table 2: Material properties of Epoxy

	C_{11}	C_{12}	C_{33}	C_{44}
Epoxy	5.53	2.97	5.53	1.28

	e_{15}	e_{31}	e_{33}	κ_{11}	κ_{33}
Epoxy	0	0	0	0.1×10^{-9}	0.1×10^{-9}

	h_{15}	h_{31}	h_{33}	μ_{11}	μ_{33}
Epoxy	0	0	0	1×10^{-6}	1×10^{-6}

In figure 10, the effective electromagnetic modulus α_{33} for fibrous three-phase magnetoelectroelastic composites (a=b; c/a=1000) is presented with respect to piezomagnetic inclusion by using Mori-Tanaka, Self Consistent and Incremental Self Consistent models. As in the two phase magnetoelectroelastic composites the convergence of the Incremental Self Consistent model is demonstrated. It is shown that the Incremental Self Consistent model with 10 steps of increments gives nearly the same results as with 20, 30, and 50 steps.

In figures 11 and 12, piezomagnetic coefficients h_{33} and h_{31} are presented respectively for fibrous three-phase magnetoelectroelastic composite materials. The Incremental Self Consistent method improves the classical Self consistent method and gives closer results to Mori-Tanaka predictions.

In figures 13 and 14, the effective magnetic coefficient μ_{33} and the electromagnetic coefficient $-\alpha_{11}$ are presented respectively for fibrous (a=b; c/a=1000) and laminated (a=b; c/a=0.001) three-phase composite materials. For these coefficients it is shown that the three micromechanical models predict the same results. Also it is shown that the effective modulus $-\alpha_{11}$ takes a maximum value at 30% of the piezomagnetic phase and piezoelectric phase. By analyzing the numerical results presented above it is shown that the electromagnetic coefficients obtained in two phase composites are higher than the electromagnetic coefficients obtained in three-phase composites. This is due to the presence of the elastic matrix in three-phase composites. Also it can be explained that in two- phase composites there is more
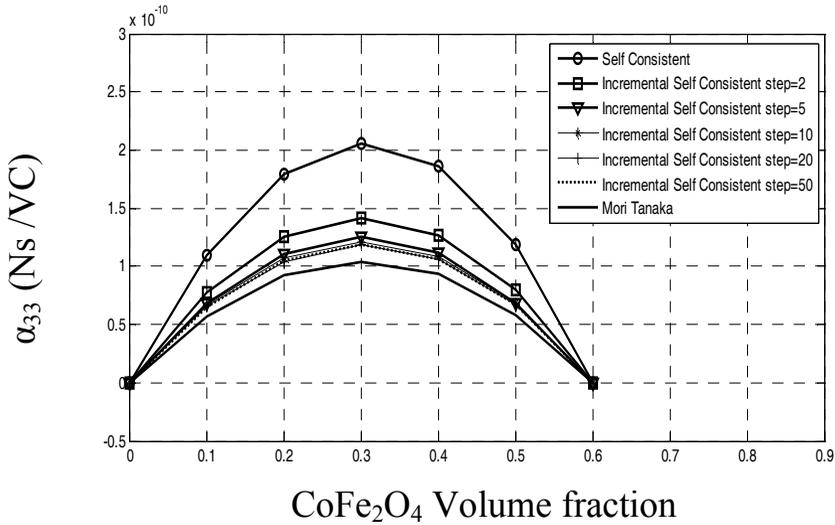
Figure 10: Effective electromagnetic modulus α_{33} for fibrous three- phase composite Epoxy/BaTiO$_3$/ CoFe$_2$O$_4$ predicted by Mori-Tanaka, Self Consistent, and Incremental Self Consistent models with the volume fraction of the matrix fixed at 40%.
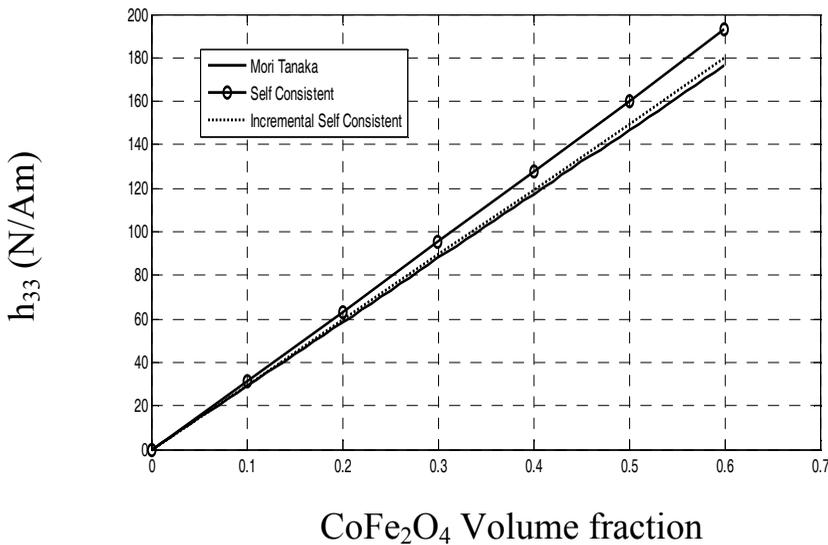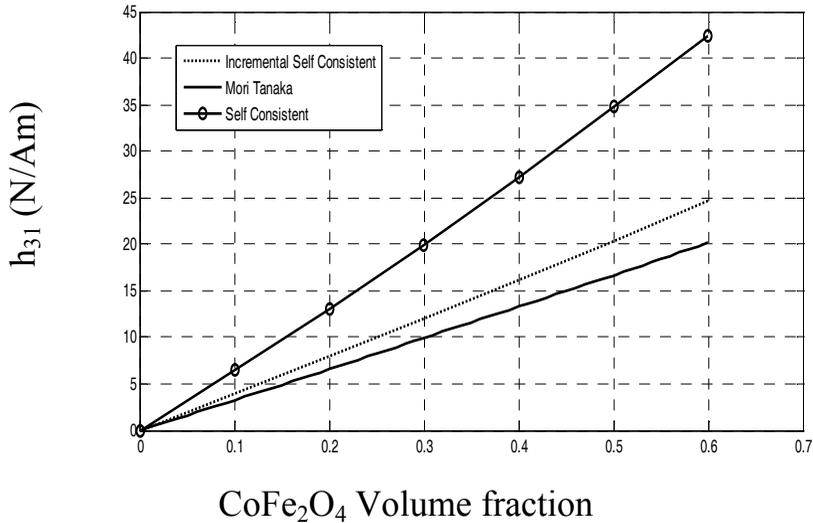

