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Orthogonal Tapered Beam Functions in the Study of Free Vibrations for Non-uniform Isotropic Rectangular Plates

M.F. Liu[1]

Abstract: A new invented Orthogonal Tapered Beam Functions (OTBFs) have been introduced in this paper and used in accordance with the Rayleigh-Ritz method to determine the natural frequencies and mode shapes of the non-uniform rectangular isotropic plates with varying thickness in one or two directions. The generation of the OTBFs is based on the static solution of a one-dimensional beam problem subjected to constant applied load, and then extends to an orthogonal or orthonomal infinite set of admissible functions by performing the three-term recurrence scheme. A wide range of non-uniform rectangular plate whose domain is referenced by a so-called truncation factor and thickness variation is presented by the so-called taper factor is investigated in the present study in order to practice the validity and efficiency of the proposed methodology. Following the Levy approach, the fourth order differential equation governing the free vibration of non-uniform isotropic plate can be transferred into an 8^{th} order eigenfrequency equation by inserting the OTBFs as shape functions. Three different combinations of clamped, simply-supported and free boundary conditions are imposed around the non-uniform plate and the aspect ratio which indicates the width over the length of the rectangular plate is ranging from half to two. The effects of taper factor together with the truncation factor, boundary condition and aspect ratio on the natural frequencies of free vibration are demonstrated for the first few modes by using the finite terms approximation of the proposed admissible functions. A general computer program has been implemented to solve the eigenfrequency equation, some numerical results are tabulated and comparisons of the results with those available in literature are also presented.

Keywords: Orthogonal Tapered Beam Functions, Non-uniform Isotropic Plate, Natural Frequencies, Taper Factor, Truncating Factor.

[1] Department of Applied Mathematics, I-Shou University, No.1, Sec. 1, Syuecheng Rd., Dashu Township, Kaohsiung County. 840, Taiwan R.O.C. E-mail: meifeng@isu.edu.tw

1 Introduction

Free vibration characteristics for mechanical structures such as beam, plate and shell are of interest to many scientists and engineers from a wide range of research fields. The natural frequencies and mode shapes for thin plate structures are among one of the most popular topics and can be evaluated by using the Energy method including Rayleigh-Ritz method and Galerkin's method. In performing these methods, an assumed deflection function, or sometimes called admissible function, should be developed according to the structures' geometric shape as well as the imposed boundary conditions. An efficient technique aiming to generate the desired admissible function which satisfies all the boundary conditions, either in Dirichlet type or in Neumann type, is well-developed in the past few years, and referred as the Boundary Characteristic Orthogonal Polynomials (BCOPs).

One dimensional BCOPs constructed by a set of beam characteristic orthogonal polynomials was employed by Bhat (1985) to obtain the natural frequencies of rectangular plates in accordance with the Rayleigh-Ritz method. The orthogonal polynomials are generated by first constructing a 4^{th} order polynomial which satisfies all the boundary conditions in the corresponding beam problems and followed by a three-term recurrence relation Bhat (1987) to generate the other members of the orthogonal set. By applying the aforementioned BCOPs as the deflection shapes of the vibrating plates in two directions, it has been shown by many researchers [Dickinson and Di Blasio (1986); Kim and Dickinson (1989); Liew and Lam (1990); Liew, Lam and Chow (1990)] that the method achieves satisfied results on the vibration characteristics of plates with complicating effects and various shapes.

Later on, the two-dimensional BCOPs was accordingly proposed by Bhat (1987) to study the polygonal plates in the Rayleigh-Ritz method. Meanwhile, Singh and Chakraverty have also developed two-dimensional BCOPs on other non-rectangular domains in the study of various shaped plates [Singh, Chakraverty (1991, 1992a, 1992b, 1994)]. The two dimensional boundary characteristic orthogonal polynomials can be produced by orthogonalizing all the previously derived orthogonal polynomials and a systematical recurrence scheme for creating the 2D BCOPs were proposed by Bhat, Chakraverty and Stiharu (1998)].

The application of boundary characteristic orthogonal polynomials, either 1D or 2D, can be found in many research papers regarding to the determination of vibration characteristics for beam or plate structures arouse from diversified fields. Chang and Wu (1997) applied the BCOPs to study the anisotropic rectangular plates with mixed boundaries and concentrated masses. Liu and Chang (2005) applied the BCOPs to perform the vibration analysis of a magneto-elastic beam with general boundary conditions subjected to axial load and external force. In addition to the geometric complexity for plate vibration, more complicated natures such as thickness non-uniformity have also attracted many research activities in the past years.

The vibration problem of rectangular plate with varying thickness in two directions has widely application in the fields of high-performance technology as well as the air vehicles, such as the design for electronic chips or airplane wings. Due to the difficulty on seeking for the closed-form solution for the vibration characteristic with plate thickness being variable, it has been found in many literatures that a variety of numerical approaches have been proposed to achieve the desired task. Among those diversified approaches, the use of boundary characteristic functions seems to be the simplest one due to the fact that it is easy to be constructed and can be applied to the more general plate with thickness variety.

Recently, Cheung and Zhou (1999) developed a set of static tapered beam function as basis function to solve the free vibration of rectangular plates with thickness varying in power form. In their study, the admissible function is generated by solving an isotropic tapered beam subjected to a static applied load which is expanded into Taylor series of order n. After combining the homogeneous solution and the particular one with respect to the corresponding order, a set of admissible function whose coefficients are determined according to the specified boundary conditions can be carried out in a swift fashion. The technique of static tapered beam function was adopted by several researchers in the study of natural frequencies for more intricate non-uniform plate problems, in which both rapid convergence and high accuracy were observed.

Meanwhile, Zhou and Chueng (2000) also pointed out that the expanding point of Taylor series expansion for the static load has an important effect on the convergence of the calculation, and indicated that optimum point is indeed the midpoint of the beam. Zhou (2002) further improved the set of static tapered beam functions by choosing the midpoint of the beam as the expansion point for the Taylor series expansion in order to achieve the optimal convergence. In his study, the refined set of static tapered beam functions in the Rayleigh-Ritz method combined with the Lagrangian multiplier technique is used to derive the eigenfrequency equation of point-supported rectangular plates with variable thickness.

Lately, Liu, Chang and Wang (2011) adopted the original set of static tapered beam functions in accordance with the Galerkin's method to investigate the vibration characteristics of orthotropic rectangular plates with tapered varying thickness resting on Wrinkler type spring foundation. In their study, the thickness of the orthotropic plate is assumed to be continuously proportional to the power function $x^s y^t$, and the boundary conditions are chosen from the combinations of clamped, simply supported and free boundary conditions. The effects of the elastic foundation together with the orthotropy aspect ratio and thickness variation on the natural frequencies are completely examined and the comparison of the results with those available in the other literatures provides the feasibility and validity of the proposed methodology.

Even though the efficiency and the accuracy of the static tapered beam functions are truly impressive, however, as it can be easily detected, the members of the set are actually not orthogonal to each other. The orthogonality for the admissible functions can effectively reduce the amount of calculation work in the procedure of performing the Rayleigh-Ritz method or Galerkin's method due to the fact that only diagonal components should be evaluated. Nevertheless, neither the original nor the refined set of static tapered beam function could accomplish this task, this gives rise to the need for proposing a new set of orthogonal tapered beam functions.

In this study, a set of refined admissible function suitable for dealing the non-uniform rectangular plate is developed based on the tapered beam function and the orthogonal recurrence scheme. The newly generated set can be referred as the set of Boundary Characteristic Orthogonal Tapered Beam Functions (BCOTBFs) and will simultaneously possess the properties of static tapered beam functions and the orthogonality of the characteristic polynomials; furthermore, the orthonormality of the set can also be achieved. In order to testify the validity and efficiency of the proposed methodology, a wide range of non-uniform rectangular plate whose domain is bounded by $x = \alpha a \sim a$ and $y = \beta b \sim b$ in the rectangular co-ordinates and thickness variation is described by the power function, $x^s y^t$ is under investigation. Some numerical results will be given and a comparison with those available in literature will be further presented.

2 The Generation of Orthogonal Tapered Beam Functions

Before proceed to the determination of the vibration characteristics for the non-uniform rectangular plates, a proper set of admissible functions which can vary appropriately with the thickness variation and simultaneously possess the orthogonal property should be implemented as a prior. In the following, we will introduce the basic idea and the procedure which can successfully generate the desired shape functions.

Let's first consider an isotropic tapered beam with unit width, length L, and varying depth $h(x)$ as shown in Fig. 1. The origin of the co-ordinate system is taken to be at the sharp end of the beam and the x-axis is set to be the neutral line of the beam. Suppose the depth of the beam can be described by a power function

$$h(x) = h_0(x/L)^r, \tag{1}$$

where h_0 denotes the depth at the right end of the beam, i.e., $h_0 = h(L)$, and the power index r refers to the taper factor of the varying depth. It should be noted that the taper factor r should be non-negative for a sharp ended beam, and could be any arbitrarily real number for a truncated beam, however the beam shape will become convex if $r > 0$, uniform if $r = 0$ and concave if $r < 0$, respectively. For simplicity, only the cases for truncated beam are considered in this paper, and a further discussion regarding the new set of orthogonal tapered beam functions for both the sharp-ended beam and the truncated beam will soon be carried out in the near future. Fig. 2 demonstrates three kinds of the truncated beam models with taper factor being nonnegative, zero and negative, the tapered beam is truncated from the left hand side at the location $x = \alpha L$, where $0 < \alpha < 1$ is the so-called truncation factor.