Figure 11: Effective piezomagnetic modulus h$_{33}$ for fibrous three- phase composite Epoxy/BaTiO$_3$/ CoFe$_2$O$_4$ predicted by Mori-Tanaka, Self Consistent and Incremental Self Consistent models with the volume fraction of the matrix fixed at 40%.
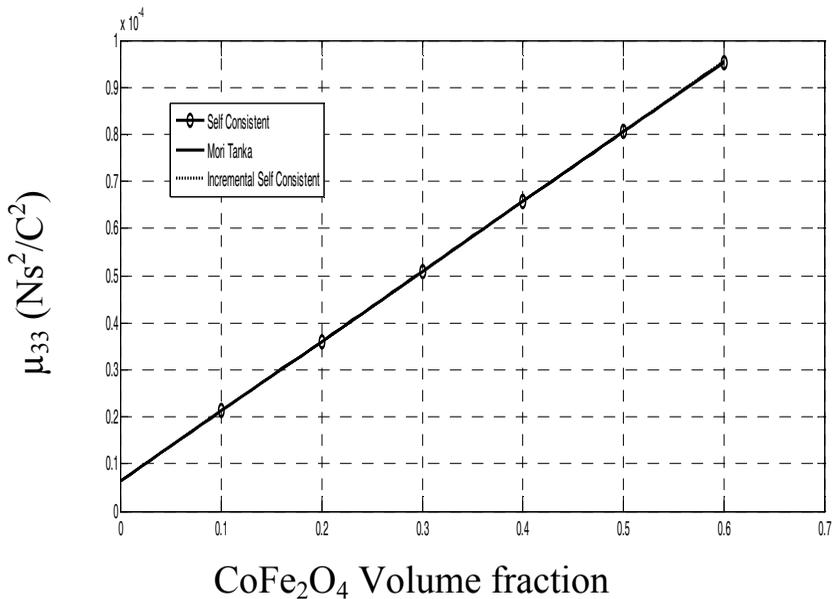
Figure 12: Effective piezomagnetic modulus h_{31} for fibrous three- phase composite Epoxy/BaTiO$_3$/ CoFe$_2$O$_4$ predicted by Mori-Tanaka, Self Consistent and Incremental Self Consistent models with the volume fraction of the matrix fixed at 40%.
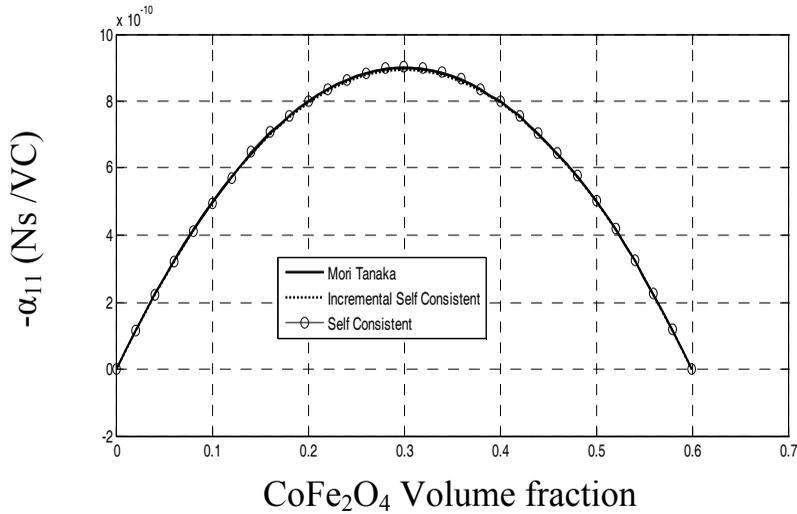