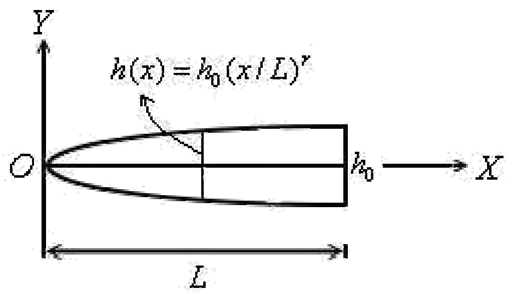


Figure 1: Diagram of the isotropic tapered beam

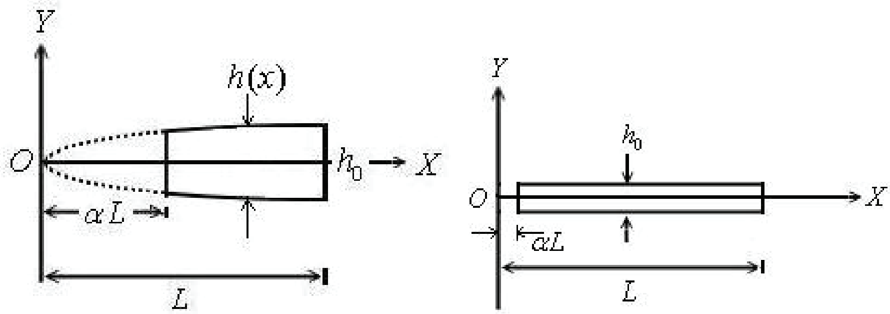
The Euler-Bernoulli beam model in the linear elasticity is adopted in the present study and thus the differential equation governing the transverse deformation of the isotropic tapered beam can be read as

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2\varphi}{dx^2}\right] = q(x), \tag{2}$$

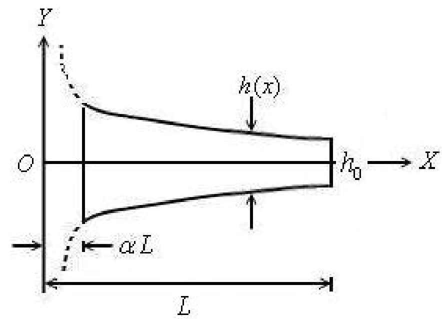
where E is the Young's modulus and $I(x)$ is the second moment of inertia, which can be written as

$$I(x) = h^3(x)/12 = (h_0^3/12)(x/L)^{3r} = I_0(x/L)^{3r}, \tag{3}$$

with I_0 being defined as the area moment of inertia at the location $x = L$. The term $q(x)$ represents the distributed applied load acting on the surface of the beam and is dependent on the x variable. By introducing the dimensionless variable $\xi \equiv x/L$,



(a) (b)



(c)

Figure 2: Thickness variations for a truncated beam (a) Convex if $r > 0$ (b) uniform if $r = 0$ (c) Concave if $r < 0$.

one can turn Eq. (2) into the following form

$$\frac{d^2}{d\xi^2}\left[\xi^{3r}\frac{d^2\varphi(\xi)}{d\xi^2}\right] = \frac{L^4}{EI_0}q(\xi L), \tag{4}$$

where $q(\xi L)$ can be expanded into Taylor series about any point in the domain $\alpha \leq \xi \leq 1$. The homogenous solution of Eq. (4) in accordance with the non-homogeneous one with respect to the i-th expansion term will form the shape function of the i-th mode and is called the static tapered beam function just as mentioned in the related papers [Cheung and Zhou (1999, 2000); Zhou (2002)].

However, differs from the previous approach, Taylor series expansion for the applied load won't be adopted in this study, instead, general solution for the beam equation subjected to a constant applied load will take over as the fundamental shape function of the tapered beam. In so doing, we will have to solve the following differential equation

$$\frac{d^2}{d\xi^2}\left[\xi^{3r}\frac{d^2\varphi(\xi)}{d\xi^2}\right] = b_4, \tag{5}$$

$b_4 \neq 0$ is constant, and the analytical solution for Eq. (5) can be easily derived to be

$$\varphi(\xi) = b_4\xi^{4-3r} + b_3\xi^{3-3r} + b_2\xi^{2-3r} + b_1\xi + b_0, \text{ if } r \neq 4/3, 1, 2/3, 1/3,$$

$$\varphi(\xi) = b_4 \ln\xi + b_3/\xi + b_2/\xi^2 + b_1\xi + b_0, \text{ if } r = 4/3,$$

$$\varphi(\xi) = b_4\xi \ln\xi + b_3 \ln\xi + b_2/\xi + b_1\xi + b_0, \text{ if } r = 1 \tag{6}$$

$$\varphi(\xi) = b_4\xi^2 + b_3\xi \ln\xi + b_2 \ln\xi + b_1\xi + b_0, \text{ if } r = 2/3,$$

$$\varphi(\xi) = b_4\xi^3 + b_3\xi^2 + b_2\xi \ln\xi + b_1\xi + b_0, \text{ if } r = 1/3,$$

where $b_j (j = 0,1,2,3,4)$ are the unknown coefficients which should be determined according to the boundary conditions of the beam. It should be noted that there are five unknowns $b_j (j = 0,1,2,3,4)$ in each shape function of Eq. (6), nevertheless, the boundary conditions such as simply-supported, clamped-clamped, or cantilever, have only 4 imposed equations on the boundary points $\xi = \alpha$ and $\xi = 1$, therefore, there is no way we can find out the unique solution of the system, i.e., these 5 coefficients are mutually dependent to each other. Since vibration mode shapes are actually describing the shape of vibration, the magnitude doesn't really matter, thus we can set one of the coefficients to be basic parameter or unity, and resolve the other four coefficients in terms of the prior one. As a result, after substituting the boundary conditions into Eq. (6), the value for each coefficient in the shape function $\varphi(\xi)$ with respect to various boundary conditions can be obtained, and the fundamental shape function can thus be constructed accordingly.

Next, by using the acquired fundamental shape function as the first member of our new set, we can proceed to generate an orthogonal set of admissible functions based on the following three-term recurrence relation as shown in Refs. [Chihara (1978); Bhat (1987)],

$$\phi_1(\xi) \equiv \varphi(\xi), \tag{7}$$

$$\phi_2(\xi) = (\xi - B_2)\phi_1(\xi), \tag{8}$$

$$\phi_k(\xi) = (\xi - B_k)\phi_{k-1}(\xi) - C_k\phi_{k-2}(\xi), \text{ for } k = 3,4,5,\cdots \tag{9}$$

where the coefficients B_k and C_k can be evaluated by imposing the orthogonality requirement,

$$\int_{\alpha}^{1} w(\xi)\phi_k(\xi)\phi_l(\xi)\, d\xi = \begin{cases} 0, & \text{if}\, k \neq l \\ \|\phi_k\| \cdot \delta_{kl}, & \text{if } k = l \end{cases}, \tag{10}$$

and found to be

$$B_k \equiv \left[\int_{\alpha}^{1} \xi w(\xi)\phi_{k-1}^{2}(\xi)d\xi\right] / \left[\int_{\alpha}^{1} w(\xi)\phi_{k-1}^{2}(\xi)d\xi\right], \tag{11}$$

$$C_k \equiv \left[\int_{\alpha}^{1} \xi w(\xi)\phi_{k-1}(\xi)\phi_{k-2}(\xi)d\xi\right] / \left[\int_{\alpha}^{1} w(\xi)\phi_{k-1}^{2}(\xi)d\xi\right] \tag{12}$$

It should be noticed that the function $w(\xi)$ appeared in the above equations is the weighting function which will be chosen to be the thickness variation of the tapered beam in the present paper, i.e., $w(\xi) \equiv \xi^r$, and the symbol $\|\phi_k\|$ represents the measured norm of the shape function defined in the Lebesgue space L^2, i.e.,

$$\|\phi_k\| \equiv \left(\int_{\alpha}^{1} w(\xi)\phi_{k}^{2}(\xi)\, d\xi\right)^{1/2}. \tag{13}$$

Even though this procedure was originally designed for developing the Boundary Characteristic Orthogonal Polynomials (BCOPs), yet it can be easily detected that the determination for B_k and C_k as well as the k-th member of the set, $\phi_k(\xi)$, can be handily performed as long as the first two members are successfully constructed no matter whether they are polynomials or not.

To this point, a new set of orthogonal tapered beam functions has been created based on the static tapered beam function and the three-term recurrence relation, moreover, a further need for normalization can also be achieved just as easy as dividing all the members by their measured norm. Before approaching to the next section for applying this new set to the vibration problem of non-uniform rectangular plates, some diagrams about the new generated functions are depicted for the first three modes with respect to various boundary conditions.

Fig. 3-10 present the first three members of the orthogonal tapered beam functions (OTBFs) for simply-simply supported (S-S), clamped-clamped (C-C) and free-clamped (F-C) beam with several varying thickness r and truncation factor $\alpha = 0.1$. In particular, the first three members of the original static tapered beam functions are reproduced for the linear varying thickness cases as presented in Figure 4, Figure 9 and Figure10 in order to further clarify the differences of these two admissible functions. As we compare the deflection shape of the OTBFs with the original static tapered beam functions presented in Ref. [Zhou (2002)], it can be detected that the first mode shapes for each set are actually identical to each other, which is reasonable because we adopt the static tapered beam function of order 0 to be our first member. However, ever since the 2^{nd} order one, corresponding to 1^{st} order of the static tapered beam functions, the newly invented OTBFs start to deform into several curved concavities while the original one still remain a single concave up or down.