Figure 13: Effective magnetic modulus μ_{33} for fibrous three-phase composite Epoxy/BaTiO$_3$/ CoFe$_2$O$_4$ predicted by Mori-Tanaka, Self Consistent and Incremental Self Consistent models with the volume fraction of the matrix fixed at 40%.

Figure 14: Effective electromagnetic modulus-$\alpha_{11}$for laminated three- phase composites Epoxy/BaTiO$_3$/ CoFe$_2$O$_4$ predicted by Mori-Tanaka, Self Consistent, and Incremental Self Consistent models with the volume fraction of the matrix fixed at 40%.
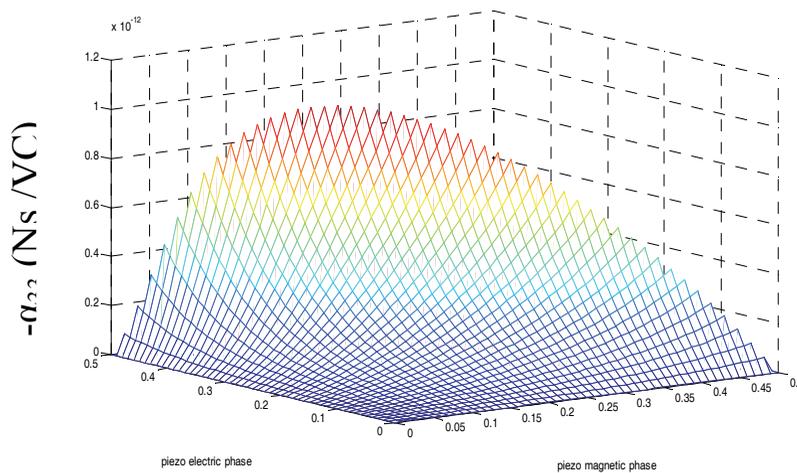