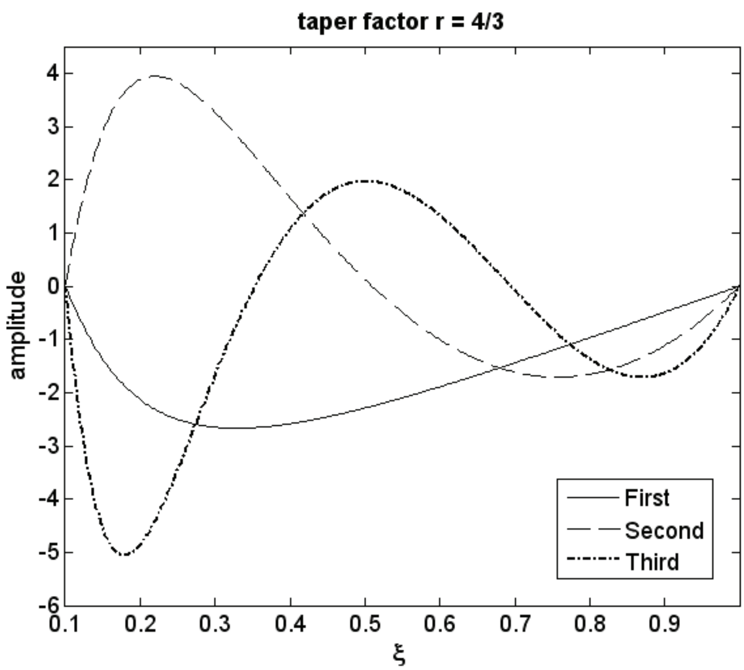


Figure 3: The first three members of the orthogonal tapered beam functions for simply-supported (S-S) beam with varying thickness ($r = 4/3$) and truncation factor $\alpha = 0.1$.

Physical meaning of mode shapes is the indication that structure deformed shape should be alike during the oscillation; therefore, for higher mode of free vibration, it is logically more rational to take the orthogonal tapered beam functions (OTBFs) as the shape functions of the vibration structure. Besides, the orthogonality, or orthomonality, could further enhance the OTBFs with the power of greatly reducing the computation amount due to the fact that only the generalized stiffness matrix should be evaluated; the off-diagonal elements of generalized mass are in fact all

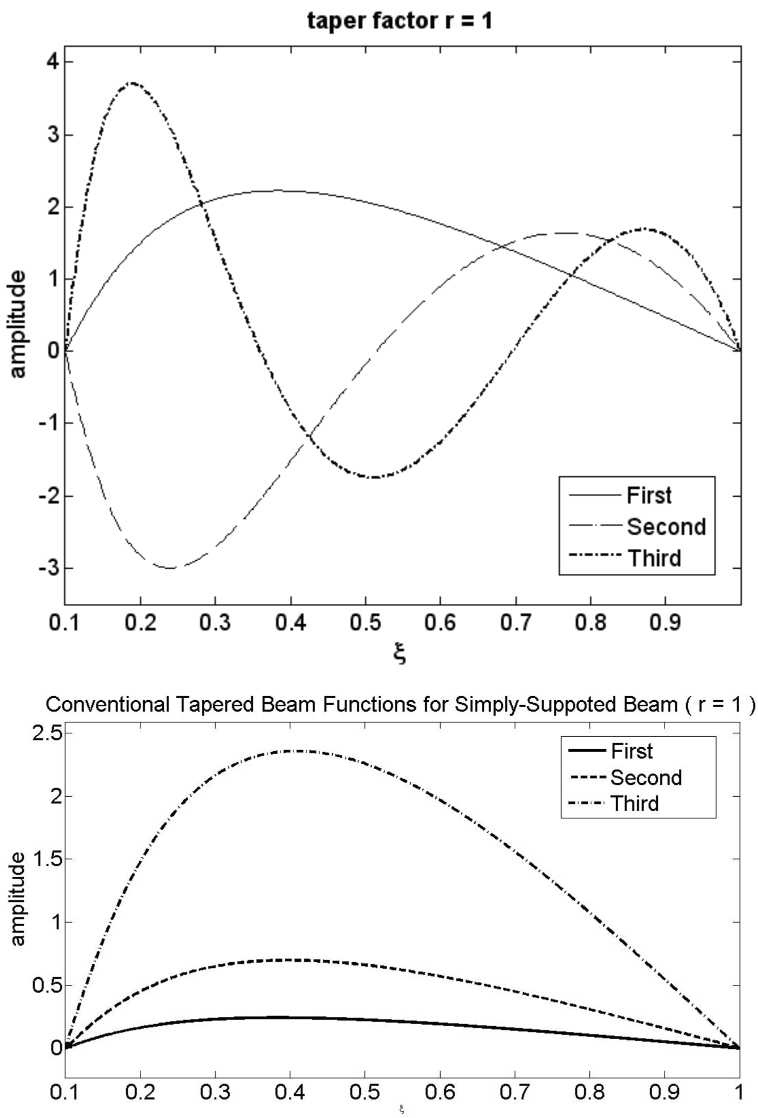


Figure 4: The first three members of the orthogonal tapered beam functions and the conventional tapered beam functions for simply-supported (S-S) beam with linear varying thickness ($r = 1$) and truncation factor $\alpha = 0.1$.

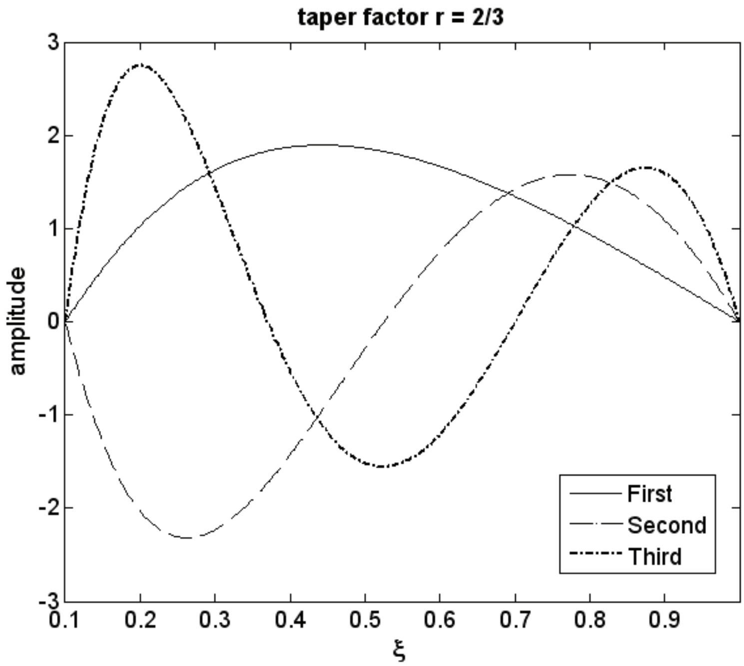


Figure 5: The first three members of the orthogonal tapered beam functions for simply-supported (S-S) beam with varying thickness ($r = 2/3$) and truncation factor $\alpha = 0.1$.

vanished. With the above OTBFs as the approximate mode shapes of structure free vibration, it is shown in the next section that the frequency equation of a wide range of non-uniform rectangular plate can be simplified, and the frequency parameters can be proved to be accurate as well.

3 Frequency Equations for Non-uniform Rectangular Plates

Consider a rectangular plate with non-uniform thickness $h = h(x,y)$ as depicted in Fig. 11, the plate is described in a system of rectangular Cartesian co-ordinates (x,y,z) with middle plane being $z = 0$ and is bounded by four edges: $x = \alpha a$, $x = a$, $y = \beta b$, and $y = b$, where $\alpha(0 < \alpha \leq 1)$ and $\beta(0 < \beta \leq 1)$ are referred to as the truncation factors of the plate along x- and y- axes. The differential equation governing the transverse free vibrations of such a non-uniform orthotropic rectangular

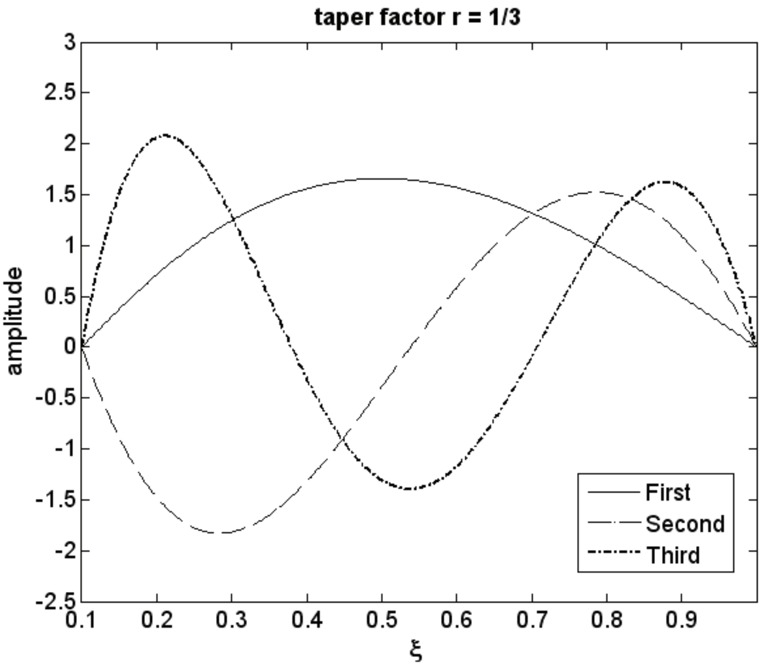


Figure 6: The first three members of the orthogonal tapered beam functions for simply-supported (S-S) beam with varying thickness ($r = 1/3$) and truncation factor $\alpha = 0.1$.