Figure 15: Effective electromagnetic modulus -$\alpha_{33}$for spherical three- phase composites Epoxy/BaTiO$_3$/ CoFe$_2$O$_4$ predicted by the Incremental Self Consistent model with the volume fraction of the matrix fixed at 50%.
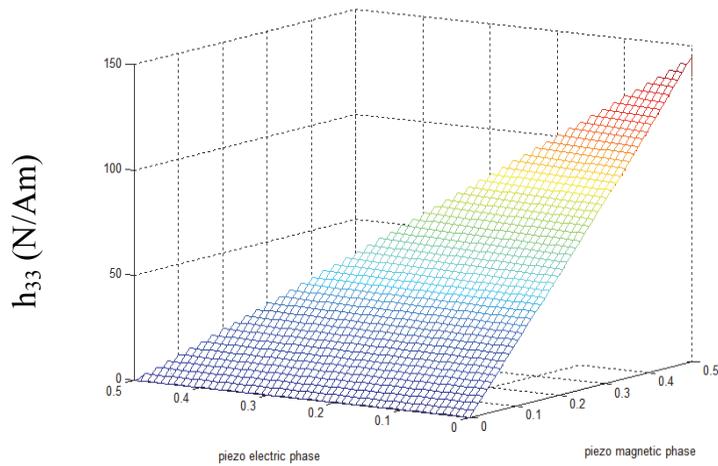
Figure 16: Effective electromagnetic modulus h_{33} for fibrous three-phase composites Epoxy/BaTiO$_3$/ CoFe$_2$O$_4$ predicted by the Incremental Self Consistent model with the volume fraction of the matrix fixed at 50%.
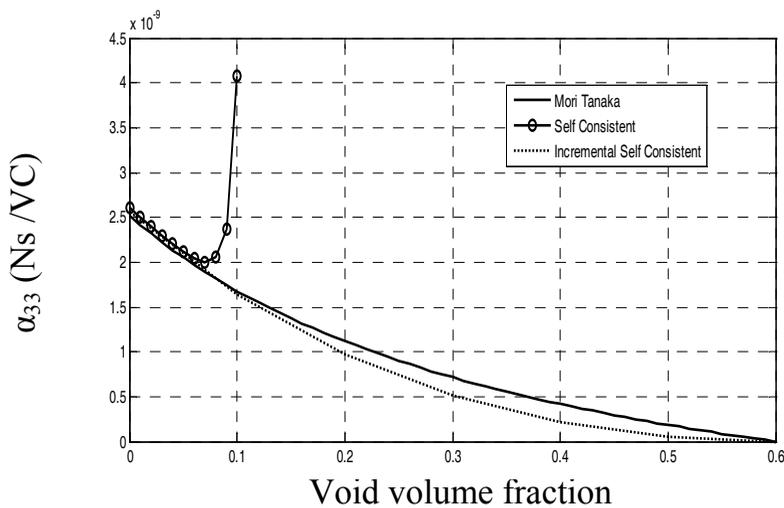

Figure 17: Effective electromagnetic modulus α_{33} for fibrous three- phase composites CoFe$_2$O$_4$ /BaTiO$_3$/ Void predicted by Mori-Tanaka and Incremental Self Consistent models with the volume fraction of the matrix fixed at 40%.

interaction between the piezoelectric phase and piezomagnetic phase than in three-phase composites.

In figures 15, 16 three dimension numerical results of a three phase composites Epoxy/BaTiO$_3$/ CoFe$_2$O$_4$ are presented to show the phases effects on the effective behavior of the composites by using the Incremental Self Consistent approach. Spherical (a=b=c) and fibrous (a=b; c/a=1000) inclusions are considered respectively in figures 15 and 16. The predictions are presented for 50% of the matrix and various concentrations of BaTiO$_3$ and CoFe$_2$O$_4$ inclusions. The variation of the effective magnetoelectric coefficient α_{33} with respect to the two spherical inclusions concentrations is clearly shown. This coefficient may be maximized with respect to concentrations of the piezoelectric and piezomagnetic phases. With the epoxy matrix, when the two phases, piezomagnetic and piezoelectric, do not coexist the magnetoeletric coupling effects are zeros. For the piezomagnetic effective coefficient h$_{33}$its variation is linear and its maximal value is obtained when the concentration of the piezomagnetic phase CoFe$_2$O$_4$is maximal and vanishes when we have only the piezoelectric phase and the epoxy matrix.