plate is given by

$$\frac{\partial^2}{\partial x^2}\left[D(x,y)\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\right]+\frac{\partial^2}{\partial y^2}\left[D(x,y)\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\right] \\ +2(1-\nu)\frac{\partial^2}{\partial x\partial y}\left(D(x,y)\frac{\partial^2 w}{\partial x\partial y}\right)+\rho_p h(x,y)\frac{\partial^2 w}{\partial t^2}=0, \tag{14}$$

herein $D=\frac{1}{12}\frac{Eh^3(x,y)}{1-\nu^2}$ is the flexural rigidity, $w\equiv w(x,y,t)$ represents the transverse deflection of the middle plane, ν is the Poison's ratio, ρ is the mass density of the plate and t is the time variable. Suppose the thickness $h(x,y)$ of the plate can be described as a power function, i.e.

$$h(x,y)=h_0(\frac{x}{a})^r(\frac{y}{b})^s=h_0H(x,y), \tag{15}$$

where h_0 stands for the thickness of the plate at the point $x=a$ and $y=b$, while r

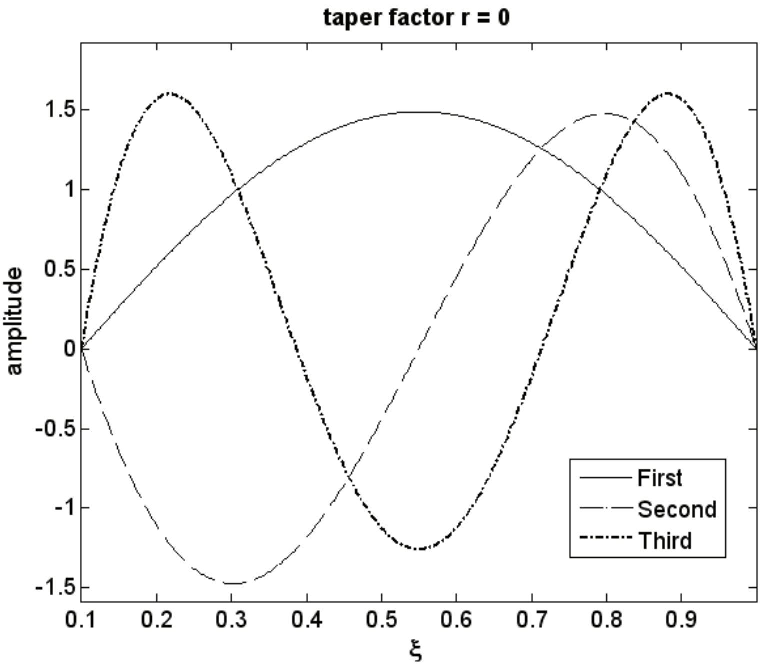


Figure 7: The first three members of the orthogonal tapered beam functions for simply-supported (S-S) beam with uniform thickness ($r = 0$) and truncation factor $\alpha = 0.1$.

and s are referred as the taper factors of the plate in the x and y directions respectively. Since only the truncated plate is under investigation, it should be noted that the taper factor r or s could be any real number in the present analysis.

In seeking for the harmonic solution to Eq. (14), Levy approach for the transverse deflection is assumed to be of the following form

$$w(x,y,t) = W(x,y)e^{i\omega t} \tag{16}$$

where $W(x,y)$ is the spatial deformation of the middle plane, ω is the oscillation frequency, t is the time variable and $i = \sqrt{-1}$. After inserting Eq. (16) into Eq. (14), the time-dependent governing equation can be easily transformed into a differential equation dependent on spatial variables x and y but independent of time variable, which can be abbreviated in the following form,

$$L(W;x,y) - \rho_P h_0 H(x,y)\omega^2 W = 0, \tag{17}$$

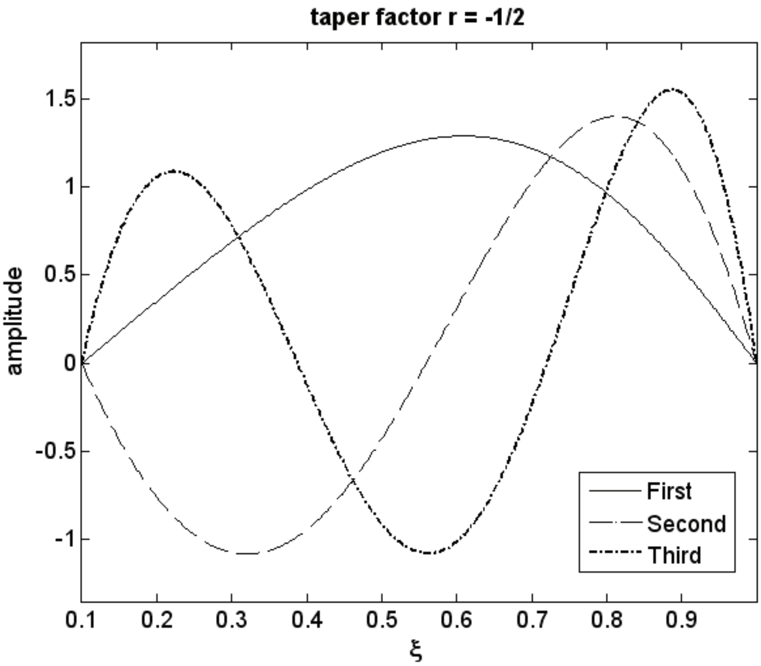


Figure 8: The first three members of the orthogonal tapered beam functions for simply-supported (S-S) beam with negative varying thickness ($r = -1/2$) and truncation factor $\alpha = 0.1$.

where $L(W;x,y)$ represents a fourth-order differential operator with respect to x and y only. By introducing the dimensionless variables $\xi \equiv x/a$ and $\eta \equiv y/b$, the maximum strain energy and the maximum kinetic energy of the plate can be written as

$$U_{\max} = \frac{b}{2a^3} D_0 \int_\alpha^1 \int_\beta^1 \xi^{3r} \eta^{3s} \left\{ \begin{array}{l} \left(\frac{\partial^2 W}{\partial \xi^2}\right)^2 + 2\gamma^2 \frac{\partial^2 W}{\partial \xi^2} \frac{\partial^2 W}{\partial \eta^2} + \gamma^4 \left(\frac{\partial^2 W}{\partial \eta^2}\right)^2 \\ -2(1-\nu)\gamma^2 \left[\frac{\partial^2 W}{\partial \xi^2} \frac{\partial^2 W}{\partial \eta^2} - \left(\frac{\partial^2 W}{\partial \eta^2}\right)^2 \right] \end{array} \right\} d\eta d\xi \tag{18}$$

$$T_{\max} = \frac{ab}{2} \rho h_0 \omega^2 \int_\alpha^1 \int_\beta^1 \xi^r \eta^s W^2(x,y) d\eta d\xi \tag{19}$$

in which $D_0 \equiv E h_0^3/[12(1-\nu^2)]$ represents the flexural rigidity of the plate at the location $\xi = 1$, $\eta = 1, \gamma \equiv \frac{a}{b} = \Gamma \frac{(1-\beta)}{(1-\alpha)}$ and $\Gamma \equiv A/B$ denotes the aspect ratio of the

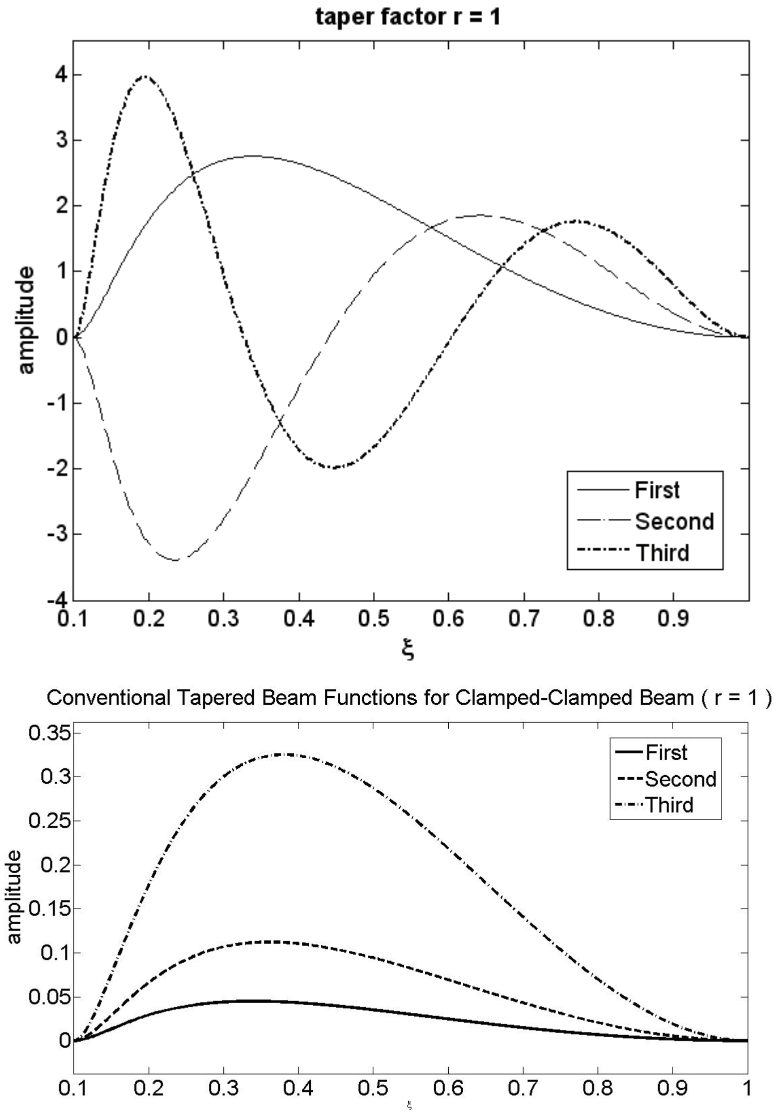


Figure 9: The first three members of the orthogonal tapered beam functions and the conventional tapered beam functions for clamped-clamped (C-C) beam with linear varying thickness ($r = 1$) and truncation factor $\alpha = 0.1$.

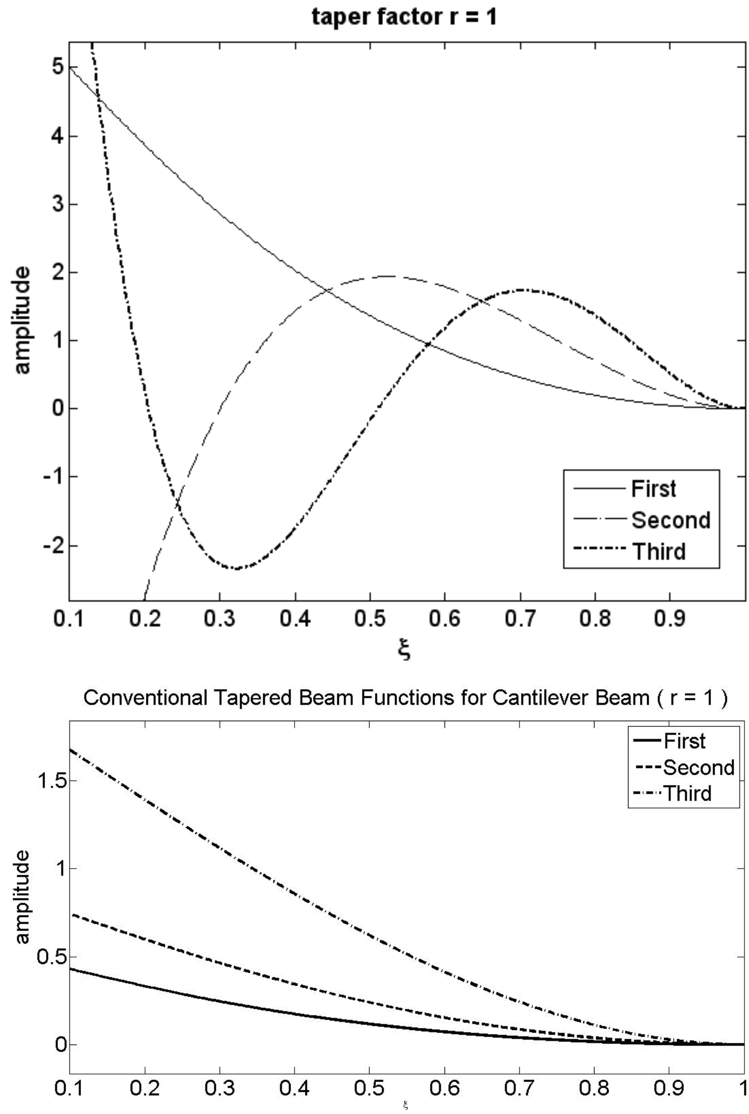


Figure 10: The first three members of the orthogonal tapered beam functions and the conventional tapered beam functions for free-clamped beam (F-C) with linear varying thickness ($r = 1$) and truncation factor $\alpha = 0.1$.

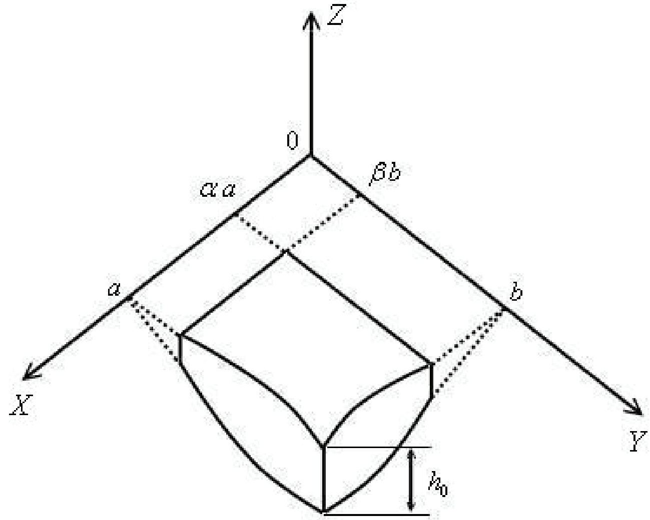


Figure 11: Rectangular plate with non-uniform thickness $h = h(x,y)$

plate. By adopting the technique of separation of variables, the spatial deformation can be expressed in terms of an infinite series as

$$W(x,y) \equiv W(\xi,\eta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\phi_m(\xi)\psi_n(\eta), \quad (20)$$

where $\phi_m(\xi)$ and $\psi_n(\eta)$ are the appropriate admissible functions that should satisfy at least the geometric boundary conditions, and if possible, all the boundary conditions. The unknown coefficients A_{mn} are indicating the distribution of each mode shape to the whole system and could be determined by the implementation of Galerkin's method. In this paper, in order to practice the usage of orthogonal tapered beam functions, the admissible functions are chosen to be the same as the newly generated functions mentioned in previous section.

Rayleigh-Ritz method requires the substitution of Eq. (20) into equations (18) and (19), then minimizing the total potential energy with respect to the coefficient A_{mn}, i.e., $\frac{\partial}{\partial A_{mn}}(U_{\max} - T_{\max}) = 0$. The procedure will result in the following eigenfrequency equation

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[C_{mnkl} - \lambda^2\overline{E}_{mk}\overline{F}_{nl}\right]A_{mn} = 0, \quad (21)$$

for each $k,l = 1,2,3,\ldots$, where $\lambda^2 \equiv \rho_P h_0 \omega^2 a^4 / D_0$ denotes the dimensionless frequency parameters, $\overline{E}_{mk} \equiv \int_\alpha^1 \xi^r \varphi_m \varphi_k d\xi = \delta_{mk}$, and $\overline{F}_{nl} \equiv \int_\beta^1 \eta^s \psi_n \psi_l d\eta = \delta_{nl}$, can be regarded as the generalized mass, and the corresponding generalized stiffness can be written as [Singh and Chakraverty (1991)]

$$C_{mnkl} \equiv E_{mk}^{(2,2)} F_{nl}^{(0,0)} + 2\gamma^2(1-\nu) E_{mk}^{(1,1)} F_{nl}^{(1,1)} + \gamma^4 E_{mk}^{(0,0)} F_{nl}^{(2,2)} + \nu\gamma^2 \left(E_{mk}^{(0,2)} F_{nl}^{(2,0)} + E_{mk}^{(2,0)} F_{nl}^{(0,2)} \right), \quad (22)$$

in which

$$E_{mk}^{(p,q)} \equiv \int_\alpha^1 \xi^{3r} \left(\frac{d^p \phi_m}{d\xi^p} \right) \left(\frac{d^q \phi_k}{d\xi^q} \right) d\xi, \quad (23)$$

$$F_{nl}^{(p,q)} \equiv \int_\beta^1 \eta^{3s} \left(\frac{d^p \psi_n}{d\eta^p} \right) \left(\frac{d^q \psi_l}{d\eta^q} \right) d\eta. \quad (24)$$

It should be noted that the orthogonal tapered beam functions are taken as the admissible functions in the present study, i.e., $\phi_m(\xi)$ is the mth orthogonal tapered beam function generated by considering an isotropic tapered beam with tapered factor r and truncation factor α along the x direction, and $\psi_n(\eta)$ is the corresponding one with tapered factor s and truncation factor β along the y direction.