In figure 17, the electromagnetic coefficient α_{33} is presented for fibrous (a=b; c/a=1000) magnetoelectroelastic three-phase composites containing voids (CoFe$_2$-O$_4$/BaTiO$_3$/Void). This figure demonstrates clearly that the Self Consistent approach can not estimate the effective electromagnetic moduli α_{33}beyond 10% concentration of voids in contrast with the Incremental Self Consistent approach which can be used until 60% voids concentration. Also it can be seen that below 10% void concentrations the three micromechanical approaches almost give the same predictions.

In figures, 18 and 19 the piezomagnetic coefficient h$_{33}$ and the dielectric coefficient $\kappa_{33}$are presented respectively for fibrous magnetoelectroelastic three-phase composites containing voids as the third inclusion by using the Incremental Self Consistent and Mori-Tanaka approaches. It is seen that the Incremental Self Consistent approach conducted far the prediction until 60% void concentration. Also for the effective coefficient h$_{33}$ the Incremental Self Consistent and the Mori-Tanaka approaches give different prediction and the prediction obtained by the Incremental Self Consistent approach is lower than that obtained by the Mori-Tanaka approach. On the other hand the numerical predictions obtained for the effective coefficient κ_{33} are almost the same.

7   Conclusion

The Incremental Self Consistent, Self Consistent, Mori-Tanaka, and Dilute micromechanical models are elaborated to predict the effective moduli of multi-phase
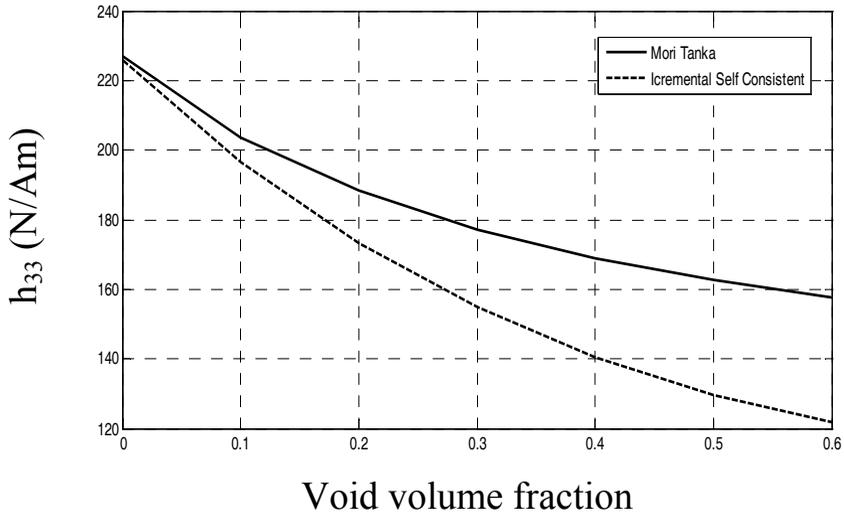
Figure 18: Effective piezomagnetic modulus h_{33} for fibrous three- phase composites $CoFe_2O_4$ /$BaTiO_3$/ Void predicted by Mori-Tanaka and Incremental Self Consistent models with the volume fraction of the matrix fixed at 40%.