Approximately the infinite series expression stated in Eq. (21) can be truncated into a finite number of double summations, i.e., $m = 1,2,\ldots,M$ and $n = 1,2,\ldots,N$, if M and N are chosen to be large enough. In so doing, the unknown coefficients A_{mn} can be presented in a matrix form with $M \times N$ components, therefore, in order to solve the unknowns, k will also need to be looped from 1 to M and l from 1 to N respectively. If we define $I \equiv (k-1) \times N + l$

and $J \equiv (m-1) \times N + n$, thus Eq. (21) can be further transformed into

$$\left[C(I,J) - \lambda^2 \delta(I,J) \right] A(J) = 0 \quad (25)$$

for $I = 1,2,\ldots,M \times N$, $J = 1,2,\ldots,M \times N$, or simply in matrix form

$$\left[\mathbf{C}_{MN\times MN} - \lambda^2 \mathbf{I}_{M\times M} \otimes \mathbf{I}_{N\times N} \right] \mathbf{A}_{MN\times 1} = \mathbf{0}_{MN\times 1}, \quad (26)$$

where $\mathbf{A} \equiv [A_{11}\, A_{12}\, \ldots\, A_{1N}\, A_{21}\, A_{22}\, \ldots\, A_{2N}\, \ldots\, A_{M1}\, A_{M2}\, \ldots\, A_{MN}]^T$ is the column vector consist of all the unknowns A_{mn}, $\mathbf{I}$ is the identity matrix and $\otimes$ is the tensor product. The component of the matrix $\mathbf{C}$, $C(I,J)$, in Eq. (25) is the same as the one evaluated in Eq. (22) but, however, is re-arranged into a proper position according to the value of k, l, m and n. Eq. (25) or (26) is obviously a typical eigenvalue problem with λ^2 being the eigenvalue and $\mathbf{A}_{MN\times 1}$ being the eigenvector. Once the related values in matrix $\mathbf{C}_{MN\times MN}$ are determined, the eigenvalue problem can be solved by any commonly used analytical or numerical method, and thus the natural frequency ω, which is included in the eigenvalue λ, can be calculated as a result. Some examples for implementing the proposed methodology are presented in the next section.

4 Numerical Results and Discussions

In order to illustrate the applicability and validity of the proposed approach, numerical computations for the eigenvalue λ_i in Eq. (21) are accomplished according to various tapered rectangular plates with the Poisson's ratio being $\nu = 0.3$. The Simpson's composite algorithm for numerical integration with 50 partitions is adopted to evaluate the related quantities stated in equations (23)-(24), nevertheless the exact values of these integrations can also be obtained analytically.

Since the proposed set of admissible functions are orthonormal to each other, we can only compute the values of each component in the matrix $\mathbf{C}$, and then use any available method to find out the corresponding eigenvalues of the matrix, either analytically or numerically. For the convenience of brevity, four capital letters are used to represent the boundary conditions of the plate, the first two indicate the boundary conditions of the plate along x direction, while the other two stand for those in the y direction. It should be noted that some special cases that cause all the coefficients b_j of the beam function to be vanished, i.e., the beam is undergoing rigid body, will not be considered in the present study, those are respectively SF/FS/FF boundary conditions along either the x- or y- direction.

The results will be transferred into a dimensionless form as $\Omega_i \equiv \lambda_i(1-\alpha)^2$ and compared with those obtained from the other available literatures in order to observe the accuracy and discrepancy of the new proposed methodology. It should be noted that any desired accuracy may be theoretically achieved by simply increasing the number of the used terms for admissible functions, however, the convergence study shows that there is a limit for accuracy on increasing the numbers of used terms especially for those plates with higher truncation factors.

We begin with the example of uniform isotropic plates aiming to verify the special cases $r = s = 0$ of the present terminology; in Table 1 the first several frequencies of the uniform square plate by using the orthogonal tapered beam functions as shape functions are listed. As we can see from Table 1, the numerical results from the proposed method show a good agreement with those in parenthesis from previous study by the original tapered beam function Cheung and Zhou (1999). However, in this table, the convergent eigenvalue for the mode number other than first is not well approaching to the reference value carried out by Cheung and Zhou especially for the simply-supported plate. The possible reason that cause the slight difference may be falling into the fact that the admissible functions generated by using the aforementioned recurrence relation are not guaranteed to satisfy both the Dirichlet type and Neumann type boundary conditions. Nevertheless, although it depends on case by case whether the Neumann type boundary conditions will be satisfied, the Dirichlet ones will be definitely assured. Therefore, by enlarging the number of series summation it is expected that the values will be convergent to an acceptable results in the end.

Tables 2-3 are the fundamental frequencies of isotropic plate with linear thickness variations in one direction and two directions subjected to various truncation factors, respectively. It is confirmed that the numerical results based on the present study pretty match those computed by the original approach conducted by Cheung and Zhou (1999). And the phenomena that higher truncation factor got more rapid convergence is also detected. It can also be observed that some of the eigenvalues

for FCFC plate in Table 2 are even lower than what were stated in the parenthesis. Same situation can be found for the SCSC and CCSC plates in Table 3 as well. The slight discrepancy may be due to the machine efficiency and the precision used in numerical integration as well as the algorithm of finding the matrix eigenvalues. However, it should be emphasized again that the new proposed orthogonal tapered beam functions will ensure the orthogonality and orthonomality in evaluating the last two matrices of Eq. (21), as a result, this at least save us the computing time for $(M \times N)^2$ operations.

Since the present study classifies the tapered factors into four special catalogs, see Eq. (6), it is for reference that we put the fundamental eigenvalues of the cases other than linear variation onto Table 4 and Table 5. Four kinds of boundary conditions imposed around the plate are considered here, they are respectively SSSS, CCCC, FCFC and SCSC, meanwhile

truncation factors are chosen to be ranging from 1/10 to 1/2 and 5/7 as stated in the referenced papers. Table 4 presents the frequencies of an isotropic square plate varying in one side only, whereas Table 5 shows the corresponding results for plate thickness varying in both sides. As we can see from these two tables, the dimensionless natural frequencies for the non-uniform plate with either one-sided or two-sided thickness variation are both increasing with the truncation factors. It can also be detected that the dimensionless natural frequencies for both the tables are decreasing with respect to the taper factors. This means that with the width and length remaining the same, the thinner of the plate will result in the lower of the natural frequency, that is, the plate will be getting more vulnerable. Meanwhile, among those four boundary conditions imposed on the plate, we can find that the

Table 1: First four frequencies of uniform isotropic square plate ($r = s = 0$, $\alpha = \beta = 0$), the values in ( ) are from reference [Cheung and Zhou (1999)].

BCs	Terms	Mode number i						
		1	2	3	4	5	6	7
SSSS	1*1	19.7476						
	2*2	19.7476	53.2411	53.2411	84.6465			
	3*3	19.7414	53.2392	53.2392	84.6465	124.2875	124.2876	152.5129
	4*4	19.7414	49.3778	49.3778	78.9994	124.2875	124.2876	150.5758
		(19.743)	(49.354)	(49.354)	(78.971)	(98.733)		
CCCC	1*1	36.0001						
	2*2	36.0001	74.2986	74.2986	108.5945			
	3*3	35.9901	74.1863	74.1863	108.5945	137.3182	138.0942	168.8435
	4*4	35.9901	73.4199	73.4199	108.2585	137.3182	138.0942	168.8383
		(36.004)	(73.432)	(73.432)	(108.27)	(131.83)		
FCFC / CFCF	1*1	7.2790						
	2*2	6.9764	26.6719	32.8414	64.9934			
	3*3	6.9455	24.1386	26.9155	48.4034	87.5273	99.7031	114.3741
	4*4	6.9352	24.0499	26.7838	48.1234	65.0541	68.0327	87.4635
		(6.9465)	(24.029)	(26.673)	(47.757)	(63.023)		

Table 2: Fundamental frequencies of isotropic square plate with linearly tapered thickness in one direction ($r = 1,\ s = 0$), the values in ( ) are from reference [Cheung and Zhou (1999)].

Boundary Conditions	Terms m*n	Truncation factor 1/10	1/5	1/3	1/2	5/7
SSSS	1*1	10.3260	11.5345	12.9954	14.7333	16.9055
	2*2	9.8567	11.1934	12.7788	14.6232	16.8737
	3*3	9.6989	11.1244	12.7586	14.6196	16.8702
	4*4	9.6946	11.1087	12.7383	14.6048	16.8651
	(5*5)	(9.6919)	(11.107)	(12.739)	(14.607)	(16.867)
CCCC	1*1	16.2578	19.2826	22.6445	26.3488	30.6750
	2*2	16.0342	19.1280	22.5524	26.3042	30.6627
	3*3	16.0212	19.1187	22.5447	26.2965	30.6541
	4*4	16.0067	19.1077	22.5373	26.2926	30.6529
	(4*4)	(16.005)	(19.117)	(22.549)	(26.306)	(30.668)
CCSS	1*1	12.9553	15.4261	18.1644	21.1695	24.6664
	2*2	12.9250	15.4049	18.1516	21.1633	24.6647
	3*3	12.9240	15.4036	18.1498	21.1608	24.6614
	4*4	12.9136	15.3967	18.1455	21.1586	24.6608
	(4*4)	(12.912)	(15.403)	(18.153)	(21.167)	(24.671)
SSCC	1*1	15.4670	17.1270	19.1809	21.6642	24.8057
	2*2	14.3207	16.2771	18.6314	21.3811	24.7232
	3*3	13.8252	16.0360	18.5510	21.3671	24.7217
	4*4	13.8247	16.0175	18.5119	21.3334	24.7092
	(4*4)	(13.814)	(16.008)	(18.512)	(21.341)	(24.719)
FCFC	1*1	5.8663	5.7268	5.7735	6.0154	6.4888
	2*2	5.4990	5.3975	5.4980	5.7821	6.2571
	3*3	5.4877	5.3835	5.4805	5.7620	6.2353
	4*4	5.4842	5.3799	5.4765	5.7571	6.2282
	(4*4)	(5.5227)	(5.4136)	(5.5042)	(5.7781)	(6.2436)

clamped around one will always attain the highest eigenfrequency, followed by the SCSC case and the simply-supported one, the cantilever plate along both sides reaches the lowest frequency. Since natural frequency is proportional to the structures rigidity, it is quite reasonable that the clamped plate is more rigid than the others and the cantilever plate plays the least role. If we compare the results on these two tables, it is clear that the frequency parameters for one-sided varying plate are always larger than those for the two-sided one, this phenomena is also

Table 3: Fundamental frequencies of isotropic rectangular plate with linearly tapered thickness in both directions ($r = 1$, $s = 1$), the values in ( ) are from reference [Cheung and Zhou (1999)].