Figure 19: Effective electromagnetic modulus κ_{33} for fibrous three- phase composites $CoFe_2O_4$ /$BaTiO_3$/ Void predicted by Mori-Tanaka and Incremental Self Consistent models with the volume fraction of the matrix fixed at 40%.
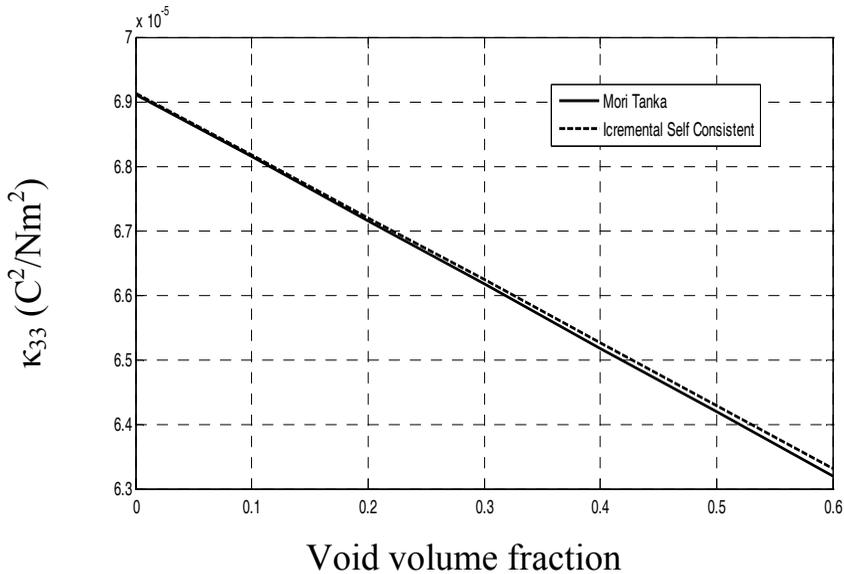
magnetoelectroelastic composite materials for different shapes, types and concentration of inclusions. The N-phase Incremental Self Consistent model is developed for magnetoelectroelastic effective properties. The expression of the effective behavior of the composite obtained by the micro mechanical models is written as function of the concentration tensors which are a function of the interaction tensors. The interaction tensor depends on the constituent properties and shape of ellipsoidal inclusions.

Numerical results have been presented for two phase composites and three phase composites with and without void by emphasizing the effect of shape and concentration inclusions. It is shown that the Self Consistent, Mori-Tanaka, and Dilute approaches lead to the same results for very low volume fraction of inclusions. However, for moderate and high volume fractions of inclusions the Self Consistent showed an over estimating especially over 20% inclusion concentration, and gives erroneous results for some coefficients. This drawback is corrected by the developed ISC model, which improves the prediction of the Self Consistent model for high volume fractions of the inclusions. In addition, it has been demonstrated from the above numerical results obtained for three-phase composites consisting of a piezoelectric phase and a void phase surrounded by a piezomagnetic matrix that the Incremental Self Consistent approach can estimate the properties of the composites for moderate volume fraction of voids. This model has been compared to the Mori-Tanaka one which is extensively used. This model will be next elaborated for predicting the behavior of disordered aggregates in nonlinear piezoelectric and magnetoelectroelastic heterogeneous media.

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Appendix

In this paper, transversely isotropic magnetoelectroelastic materials are used. The used magnetoelectroelastic matrix is given by:

$$E = \begin{bmatrix} c & e^t & h^t \\ e & -\kappa & -\alpha \\ h & -\alpha & -\mu \end{bmatrix}$$

with

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \qquad e^t = \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$h^t = \begin{bmatrix} 0 & 0 & h_{31} \\ 0 & 0 & h_{31} \\ 0 & 0 & h_{33} \\ 0 & h_{15} & 0 \\ h_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \kappa = \begin{bmatrix} -\kappa_{11} & 0 & 0 \\ 0 & -\kappa_{11} & 0 \\ 0 & 0 & -\kappa_{33} \end{bmatrix}$$

$$\mu = \begin{bmatrix} -\mu_{11} & 0 & 0 \\ 0 & -\mu_{11} & 0 \\ 0 & 0 & -\mu_{33} \end{bmatrix} \qquad \alpha = \begin{bmatrix} -\alpha_{11} & 0 & 0 \\ 0 & -\alpha_{11} & 0 \\ 0 & 0 & -\alpha_{33} \end{bmatrix}$$