Boundary Conditions	Terms	Truncation factor $\alpha = \beta$				
		1/10	1/5	1/3	1/2	5/7
SSSS	1*1	5.6267	6.8549	8.5974	11.0035	14.4734
	2*2	5.1199	6.4395	8.3033	10.8366	14.4187
	3*3	4.9196	6.3445	8.2755	10.8341	14.4172
	4*4	4.9043	6.3263	8.2488	10.8120	14.4086
		(4.9012)	(6.3235)	(8.2490)	(10.814)	(14.410)
CCCC	1*1	7.5106	10.4022	14.2696	19.2911	26.1384
	2*2	7.2187	10.1913	14.1364	19.2215	26.1167
	3*3	7.1690	10.1721	14.1280	19.2152	26.1093
	4*4	7.1529	10.1605	14.1188	19.2095	26.1073
		(7.1689)	(10.180)	(14.137)	(19.225)	(26.122)
SSSC	1*1	6.3853	8.0080	10.2296	13.2086	17.4099
	2*2	5.8879	7.6125	9.9611	13.0640	17.3616
	3*3	5.6713	7.5069	9.9273	13.0591	17.3613
	4*4	5.6502	7.4924	9.9053	13.0404	17.3542
		(5.6508)	(7.4916)	(9.9075)	(13.045)	(17.359)
SSCC	1*1	7.0874	9.2043	12.0617	15.8455	21.1314
	2*2	6.5136	8.7104	11.6960	15.6305	21.0594
	3*3	6.2570	8.5679	11.6425	15.6202	21.0582
	4*4	6.2340	8.5529	11.6156	15.5940	21.0470
		(6.2352)	(8.5537)	(11.620)	(15.602)	(21.056)
SCSC	1*1	6.3250	8.3293	11.0616	14.6830	19.7207
	2*2	6.0481	8.1540	10.9791	14.6634	19.7195
	3*3	5.9969	8.1394	10.9753	14.6613	19.7177
	4*4	5.9820	8.1262	10.9632	14.6538	19.7151
		(5.9928)	(8.1347)	(10.972)	(14.662)	(19.723)
CCSC	1*1	6.8981	9.3359	12.6398	16.9956	23.0299
	2*2	6.6090	9.1359	12.5268	16.9516	23.0247
	3*3	6.5577	9.1185	12.5204	16.9472	23.0203
	4*4	6.5421	9.1052	12.5082	16.9391	23.0173
		(6.5551)	(9.1185)	(12.521)	(16.951)	(23.029)

rational because the plate thickness of the latter one is reduced.

Table 4: Fundamental frequencies of isotropic square plate with other tapered thickness variation in one direction ($s = 0$), number of summation is taken to be M=N=3.

Taper Factor	BCs	Truncation factor $\alpha = \beta$				
		1/10	1/5	1/3	1/2	5/7
	SSSS	15.4945	16.2262	17.0138	17.8333	18.7263
	CCCC	27.6491	29.2537	30.8513	32.4407	34.1218
	FCFC	6.0999	6.1737	6.3061	6.4778	6.6883
	SCSC	21.1527	22.1877	23.2936	24.4351	25.6687
	SSSS	12.2259	13.4045	14.7111	16.1346	17.7705
	CCCC	21.1025	23.6893	26.3966	29.2186	32.3445
	FCFC	5.7022	5.6967	5.8353	6.0885	6.4523
	SCSC	16.5049	18.2021	20.0663	22.0749	24.3531
	SSSS	7.8600	9.3125	11.1088	13.2684	16.0220
	CCCC	12.2027	15.4116	19.2302	23.6505	29.0467
	FCFC	5.2842	5.1336	5.1969	5.4840	6.0355
	SCSC	10.0957	12.2798	14.9166	18.0340	21.9273

In the following examples, we will focus on discussing the plate characteristics for the most commonly used boundary conditions in the real world, that is, the natural frequencies of plate structures with four edges being simply-supported. The first four eigen-frequencies of non-uniform simply-supported square plate with several thickness variations along x direction subjected to various truncation factors are given in Table 6. Three terms of the orthogonal tapered beam functions are used in each direction, four kinds of special cases in thickness variations are discussed and the truncation factors along both sides are assumed to be the same. It can be shown that the eigenfrequency parameters will be increasing with respect to the truncation factors, yet be decreasing with respect to the taper factors. These phenomena have also been pointed out by previous researchers while adopting the original admissible functions. The first four eigen-frequencies of non-uniform simply-supported square plate with several thickness variations in both directions and with various truncation factors are given in Table 7. Three terms of series summation are again implemented, four kinds of thickness variations are also investigated with taper factors and truncation factors set to be the same in both sides. As we can see from these two tables, results obtained in the present study can be compared with those

Table 5: Fundamental frequencies of isotropic square plate with other tapered thickness variation in both directions ($r = s$), number of summation is taken to be M=N=3.

Taper factor	BCs	Truncation factor $\alpha = \beta$				
		1/10	1/5	1/3	1/2	5/7
	SSSS	12.1713	13.3405	14.6641	16.1098	17.7634
	CCCC	21.2088	23.7670	26.4432	29.2408	32.3505
	FCFC	5.1481	5.3538	5.6501	6.0067	6.4313
	SCSC	16.5409	18.1893	20.0461	22.0615	24.3487
	SSSS	7.6034	9.1151	10.9671	13.1878	15.9965
	CCCC	12.3630	15.5877	19.3586	23.7208	29.0682
	FCFC	3.9920	4.2321	4.6482	5.2149	5.9594
	SCSC	10.0188	12.1974	14.8454	17.9884	21.9114
	SSSS	3.3525	4.5524	6.3338	8.9434	13.0058
	CCCC	4.0836	6.6018	10.2835	15.5450	23.4432
	FCFC	1.8524	2.4777	3.1276	3.9437	5.1242
	SCSC	3.4749	5.3898	8.0979	11.9428	17.7424

from other available literatures, see Ref. [Cheung and Zhou (1999)] and Ref. [Bert and Malik (1996)], and good agreement has been observed.

In order to see the effects of the aspect ratio on the frequency parameters of the isotropic plate, the first four frequencies of simply-supported isotropic rectangular plate with varying thickness in one direction are tabulated in Table 8, followed by the corresponding ones with varying thickness in both directions presented on Table 9. In Table 8, values stated in parentheses are comparable data cited from Ref. [Cheung and Zhou (1999)] and for the first few frequencies, accuracy can be observed. The phenomena of increasing with truncation factors and decreasing with respect to taper factors are detected again. Furthermore, if we fix the taper factor to be unary and the truncation factor to be 5/7, the fundamental frequency for the rectangular plate with aspect ratio being 1/2 is less than the corresponding one with aspect ratio being 1 and the case of aspect ratio being 2 can be found to be much higher than the other two. Since the taper factor along y-direction for the case in Table 8 is set to be $s = 0$, dissymmetry of the values is expected, and the larger aspect ratio can be interpreted as the wider plate, which will be more rigid and receive higher frequency as a result.

Table 6: First four frequencies of non-uniform isotropic SSSS square plate ($\alpha = \beta$) with tapered thickness variation in one direction ($s = 0$), values in [] are from [Cheung and Zhou (1999)]., in () are from [Bert and Malik (1996)], number of summation is taken to be M=N=3.

r	$\alpha = \beta$	Ω_1	Ω_2	Ω_3	Ω_4
4/3	1/10	7.8600	17.1150	18.9695	32.7278
	1/5	9.3125	21.6105	22.4750	40.2571
	1/3	11.1088	27.4877	27.5242	46.6331
	1/2	13.2684	34.1580	34.4719	55.6566
	5/7	16.0220	42.6412	42.8230	68.1400
1	1/10	9.6989	23.0237	23.4524	42.0503
	1/5	11.1244 [11.107]	27.4039	27.5582	46.8568
	1/3	12.7586 [12.729]	32.3933	32.8521	53.3766
	1/2	14.6196 [14.607]	38.2578	38.6615	61.6501
	5/7	16.8702 [16.863] (16.864)	45.1311	45.2828	71.9619
2/3	1/10	12.2259	30.2777	31.1693	51.1780
	1/5	13.4045	34.0505	34.9173	55.9981
	1/3	14.7111	38.3063	38.9656	61.8772
	1/2	16.1346	42.8215	43.1876	68.4959
	5/7	17.7705	47.7299	47.8382	75.9904
1/3	1/10	15.4945	40.0683	41.4127	64.8279
	1/5	16.2262	42.6763	43.5432	68.4882
	1/3	17.0138	45.2882	45.7728	72.3268
	1/2	17.8333	47.8283	48.0465	76.1792
	5/7	18.7263	50.4339	50.4901	80.2199

The fundamental frequencies of simply-supported rectangular plate with varying thickness in both sides are listed in Table 9 for three kinds of aspect ratios subjected to various taper factors and truncation factors. As we take a glance on the table, it can be easily found that the values are actually not symmetric to the aspect ratio as we originally predicted, however, this is definitely not wrong. Due to the definition of non-dimensional frequency parameters, $\lambda^2 \equiv \rho_P h_0 \omega^2 a^4 / D_0$, we can see that only the plate length along x- direction, a, is included, i.e., we should take the value of a into account when the dimensional frequency parameters, ω, is under consideration. For example, we can deal with two identical non-uniform rectangular plates, one has the dimensions $a = \ell$, $b = \ell/2$ with $\gamma = 1/2$, and the other is $a = 2\ell$, $b = \ell$ with $\gamma = 2$, their taper factors and truncation factors on both sides are set to be the same as shown in Table 9. It can be easily derived

Table 7: First four frequencies of non-uniform isotropic SSSS square plate ($\alpha = \beta$) with tapered thickness variation in both directions ($r = s$), values in [] are from [Cheung and Zhou (1999)], numbers of summation: M=N=3.

$r=s$	$\alpha=\beta$	Ω_1	Ω_2	Ω_3	Ω_4
4/3	1/10	3.3525	6.0558	7.5857	13.8920
	1/5	4.5524	8.6559	10.5553	18.0876
	1/3	6.3338	13.4901	15.4118	25.0859
	1/2	8.9434	21.4758	22.8996	36.3803
	5/7	13.0058	34.0234	34.5859	54.8280
1	1/10	4.9196 [4.9012]	9.5985	11.6320	19.7145
	1/5	6.3445 [6.3235]	13.4407	15.4769	25.2089
	1/3	8.2755 [8.2490]	19.2641	20.9399	33.3217
	1/2	10.8341 [10.814]	27.2898	28.3305	44.8035
	5/7	14.4172 [14.410]	38.2105	38.5667	61.1691
2/3	1/10	7.6034	16.9307	18.9421	30.3084
	1/5	9.1151	21.6124	23.2475	36.7327
	1/3	10.9671	27.5136	28.6242	45.1170
	1/2	13.1878	34.4488	35.0386	55.3993
	5/7	15.9965	42.7996	42.9771	68.2173
1/3	1/10	12.1713	30.6910	31.7112	49.5411
	1/5	13.3405	34.5749	35.2575	55.3728
	1/3	14.6641	38.7427	39.1377	61.7878
	1/2	16.1098	43.0717	43.2561	68.5565
	5/7	17.7634	47.8048	47.8544	76.0245

that the dimensional natural frequency can be retrieved as $\omega^2 \equiv \lambda^2 D_0/\rho_P h_0 a^4$, and thus for the first case we have $\omega \equiv \lambda_{\gamma=\frac{1}{2}}\sqrt{D_0}/\sqrt{\rho_P h_0}\ell^2$, for the second one, it is $\omega \equiv \lambda_{\gamma=2}\sqrt{D_0}/\sqrt{\rho_P h_0}\,(2\ell)^2$, it's quite obviously that the relation $\lambda_{\gamma=\frac{1}{2}} = \lambda_{\gamma=2}/4$ should be hold since the shapes of these two plates are actually alike. As we can observe from Table 9, the value of dimensionaless frequency parameters λ for $\gamma = 2$ is always 4 times the value of that for $\gamma = 1/2$, with all the cases subjected to various taper factors and truncation factors, this phenomena directly confirm the validity and symmetry.

5 Conclusions

A new set of admissible functions which are based on the static transverse deformation of an isotropic tapered beam, and orthogonal to each other via recurrence

Table 8: First four frequencies of simply-supported isotropic rectangular plate with variable thickness in one direction ($s = 0$), values in ( ) are from reference [Cheung and Zhou (1999)], numbers of summation: M=N=3.

Γ(A/B)	r		Ω_1	Ω_2	Ω_3	Ω_4
1/2	1/2	1/10	8.5221	14.2757	25.9412	29.9249
		1/5	9.1626	15.2874	27.9228	32.9590
			(9.1607)	(14.7317)	(23.821)	(31.085)
		1/3	9.8557	16.3968	30.0558	36.1362
			(9.8543)	(15.808)	(25.623)	(33.474)
		1/2	10.5867	17.5808	32.2938	39.3206
			(10.586)	(16.957)	(27.527)	(35.978)
		5/7	11.3976	18.9077	34.7698	42.6502
			(11.397)	(18.241)	(29.639)	(38.745)
	1	1/10	5.6757	10.0462	17.2757	19.5629
		1/5	6.6822	11.5258	20.3068	23.2937
			(6.6773)	(11.107)	(17.566)	(22.748)
		1/3	7.8060	13.2220	23.7765	27.8679
			(7.8023)	(12.739)	(20.402)	(26.551)
		1/2	9.0529	15.1523	27.6125	33.1289
			(9.0507)	(14.607)	(23.589)	(30.782)
		5/7	10.5199	17.4858	32.0944	39.2016
			(10.518)	(16.867)	(27.371)	(35.766)
1	1/2	1/10	13.7604	34.7559	36.0576	57.3072
		1/5	14.7411	38.1057	39.0862	61.7788
			(14.732)	(36.275)	(36.591)	(58.765)
		1/3	15.8149	41.6663	42.2856	66.8556
			(15.808)	(39.207)	(39.391)	(64.146)
		1/2	16.9596	45.2716	45.5755	72.2357
		5/7	18.2412	49.0690	49.1520	78.0803
	1	1/10	9.6989	23.0237	23.4524	42.0503
		1/5	11.1244	27.4039	27.5582	46.8568
			(11.107)	(25.773)	(26.960)	(46.569)
		1/3	12.7586	32.3933	32.8521	53.3766
			(12.729)	(30.588)	(31.353)	-
		1/2	14.6196	38.2578	38.6615	61.6501
		5/7	16.8702	45.1311	45.2828	71.9619
2	1/2	1/10	33.5456	55.1405	111.4062	113.8784
		1/5	36.3119	59.5704	116.7079	127.2079
			(36.274)	(58.765)	(116.05)	(148.30)
		1/3	39.2433	64.5577	123.0805	141.5505
			(39.206)	(63.146)	(128.96)	-
		1/2	42.2694	69.8097	130.4886	155.8090
		5/7	42.2694	69.8097	130.4886	155.8090
	1	1/10	21.6750	40.1536	65.1472	95.8089
		1/5	25.8256	44.9216	81.3052	101.0557
			(25.773)	-	-	(106.39)
		1/3	30.6449	51.3710	102.0052	107.8127
			(30.587)	-	-	-
		1/2	35.9379	59.4989	116.9509	126.5482
		5/7	42.0066	69.5531	130.2687	154.7741

Table 9: Fundamental frequencies of simply supported isotropic rectangular plate with variable thickness in both directions ($r = s$), the values in ( ) are from reference [Cheung and Zhou (1999)], numbers of summation: M=N=3.

Γ(A/B)	r	Truncation factor $\alpha = \beta$				
		1/10	1/5	1/3	1/2	5/7
1/2	4/3	1.7091	2.3994	3.5547	5.3215	8.0303
	1	2.6363	3.5672	4.8735	6.5984	8.9541
	2/3	4.4333	5.4562	6.7001	8.1634	9.9748
	1/2	5.8029	6.7454	7.8377	9.0671	10.5238
	1/3	7.5445	8.2993	9.1442	10.0594	11.1000
1	4/3	3.3525	4.5524	6.3338	8.9434	13.0058
	1	4.9196	6.3445	8.2755	10.8341	14.4172
		(4.9012)	(6.3235)	(8.2490)	(10.814)	(14.410)
	2/3	7.6034	9.1151	10.9671	13.1878	15.9965
	1/2	9.6027	11.0117	12.6707	14.5700	16.8551
	1/3	12.1713	13.3405	14.6641	16.1098	17.7634
2	4/3	6.8366	9.5974	14.2190	21.2859	32.1211
	1	10.5453	14.2686	19.4940	26.3937	35.8166
	2/3	17.7331	21.8250	26.8002	32.6538	39.8990
	1/2	23.2115	26.9816	31.3509	36.2684	42.0950
	1/3	30.1779	33.1973	36.5768	40.2376	44.3999

relation is established for the free vibration of a wide range of non-uniform rectangular plates. The first member of the new set is classified into four categories according to the taper factor which directly indicates the thickness non-uniformity of the tapered beam, whereas the rest members are generated in a form that orthogonality with each other can be achieved. It can be found that this new set of orthogonal tapered beam functions can appropriately vary with the thickness variation of the plate and also simultaneously posses the important property of orthogonality, or furthermore, the orthonomality.

In practicing the proposed admissible function, rectangular plates with varying thickness along both sides and being truncated from the left edges are under investigation, in particular, the natural frequencies resulting from the implementation of Rayleigh-Ritz method are carried out in a dimensionless form. Numerical results with respect to various taper factors and truncation factors are tabulated and compared with those available from previous literature by using the non-orthogonal shape functions. In addition, aspect ratio of the rectangular plate is further examined in order to see the variation of frequency parameter with respect to different plate dimensions.

This paper presents a simple and systematic way to generate the set of orthogonal admissible functions which not only satisfy the boundary conditions but also matches the structure non-uniformity. The cost of evaluation and computation for the vibration characteristics of plates can thus be greatly reduced based on the orthogonal admissible functions. The basic concept of constructing the proposed admissible functions is clear, and the procedure for orthogonalization process is also well organized, indeed no complicated mathematical knowledge is required. Other application of the proposed methodology can be extended to the determination of approximate solutions for eigenvalue problems arose from structure vibrations, fluid mechanics and diffusion problems etc.

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